Mixed Integer Estimation and Validation for Next Generation GNSS

Peter J.G. Teunissen

Contents

Abstract

The coming decade will bring a proliferation of Global Navigation Satellite Systems (GNSS) that are likely to revolutionize society in the same way as the mobile phone has done. The promise of a broader multifrequency, multi-signal GNSS "system of systems" has the potential of enabling a much wider range of demanding applications compared to the current GPS-only situation.

P.J.G. Teunissen (✉)
Department of Spatial Sciences, Curtin University of Technology, Perth, Australia

Department of Geoscience and Remote Sensing, Delft University of Technology, Delft, Netherlands

W. Freeden et al. (eds.), *Handbook of Geomathematics*,
DOI 10.1007/978-3-642-54551-1_37

In order to achieve the highest accuracies, one must exploit the unique properties of the received carrier signals. These properties include the multi-satellite system tracking, the mm-level measurement precision, the frequency diversity, and the integer ambiguities of the carrier phases. Successful exploitation of these properties results in an accuracy improvement of the estimated GNSS parameters of two orders of magnitude. The theory that underpins this ultraprecise GNSS parameter estimation and validation is the theory of integer inference. This theory is the topic of the present chapter.

1 Introduction

1.1 Next Generation GNSS

Background

Global Navigation Satellite Systems (GNSSs) involve satellites, ground stations, and user receiver equipment and software to determine positions anywhere around the world at any time. The global positioning system (GPS) from the United States is the best-known and currently fully operational GNSS. Fueling growth during the next decade will be the next generation GNSSs that are currently being deployed and developed. Current and prospective providers of GNSS systems are the United States, Russia, the European Union, China, Japan, and India. The United States is modernizing its dual-frequency GPS. A third civil frequency will be added, with expected 24-satellite full constellation capability (FOC) around 2015. Russia is revitalizing its GLONASS system, from a current only partially functioning system to 24-satellite FOC reached by 2010. The European Union is developing a complete new multifrequency GNSS, called Galileo, which is currently in orbit validation phase and which will have its 30-satellite FOC by 2012. China is developing its own 30-satellite GNSS, called Compass, of which the first satellite was launched in April 2007. Finally, India and Japan are developing GNSS augmentation systems. India's 7-satellite IRNSS (Indian Regional Navigational Satellite System) is expected operational in 2012 and Japan will soon launch its first of three QZSS (Quasi-Zenith Satellite System) satellites. QZSS is designed to increase the number of satellites available at high-elevation angles over Japan.

Benefits of GNSS

The promise of a broader and more diverse system of GNSSs has enormous potential for improving the accuracy, integrity, and efficiency of positioning worldwide, the importance of which can hardly be overstated.

The availability of many more satellites and signals creates exciting opportunities to extend current GPS applications and to enable new applications in areas where the GPS-only situation has been a hindrance to market growth. Extending the operational range of precise carrier-phase GNSS, currently restricted to about

15 km, will allow instantaneous cm-level accuracy positioning at remote locations on land and offshore. Improved integrity will service various industries having high marginal costs (e.g., mining, agriculture, machine-guided construction). Single-frequency tracking of more satellites will create opportunities for the low-cost receiver market of precise real-time location devices, such as handheld or in moving vehicles. In addition, environmental- and spaceborne GNSS will benefit enormously from tracking multiple satellites on multiple frequencies. Environmental GNSS, in general, benefits from denser atmospheric profiling, while short-term weather prediction in particular benefits from a reduction in the latency of GNSS-integrated water vapor estimates. The benefits for spaceborne GNSS are highly accurate orbit determinations of Earth-orbiting space platforms, possibly even in real time, thus offering increased spacecraft autonomy, simplification of spacecraft operations, and support to rapid delivery of end-user data products such as atmospheric profiles from occultation or synthetic aperture radar images for deformation monitoring. An overview of this great variety of GNSS models and their applications can be found in textbooks like Parkinson and Spilker (1996), Strang and Borre (1997), Teunissen and Kleusberg (1998), Farell and Barth (1999), Leick (2004), Misra and Enge (2006), and Hofmann-Wellenhof et al. (2008).

Theory

Several key issues need to be addressed in order to achieve the fullest exploitation of the opportunities created by future GNSSs. The highest possible accuracies can only be achieved if one is able to exploit the unique properties of the received carrier signals. These properties include the mm-level precision with which the carrier phases can be tracked, the frequency diversity of the carriers, and the knowledge that certain functions of the carriers are integer valued. The process of exploiting these properties is known as integer ambiguity resolution (IAR). IAR improves the precision of the estimated GNSS model parameters by at least two orders of magnitude. For positioning, successful IAR effectively transforms the estimated fractional carrier phases into ultraprecise receiver-satellite ranges, thus making high-precision (cm- to mm-level) positioning possible. As a beneficial by-product, it also improves other GNSS model parameters, such as atmospheric parameters, and it enables reduction of GNSS parameter-estimation space, sometimes up to 50 %, thus simplifying computations considerably and accelerating the time to position. However, the success of IAR depends on the strength of the underlying GNSS model. The weaker the model, the more data needs to be accumulated before IAR can be successful and the longer it therefore takes before one can profit from the ultraprecise carrier signals. Clearly, the aim is to have short times-to-convergence, preferably zero, thereby enabling truly instantaneous GNSS positioning.

The theory that underpins ultraprecise GNSS parameter estimation is the theory of integer inference. This theory of estimation and validation is the topic of the present chapter. Although a large part of the theory has been developed since the 1990s for GPS, the theory has a much wider range of applicability.

1.2 Mixed Integer Model

Central in the theory of integer inference is the mixed integer model. To introduce this model, we use the GNSS pseudorange and carrier-phase observables as leading example. If we denote the j-frequency pseudorange and carrier phase for the $r-s$ receiver-satellite combination at epoch t as $p^s_{r,j}(t)$ and $\varphi^s_{r,j}(t)$, respectively, then their observation equations can be formulated as

$$
\begin{aligned}
p^s_{r,j}(t) &= \rho^s_r(t) + T^s_r(t) + \mu_j I^s_r(t) + c\,dt^s_r(t) + e^{s,p}_{r,j}(t), \\
\phi^s_{r,j}(t) &= \rho^s_r(t) + T^s_r(t) - \mu_j I^s_r(t) + c\,\delta t^s_r(t) + \lambda_j M^s_{r,j} + e^{s,\phi}_{r,j}(t),
\end{aligned} \tag{1}
$$

where ρ^s_r is the receiver-satellite range, T^s_r is the tropospheric delay, I^s_r is the ionospheric delay, dt^s_r and δt^s_r are the pseudorange and carrier-phase receiver-satellite clock biases, $M^s_{r,j}$ is the time-invariant carrier-phase ambiguity, c is the speed of light, λ_j is the j-frequency wave length, $\mu_j = (\lambda_j/\lambda_1)^2$, and $e^{s,p}_{r,j}$ and $e^{s,\varphi}_{r,j}$ are the remaining error terms, respectively. The real-valued carrier-phase ambiguity $M^s_{r,j} = \varphi_{r,j}(t_0) + \varphi^s_j(t_0) + N^s_{r,j}$ is the sum of the initial receiver-satellite phases and the integer ambiguity $N^s_{r,j}$.

Through differencing of the observation equations, one can eliminate the initial phases and the clock biases. The so-called double differenced (DD) observation equations then take the form

$$
\begin{aligned}
p^{ts}_{qr,j}(t) &= \rho^{ts}_{qr}(t) + T^{ts}_{qr}(t) + \mu_j I^{ts}_{qr}(t) + e^{ts,p}_{qr,j}(t), \\
\phi^{ts}_{qr,j}(t) &= \rho^{ts}_{qr}(t) + T^{ts}_{qr}(t) - \mu_j I^{ts}_{qr}(t) + \lambda_j N^{ts}_{qr,j} + e^{ts,\phi}_{qr,j}(t),
\end{aligned} \tag{2}
$$

where $p^{ts}_{qr,j}(t) = [p^s_{r,j}(t) - p^t_{r,j}(t)] - [p^s_{q,j}(t) - p^t_{q,j}(t)]$, with a similar notation for the other DD variates. The DD tropospheric slant delays are usually further reduced to a single DD vertical delay T^{vert}_{qr} by means of mapping functions. Furthermore, the need for having the ionospheric delays present depends very much on the baseline length between receivers. These delays can usually be neglected for distances less than 15 km.

If we assume the error terms $e^{ts,\phi}_{qr,j}(t)$ and $e^{ts,\phi}_{qr,j}(t)$ in Eq. 2 to be zero-mean random variables, the observation equations can be used to set up a linear model in which some of the unknown parameters are reals and others are integer $N^{ts}_{qr,j}$. Such a model is an example of a mixed integer linear model.

The observation equations of Eq. 2 are parametrized in the DD ranges $\rho^{ts}_{qr}(t)$. These ranges depend on the receiver positions and on the satellite positions. Assuming the satellite orbits are known, these ranges are usually further linearized with respect to the unknown receiver coordinates. As a result one obtains linearized equations that are parametrized in the between-receiver baseline vector increments. Such a model is an example of a mixed integer *linearized* model. These linearized GNSS models can usually be treated as if they are linear, since the nonlinearities are small.

We now define the general form of a mixed integer linear model.

Definition 1 (Mixed integer linear model). Let (A, B) be a given $m \times (n + p)$ matrix of full rank and let Q_{yy} be a given $m \times m$ positive definite matrix. Then

$$y \sim \mathrm{N}(Aa + Bb, Q_{yy}), a \in \mathbb{Z}^n, b \in \mathbb{R}^p \tag{3}$$

will be referred to as the mixed integer linear model.

The notation "$\sim$" is used to describe "distributed as." In a GNSS context, the m-vector y contains the pseudorange and carrier-phase observables, the n-vector a the integer DD ambiguities, and the real-valued p-vector b the remaining unknown parameters, such as baseline components (coordinates) and possibly atmospheric delay parameters (troposphere, ionosphere). As in most GNSS applications, the underlying distribution of the above mixed integer model is assumed to be a multivariate normal distribution. Various results in the following sections are also valid, however, for general distributions.

1.3 Chapter Overview

The mixed integer model (Eq. 3) is usually solved and validated in a number of steps. We now briefly present the contributions of this chapter in relation to these steps.

In the first step, the integer nature of a is discarded. The parameters a and b are estimated using least-squares (LS) estimation, which in the present case is equivalent to using maximum likelihood (ML) or best linear unbiased estimation (BLUE). As a result one obtains the so-called float solution:

$$\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix} \sim \mathrm{N}\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} Q_{\hat{a}\hat{a}} & Q_{\hat{a}\hat{b}} \\ Q_{\hat{b}\hat{a}} & Q_{\hat{b}\hat{b}} \end{bmatrix} \right). \tag{4}$$

In this first step, one usually also tests the data and GNSS model for possible model misspecifications, e.g., outliers, cycle slips, or other modeling errors. This can be done with the standard theory of hypothesis testing (Baarda 1968; Koch 1999; Teunissen 2006).

In the second step, a mapping $\mathcal{I} : \mathbb{R}^n \mapsto \mathbb{Z}^n$ is introduced that takes the integer constraints $a \in \mathbb{Z}^n$ into account:

$$\check{a} = \mathcal{I}(\hat{a}). \tag{5}$$

There are many ways in which the mapping $\mathcal{I}$ can be defined. In Sect. 2, we introduce three different classes of such estimators. They are the class of integer estimators (I), the class of integer aperture estimators (IA), and the class of integer equivariant estimators (IE). These three classes were introduced in Teunissen (1999a, 2003a,b). They are subsets of one another and related as

$$I \subset IA \subset IE. \tag{6}$$

Each class consists of a multitude of estimators. For each class we present the optimal estimator. As optimality criterion we either use the maximization of the probability of correct integer estimation or the minimization of the mean squared error.

Since estimators from the class of integer estimators are most often used, we present in Sect. 3 an analysis of the properties of the three most popular integer estimators. They are the estimators of integer rounding (IR), integer bootstrapping (IB), and integer least-squares (ILS). Special attention is given to computational issues and to their success-rates, i.e., the probabilities of correct integer estimation. It is shown that the performances of these three integer estimators are related as

$$P(\breve{a}_{\mathrm{IR}} = a) \leq P(\breve{a}_{\mathrm{IB}} = a) \leq P(\breve{a}_{\mathrm{ILS}} = a). \tag{7}$$

Knowing that a is integer strengthens the model and allows one, in principle, to re-evaluate the validation of the model as compared to the first step validation. However, the standard theory of hypothesis testing is then not applicable anymore. This validation problem will be addressed in Sect. 4, where we also present a cross-validation method for the mixed integer model.

In the final step, once $\breve{a}$ of Eq. 5 is computed and accepted, the float estimator $\hat{b}$ is readjusted to obtain the so-called fixed estimator

$$\breve{b} = \hat{b} - Q_{\hat{b}\hat{a}} Q_{\hat{a}\hat{a}}^{-1}(\hat{a} - \breve{a}). \tag{8}$$

Whether or not $\breve{b}$ is an improvement over $\hat{b}$ depends in a large part on the probabilistic properties of $\breve{a}$. We therefore present in Sect. 4 the probability density function (PDF) of $\breve{b}$ and show how it is influenced by the probability mass function (PMF) of $\breve{a}$.

2 Principles of Integer Inference

In this section, we present three different classes of integer parameter estimators. They are the integer estimators, the integer aperture estimators, and the integer equivariant estimators. Within each class we determine the optimal estimator.

2.1 Integer Estimation

Pull-In Regions

We start with the requirement that the estimator $\breve{a}$ needs to be integer, $\breve{a} = \mathcal{I}(\hat{a}) \in \mathbb{Z}^n$. Then $\mathcal{I} : \mathbb{R}^n \mapsto \mathbb{Z}^n$ is a many-to-one map, instead of a one-to-one map. Different real-valued vectors will be mapped to one and the same integer vector. One can therefore assign a subset, say $\mathcal{P}_z \subset \mathbb{R}^n$, to each integer vector $z \in \mathbb{Z}^n$,

$$\mathcal{P}_z = \{x \in \mathbb{R}^n | z = \mathcal{I}(x)\},\ z \in \mathbb{Z}^n. \tag{9}$$

This subset is referred to as the *pull-in region* of z. It is the region in which all vectors are pulled to the same integer vector z.

The concept of pull-in regions can be used to define integer estimators.

Definition 2 (Integer estimators). The mapping $\breve{a} = \mathcal{I}(\hat{a})$ is said to be an integer estimator if its pull-in regions satisfy

1. $\bigcup_{z\in\mathbb{Z}^n} \mathcal{P}_z = \mathbb{R}^n$,
2. $\text{Int}(\mathcal{P}_u) \cap \text{Int}(\mathcal{P}_v) = \emptyset, \forall u,\ v \in \mathbb{Z}^n,\ u \neq v$,
3. $\mathcal{P}_z = z + \mathcal{P}_0,\ \forall z \in \mathbb{Z}^n$.

According to this definition an integer estimator is completely specified once the pull-in region $\mathcal{P}_0$ is given. The following explicit expression can be given for an integer estimator:

$$\check{a} = \sum_{z\in\mathbb{Z}^n} z p_z(\hat{a}) \tag{10}$$

with the indicator function, $p_z(x)$, defined as

$$p_z(x) = \begin{cases} 1 \text{ if } x \in \mathcal{P}_z \\ 0 \text{ if } x \notin \mathcal{P}_z. \end{cases}$$

Note, since $\sum_{z\in\mathbb{Z}^n} p_z(x) = 1, \forall x \in \mathbb{R}^n$, that the $p_z(\hat{a})$ can be interpreted as weights. The integer estimator $\breve{a}$ is therefore equal to a weighted sum of integer vectors with binary weights. Examples of I-estimators are integer rounding, integer bootstrapping, and integer least-squares. Their pull-in regions are the multivariate versions of a square, a parallelogram, and a hexagon. The properties of these popular integer estimators will be further detailed in Sect. 3.

PMF and Success-Rate

The outcome of an integer estimator should only be used if one has enough confidence in its solution. To evaluate one's confidence in $\breve{a}$, one needs its PMF. The PMF of $\breve{a}$ is obtained by integrating the PDF of $\hat{a}$, $f_{\hat{a}}(x|a)$, over the pull-in regions $\mathcal{P}_z \subset \mathbb{R}^n$,

$$\text{P}(\check{a} = z) = \text{P}(\hat{a} \in \mathcal{P}_z) = \int_{\mathcal{P}_z} f_{\hat{a}}(x|a)dx, \quad z \in \mathbb{Z}^n. \tag{11}$$

In case $\hat{a} \sim \text{N}(a, Q_{\hat{a}\hat{a}})$, the PDF is given as $f_{\hat{a}}(x|a) = C \exp\left\{-\frac{1}{2}||x-a||^2_{Q_{\hat{a}\hat{a}}}\right\}$, where C is a normalizing constant and $||\cdot||^2_M = (\cdot)^T M^{-1}(\cdot)$.

The PMF of $\breve{a}$ depends, of course, on the pull-in regions $\mathcal{P}_z$ and therefore on the chosen integer estimator. Since various integer estimators exist, some may be better than others. Having the problem of GNSS ambiguity resolution in mind, one is particularly interested in the probability of correct integer estimation. This probability is referred to as the *success-rate*, $\mathrm{P}_s = \mathrm{P}(\breve{a} = a)$. Its complement is referred to as the *fail-rate*, $\mathrm{P}_f = 1 - \mathrm{P}_s$. The success-rate is computed as

$$\mathrm{P}_s = \int_{\mathcal{P}_a} f_{\hat{a}}(x|a)dx = \int_{\mathcal{P}_0} f_{\hat{a}}(x + a|a)dx. \tag{12}$$

This shows, if the PDF has the translational property $f_{\hat{a}}(x+a|a) = f_{\hat{a}}(x|0)$, that the success-rate can be computed without knowledge of the unknown integer vector $a \in \mathbb{Z}^n$. Obviously, this is the case for the PDF of the multivariate normal distribution.

Equation 12 is very important for GNSS applications. It allows the GNSS user to evaluate (often even before the actual measurements are taken) whether or not the strength of the underlying GNSS model is such that one can expect successful integer ambiguity resolution. The evaluation of the multivariate integral of Eq. 12 can generally be done through Monte Carlo integration (Robert and Casella 1999). For some important integer estimators, we also have easy-to-compute expressions and/or sharp (lower and upper) bounds of their success-rates available (cf. Sect. 3).

Optimal Integer Estimation

Since the success-rate depends on the pull-in region and therefore on the chosen integer estimator, it is of importance to know which integer estimator maximizes the probability of correct integer estimation (Teunissen 1999b).

Theorem 1 (Optimal integer estimation). *Let $f_{\hat{a}}(x|a)$ be the PDF of $\hat{a}$ and let the integer maximum likelihood (IML) estimator*

$$\breve{a}_{\mathrm{IML}} = \arg\max_{z \in \mathbb{Z}^n} f_{\hat{a}}(\hat{a}|z) \tag{13}$$

be an integer estimator. Then

$$\mathrm{P}(\breve{a}_{\mathrm{IML}} = a) \geq \mathrm{P}(\breve{a} = a) \tag{14}$$

for any integer estimator $\breve{a}$.

This result shows that of all integer estimators, the IML estimator has the largest success-rate. The theorem holds true for an arbitrary PDF of $\hat{a}$. In case $\hat{a} \sim \mathrm{N}(a, Q_{\hat{a}\hat{a}})$, the optimal I estimator is the integer least-squares (ILS) estimator

$$\breve{a}_{\mathrm{ILS}} = \arg\min_{a \in \mathbb{Z}^n} ||\hat{a} - a||^2_{Q_{\hat{a}\hat{a}}}. \tag{15}$$

The above theorem therefore gives a probabilistic justification for using the ILS estimator when the PDF is Gaussian. We will have more to say about the ILS estimator in Sect. 3.3.

2.2 Integer Aperture Estimation

Aperture Pull-In Regions

The outcome of an integer (I) estimator is always an integer, whether the fail-rate is large or small. Since the user has no direct control over this fail-rate (other than strengthening the underlying model a priori), the user has no direct influence on the confidence of its integer solution.

To give the user control over the fail-rate, the class of integer aperture (IA) estimators was introduced. This class is larger than the I class. The IA class is defined by dropping one of the three conditions in Definition 2, namely, the condition that the pull-in regions should cover $\mathbb{R}^n$ completely. We thus allow the IA pull-in regions to have gaps.

Definition 3 (Integer aperture estimators). Let $\Omega \subset \mathbb{R}^n$ and $\Omega_z = \Omega \cap \mathcal{P}_z$, where $\mathcal{P}_z$ is a pull-in region of an arbitrary I estimator. Then the estimator

$$\breve{a} = \begin{cases} \hat{a} \text{ if } \hat{a} \notin \Omega \\ z \text{ if } \hat{a} \in \Omega_z \end{cases} \tag{16}$$

is said to be an integer aperture estimator if its pull-in regions satisfy

1. $\bigcup_{z \in \mathbb{Z}^n} \Omega_z = \Omega \subset \mathbb{R}^n$,
2. $\text{Int}(\Omega_u) \bigcap \text{Int}(\Omega_v) = \emptyset, \ \forall u, \ v \in \mathbb{Z}^n, \ u \neq v$,
3. $\Omega_z = z + \Omega_0, \ \forall z \in \mathbb{Z}^n$.

If we compare it with Definition 2, we note that the role of the complete space $\mathbb{R}^n$ has been replaced by the subset $\Omega \subset \mathbb{R}^n$. It is easily verified from the above conditions that Ω is z-translational invariant, $\Omega = \Omega + z, \forall z \in \mathbb{Z}^n$. Also note that the I class is a subset of the IA class. Thus I estimators are IA estimators, but the converse is not necessarily true.

An IA estimator maps the float solution $\hat{a}$ to the integer vector z if $\hat{a} \in \Omega_z$ and it maps the float solution to itself if $\hat{a} \notin \Omega$. An IA estimator can therefore be expressed explicitly as

$$\breve{a}_{\text{IA}} = \hat{a} + \sum_{z \in \mathbb{Z}^n} (z - \hat{a}) \omega_z(\hat{a}) \tag{17}$$

with $\omega_z(x)$ the indicator function of Ω_z.

Note that the IA estimator is completely determined once $\Omega_0 \subset \mathcal{P}_0$ is given. Thus Ω_0 plays the same role for the IA estimators as $\mathcal{P}_0$ does for the I estimators. By changing the size and shape of Ω_0, one changes the outcome of the IA estimator. The subset Ω_0 can therefore be seen as an adjustable pull-in region with two limiting cases: the limiting case in which Ω_0 is empty and the limiting case when Ω_0 equals $\mathcal{P}_0$. In the first case the IA estimator becomes identical to the float solution $\hat{a}$, and in the second case the IA estimator becomes identical to an I estimator. The subset Ω_0 therefore determines the *aperture* of the pull-in region.

Probability Distribution and Successful Fix Rate

In order to evaluate the performance of an IA estimator, the following three outcomes need to be distinguished: $\hat{a} \in \Omega_a$ for success (correct integer estimation), $\hat{a} \in \Omega \setminus \Omega_a$ for failure (incorrect integer estimation), and $\hat{a} \notin \Omega$ for undecided (a not estimated as integer). The corresponding probabilities of success (s), failure (f), and undecided (u) are given as

$$\begin{aligned} \mathrm{P}_s &= \mathrm{P}(\breve{a}_{\mathrm{IA}} = a) = \int_{\Omega_a} f_{\hat{a}}(x|a)dx, \\ \mathrm{P}_f &= \sum_{z \in \mathbb{Z}^n \setminus \{a\}} \mathrm{P}(\breve{a}_{\mathrm{IA}} = z) = \sum_{z \in \mathbb{Z}^n \setminus \{a\}} \int_{\Omega_z} f_{\hat{a}}(x|a)dx, \\ \mathrm{P}_u &= 1 - \mathrm{P}_s - \mathrm{P}_f = 1 - \int_{\Omega_0} f_{\breve{\delta}}(x|a)dx, \end{aligned} \tag{18}$$

where

$$f_{\breve{\delta}}(x|a) = \sum_{z \in \mathbb{Z}^n} f_{\hat{a}}(x + z|a) p_0(x) \tag{19}$$

is the PDF of the residual $\breve{\delta} = \hat{a} - \hat{a}$ (Teunissen 2002; Verhagen and Teunissen 2004, 2005).

Since $\Omega_a \subset \mathcal{P}_a$, it follows that the success-rate of an IA estimator will never be larger than that of the corresponding I estimator. So what have we gained? What we have gained is that we can now control the fail-rate and in particular control the successful fix rate, i.e., the probability of successful fixing. Note that the complement of the undecided probability $1 - \mathrm{P}_u = \mathrm{P}_s + \mathrm{P}_f$ is the fix probability, i.e., the probability that the outcome of the IA estimator is integer. The probability of successful fixing is therefore given by the ratio

$$\mathrm{P}_{sf} = \frac{\mathrm{P}_s}{\mathrm{P}_s + \mathrm{P}_f}. \tag{20}$$

To have confidence in the integer outcomes of IA-estimation, a user would like to have P_{sf} close to 1. This can be achieved by setting the fail-rate P_f at a small-enough level. Thus the user chooses the level of fail-rate he finds acceptable and then determines the size of the aperture pull-in region that corresponds with this fail-rate level. With such a setting, the user has the guarantee that the fail-rate of his IA estimator will never become unacceptably large.

As with I estimation, the user can choose from a whole class of IA estimators simply by using different shape definitions for the aperture pull-in regions. Various examples have been given in Verhagen and Teunissen (2006). The ILS estimator combined with the popular GNSS *Ratio-Test* (Leick 2004) is such an example. Unfortunately one can still find various incorrect interpretations of the Ratio-Test in the literature. So is its use often motivated by stating that it determines whether the ILS solution is true or false. This is not correct. Also the current ways of choosing the tolerance value τ are ad hoc or based on false theoretical grounds. Often a fixed value of $\frac{1}{2}$ or $\frac{1}{3}$ is used. However, as shown in Verhagen and Teunissen (2006) and Teunissen and Verhagen (2009), instead of using a fixed τ-value, one should use the *fixed fail-rate* approach. From the fixed fail-rate, one can then compute the variable τ-value (it varies with varying strength of the underlying GNSS model).

Optimal Integer Aperture Estimation

So far we considered IA estimation with a priori chosen aperture pull-in shapes. Now we determine which of the IA estimators performs best. As the optimal IA estimator we choose the one which maximizes the success-rate subject to a given fail-rate. The optimal IA estimator is given by the following theorem (Teunissen 2005).

Theorem 2 (Optimal integer aperture estimation). *Let $f_{\hat{a}}(x|a)$ and $f_{\breve{\delta}}(x|a)$ be the PDFs of $\hat{a}$ and $\breve{\delta} = \hat{a} - \breve{a}_{\text{IML}}$, respectively, and let P_s and P_f be the success-rate and the fail-rate of the IA estimator. Then the solution to*

$$\max_{\Omega_0 \subset \mathcal{P}_0} \text{P}_s \text{subject to given } \text{P}_f \tag{21}$$

is given by the aperture pull-in region

$$\Omega_0 = \{x \in \mathcal{P}_0 | f_{\breve{\delta}}(x|a) \leq \lambda f_{\hat{a}}(x + a|a)\}, \tag{22}$$

where

$$\mathcal{P}_0 = \{x \in \mathbb{R}^n | 0 = \arg\max_{z \in \mathbb{Z}^n} f_{\hat{a}}(x|z)\}$$

and with the aperture parameter λ chosen so as to satisfy the a priori fixed fail-rate P_f.

The steps in computing the optimal IA estimator are therefore as follows: (1) Compute the optimal I estimator $\breve{a} = \arg\max_{z \in \mathbb{Z}^n} f_{\hat{a}}(\hat{a}|z)$. (2) Determine the aperture parameter λ from the user-defined fail-rate P_f. Ways of doing this are discussed in Verhagen (2005a,b) and Teunissen and Verhagen (2009). (3) Check whether $\breve{\delta} = \hat{a} - \breve{a}$ lies in Ω_0. If $\breve{\delta} \in \Omega_0$, then $\breve{a}$ is the outcome of the optimal IA estimator; otherwise, the outcome is $\hat{a}$.

2.3 Integer Equivariant Estimation

A Larger Class of Estimators

The class of IA estimators includes the class of I estimators. Now we introduce an even larger class. This larger class is obtained by dropping another condition of Definition 2. Since we would at least like to retain the integer remove-restore property, we keep the condition that the estimators must be z-translational invariant. Such estimators will be called *integer equivariant* (IE) estimators.

Definition 4 (Integer equivariant estimators). The estimator $\hat{\theta}_{\mathrm{IE}} = F_\theta(\hat{a})$, with $F_\theta : \mathbb{R}^n \mapsto \mathbb{R}$, is said to be an integer equivariant estimator of the linear function $\theta = l^T a$ if

$$F_\theta(x+z) = F_\theta(x) + l^T z, \;\; \forall x \in \mathbb{R}^n, \; z \in \mathbb{Z}^n. \tag{23}$$

It will be clear that I estimators and IA estimators are also IE estimators. The converse, however, is not necessarily true.

The class of IE estimators is also larger than the class of linear unbiased estimators, assuming that the float solution is unbiased. Let $F_\theta^T \hat{a}$, for some $F_\theta \in \mathbb{R}^n$, be the linear estimator of $\theta = l^T a$. For it to be unbiased one needs, using $E\{\hat{a}\} = a$, that $F_\theta^T E\{\hat{a}\} = l^T \; a, \; \forall a \in \mathbb{R}^n$ holds true or that $F_\theta = l$. But this is equivalent to stating that $F_\theta^T(\hat{a} + a) = F_\theta^T \hat{a} + l^T a, \; \forall \hat{a} \in \mathbb{R}^n, \; a \in \mathbb{R}^n$. Comparison with (23) shows that the condition of linear unbiasedness is more restrictive than the condition of integer equivariance. The class of linear unbiased estimators is therefore a subset of the IE class. This implies that a "best" IE estimator must be at least as good as the BLUE $\hat{a}$. After all the float solution $\hat{a}$ is an IE estimator as well.

Best Integer Equivariant Estimation

We denote the best integer equivariant (BIE) estimator of θ as $\hat{\theta}_{\mathrm{BIE}}$ and use the mean squared error (MSE) as our criterion of "best." The BIE estimator of $\theta = l^T a$ is therefore defined as

$$\hat{\theta}_{\mathrm{BIE}} = \arg \min_{F_\theta \in \mathrm{IE}} \mathrm{E}\{(F_\theta(\hat{a}) - \theta)^2\} \tag{24}$$

in which IE stands for the class of IE estimators. The minimization is thus taken over all integer equivariant functions that satisfy the condition of Definition 4. Thus the BIE estimator is the optimal IE estimator in the MSE sense.

The reason for choosing the MSE criterion is twofold. First, it is a well-known probabilistic criterion for measuring the closeness of an estimator to its target value, in our case $\theta = l^T a$. Second, the MSE criterion is also often used as measure for the quality of the float solution itself. It should be kept in mind however that the MSE criterion is a weaker criterion than the probabilistic criterion used in the previous two sections.

The BIE estimator is given by the following theorem (Teunissen 2003b).

Theorem 3 (Best integer equivariant estimation). *Let $f_{\hat{a}}(x|a)$ be the PDF of $\hat{a}$ and let $\hat{\theta}_{\text{BIE}}$ be the best integer equivariant estimator of $\theta = l^T a$. Then $\hat{\theta}_{\text{BIE}} = l^T \hat{a}_{\text{BIE}}$, where*

$$\hat{a}_{\text{BIE}} = \sum_{z \in \mathbb{Z}^n} z w_z(\hat{a}) \tag{25}$$

with the weighting functions $w_z(x)$ given as

$$w_z(x) = \frac{f_{\hat{a}}(x + a - z|a)}{\sum_{u \in \mathbb{Z}^n} f_{\hat{a}}(x + a - u|a)}. \tag{26}$$

As the I estimator, the BIE estimator is also a weighted sum of all integer vectors in $\mathbb{Z}^n$. In the present case, however, the weights are not binary. They vary between 0 and 1, and their values are determined by the float solution and its PDF. As a consequence the BIE estimator will be real valued in general, instead of integer valued.

An important consequence of the above theorem is that the BIE estimator is always better than or at least as good as any integer estimator as well as any linear unbiased estimator. After all the class of integer estimators and the class of linear unbiased estimators are both subsets of the class of IE estimators. The nonlinear BIE estimator is therefore also better than the best linear unbiased estimator (BLUE):

$$\text{MSE}(\hat{\theta}_{\text{BIE}}) \leq \text{MSE}(\hat{\theta}_{\text{BLUE}}). \tag{27}$$

The BLUE is the minimum variance estimator of the class of linear unbiased estimators and it is given by the well-known Gauss-Markov theorem. The two estimators $\hat{\theta}_{\text{BIE}}$ and $\hat{\theta}_{\text{BLUE}}$ therefore both minimize the mean squared error, albeit within a different class.

The above theorem holds true for any PDF the float solution $\hat{a}$ might have. In the Gaussian case $\hat{a} \sim \text{N}(a, Q_{\hat{a}\hat{a}})$, the weighting function of Eq. 26 becomes

$$w_z(x) = \frac{\exp\left\{-\frac{1}{2}\|x - z\|^2_{Q_{\hat{a}\hat{a}}}\right\}}{\sum_{u \in \mathbb{Z}^n} \exp\left\{-\frac{1}{2}\|x - u\|^2_{Q_{\hat{a}\hat{a}}}\right\}}. \tag{28}$$

Since the space of integers $\mathbb{Z}^n$ can be seen as a certain discretized version of the space of real numbers $\mathbb{R}^n$, one would expect, if the integer grid size gets smaller in relation to the size and extend of the PDF, that the difference between the two estimators, $\hat{a}_{\text{BIE}}$ and $\hat{a}$, gets smaller as well. Similarly, if the PDF gets more peaked in relation to the integer grid size, one would expect that the BIE estimator $\hat{a}_{\text{BIE}}$ tends to an integer estimator. This is made precise in the following lemma.

Lemma 1 (Limits of the BIE estimator).

(i) *If we replace* $\sum_{z\in\mathbb{Z}^n}$ *by* $\int_{\mathbb{R}^n} dz$ *in Eqs. 25 and 28, then*

$$\hat{a}_{\text{BIE}} = \hat{a}.$$

(ii) *Let* $\hat{a} \sim \text{N}(a, Q_{\hat{a}\hat{a}} = \sigma^2 G)$*. Then*

$$\lim_{\sigma\to 0} \hat{a}_{\text{BIE}} = \breve{a}_{\text{ILS}}.$$

A probabilistic performance comparison between $\hat{a}$, $\breve{a}_{\text{ILS}}$, and $\hat{a}_{\text{BIE}}$ can be found in Verhagen and Teunissen (2005).

It is interesting to observe that the above given Gaussian-based expression for $\hat{a}_{\text{BIE}}$ is identical to its Bayesian counterpart as given in Betti et al. (1993), Gundlich and Koch (2002), and Gundlich and Teunissen (2004). This is not quite true for the general case however. Still, this Gaussian equivalence nicely bridges the gap between the current theory of integer inference and the Bayesian approach.

3 Three Popular Integer Estimators

In this section, we discuss integer rounding, integer bootstrapping, and integer least-squares, with special attention to computational issues and the success-rates. We assume $\hat{a} \sim \text{N}(a \in \mathbb{Z}^n, Q_{\hat{a}\hat{a}})$.

3.1 Integer Rounding

Scalar and Vectorial Rounding

The simplest integer estimator is "rounding to the nearest integer." In the scalar case, its pull-in regions (intervals) are given as

$$\mathcal{R}_z = \{x \in \mathbb{R} \mid |x - z| \leq 1/2\}, \ z \in \mathbb{Z}. \tag{29}$$

Any outcome of $\hat{a} \sim \mathrm{N}(a \in \mathbb{Z}, \sigma_{\hat{a}}^2)$ that satisfies $|\hat{a} - z| \leq 1/2$ will thus be pulled to the integer z. We denote the rounding estimator as $\breve{a}_{\mathrm{R}}$ and the operation of integer rounding as $\lceil . \rfloor$. Thus $\breve{a}_{\mathrm{R}} = \lceil \hat{a} \rfloor$ and $\breve{a}_{\mathrm{R}} = z$ if $\hat{a} \in \mathcal{R}_z$.

The PMF of $\breve{a}_{\mathrm{R}} = \lceil \hat{a} \rfloor$ is given as

$$\mathrm{P}(\breve{a}_{\mathrm{R}} = z) = \Phi\left(\frac{1 - 2(a - z)}{2\sigma_{\hat{a}}}\right) + \Phi\left(\frac{1 + 2(a - z)}{2\sigma_{\hat{a}}}\right) - 1, \; z \in \mathbb{Z}, \tag{30}$$

where $\Phi(x)$ denotes the normal distribution function, $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}v^2\right\} \, dv$.

Note that the PMF is symmetric about a. Thus integer rounding provides for an unbiased integer estimator $\mathrm{E}(\breve{a}_{\mathrm{R}}) = a \in \mathbb{Z}$. Also note that the PMF becomes more peaked when $\sigma_{\hat{a}}$ gets smaller.

For GNSS ambiguity resolution, the success-rate is of particular importance. The success-rate of integer rounding follows from Eq. 30 by setting z equal to a:

$$\mathrm{P}(\breve{a}_{\mathrm{R}} = a) = 2\Phi\left(\frac{1}{2\sigma_{\hat{a}}}\right) - 1. \tag{31}$$

Thus the smaller the standard deviation, the larger the success-rate. A success-rate better than 0.99 requires $\sigma_{\hat{a}} \leq 0.15$ cycle.

Scalar rounding is easily generalized to the vectorial case. It is defined as the component-wise rounding of $\hat{a} = (\hat{a}_1, \ldots, \hat{a}_n)^T$, $\breve{a}_{\mathrm{R}} = (\lceil \hat{a}_1 \rfloor, \lceil \hat{a}_2 \rfloor, \ldots, \lceil \hat{a}_n \rfloor)^T$. The pull-in regions of vectorial rounding are the multivariate versions of the scalar pull-in intervals:

$$\mathcal{R}_z = \left\{ x \in \mathbb{R}^n \,|\, \left| c_i^T (x - z) \right| \leq 1/2, \; i = 1, \ldots, n \right\}, \; z \in \mathbb{Z}^n, \tag{32}$$

where c_i denotes the unit vector having a 1 as its ith entry and 0s otherwise. Thus the pull-in regions of rounding are unit squares in 2D, unit cubes in 3D, etc.

Rounding Success-Rate

To determine the joint PMF of the components of $\breve{a}_{\mathrm{R}}$, we have to integrate the PDF of $\hat{a} \sim \mathrm{N}(a, Q_{\hat{a}\hat{a}})$ over the pull-in regions $\mathcal{R}_Z$. These n-fold integrals are unfortunately difficult to evaluate, unless the variance matrix $Q_{\hat{a}\hat{a}}$ is diagonal, in which case the components of $\breve{a}_{\mathrm{R}}$ are independent and their joint PMF follows as the product of the univariate PMFs of the components. The corresponding success-rate is then given by the n-fold product of the univariate success-rates.

In case of GNSS, the ambiguity variance matrix will usually be fully populated, meaning that one will have to resort to methods of Monte Carlo simulation for computing the joint PMF. In the case of the success-rate, one can alternatively make use of the following bounds (Teunissen 1998b).

Theorem 4 (Rounding success-rate bounds). *Let* $\hat{a} \sim \mathrm{N}(a \in \mathbb{Z}^n, Q_{\hat{a}\hat{a}})$. *Then the rounding success-rate can be bounded from below and from above as*

$$\prod_{i=1}^{n}\left[2\Phi\left(\frac{1}{2\sigma_{\hat{a}_i}}\right)-1\right] \le \mathrm{P}(\breve{a}_\mathrm{R}=a) \le \left[2\Phi\left(\frac{1}{2\sigma_{\max}}\right)-1\right], \qquad (33)$$

where $\sigma_{\max} = \max_{i=1,\ldots,n} \sigma_{\hat{a}_i}$.

These easy-to-compute bounds are very useful for determining the expected success of GNSS ambiguity rounding. The upper bound is useful to quickly decide against such ambiguity resolution. It shows that ambiguity resolution based on vectorial rounding can not be expected successful, if already one of the scalar rounding success-rates is too low.

The lower bound is useful to quickly decide in favor of vectorial rounding. If the lower bound is sufficiently close to 1, one can be confident that vectorial rounding will produce the correct integer ambiguity vector. Note that this requires each of the individual probabilities in the product of the lower bound to be sufficiently close to 1.

Z-Transformations

Although $\breve{a}_\mathrm{R}$ is easy to compute, the rounding estimator suffers from a lack of invariance against integer reparametrizations or the so-called Z-transformations. A matrix is called a Z-transformation if it is one-to-one (i.e., invertible) and integer (Teunissen 1995a). Such transformations leave the integer nature of the parameters in tact.

By saying that the rounding estimator lacks Z-invariance, we mean that if the float solution is Z-transformed, the integer solution does not transform accordingly. That is, rounding and transforming do not commute

$$\breve{z}_\mathrm{R} \neq Z\breve{a}_\mathrm{R} \text{ if } \hat{z} = Z\hat{a}. \qquad (34)$$

Only in case Z is a permutation matrix, $Z = \Pi$, do we have $\breve{z}_\mathrm{R} = \Pi\breve{a}_\mathrm{R}$. In this case, the transformation is a simple reordering of the ambiguities.

Also the success-rate lacks Z-invariance. Since the pull-in regions of rounding remain unaffected by the Z-transformation, while the distribution of the float solution changes to $\hat{z} \sim \mathrm{N}(z = Za, Q_{\hat{z}\hat{z}} = ZQ_{\hat{a}\hat{a}}Z^T)$, we have, in general,

$$\mathrm{P}(\breve{z}_\mathrm{R} = z) \neq \mathrm{P}(\breve{a}_\mathrm{R} = a). \qquad (35)$$

This lack of invariance implies that integer rounding is not optimal in the vectorial case. The lack of invariance does not occur in the scalar case, since multiplication by ± 1 is then the only admissible Z-transformation.

Does the mentioned lack of invariance mean that rounding is unfit for GNSS integer ambiguity resolution? No, by no means. Integer rounding is a valid ambiguity estimator, since it obeys the principle of integer equivariance, and it is an attractive estimator, because of its computational simplicity. Whether or not it can be successfully applied in any concrete situation depends solely on the value of its success-rate for that particular situation. What the lack of invariance shows is the nonoptimality of rounding. Despite being nonoptimal, rounding can achieve high success-rates, provided the underlying GNSS model is of sufficient strength and provided the proper ambiguity parametrization is chosen. In section "The ILS Search", we come back to this issue and see how we can use the existing degrees of freedom of integer parametrization to our advantage.

3.2 Integer Bootstrapping

The Bootstrapping Principle

Integer bootstrapping is a generalization of integer rounding; it combines integer rounding with sequential conditional least-squares estimation and as such takes some of the correlation between the components of the float solution into account. The method goes as follows. If $\hat{a} = (\hat{a}_1, \ldots, \hat{a}_n)^T$, one starts with $\hat{a}_1$ and as before rounds its value to the nearest integer. Having obtained the integer of the first component, the real-valued estimates of all remaining components are then corrected by virtue of their correlation with $\hat{a}_1$. Then the second, but now corrected, real-valued component is rounded to its nearest integer. Having obtained the integer value of this second component, the real-valued estimates of all remaining $n-2$ components are then again corrected by virtue of their correlation with the second component. This process is continued until all n components are taken care of. We have the following definition.

Definition 5 (Integer bootstrapping). Let $\hat{a} = (\hat{a}_1, \ldots, \hat{a}_n)^T \in \mathbb{R}^n$ be the float solution and let $\breve{a}_\text{B} = (\breve{a}_{\text{B},1}, \ldots, \breve{a}_{\text{B},n})^T \in \mathbb{Z}^n$ denote the corresponding integer bootstrapped solution. Then

$$\begin{aligned} \breve{a}_{\text{B},1} &= \lceil \hat{a}_1 \rfloor, \\ \breve{a}_{\text{B},2} &= \lceil \hat{a}_{2|1} \rfloor = \lceil \hat{a}_2 - \sigma_{21}\sigma_1^{-2}(\hat{a}_1 - \breve{a}_{\text{B},1}) \rfloor, \\ &\vdots \\ \breve{a}_{\text{B},n} &= \lceil \hat{a}_{n|N} \rfloor = \lceil \hat{a}_n - \textstyle\sum_{j=1}^{n-1} \sigma_{n,j|J}\sigma_{j|J}^{-2}(\hat{a}_{j|J} - \breve{a}_{\text{B},j}) \rfloor, \end{aligned} \tag{36}$$

where $\hat{a}_{i|I}$ is the least-squares estimator of a_i conditioned on the values of the previous $I = \{1, \ldots, (i-1)\}$ sequentially rounded components, $\sigma_{i,j|J}$ is the covariance between $\hat{a}_i$ and $\hat{a}_{j|J}$, and $\sigma^2_{j|J}$ is the variance of $\hat{a}_{j|J}$. For $i = 1$, $\hat{a}_{i|I} = \hat{a}_1$.

As the definition shows, the bootstrapped estimator can be seen as a generalization of integer rounding. The bootstrapped estimator reduces to integer rounding in case correlations are absent, i.e., in case the variance matrix $Q_{\hat{a}\hat{a}}$ is diagonal.

The bootstrapped estimator combines sequential conditional least-squares estimation with integer rounding. If we replace the "least-squares estimation" part by "linear estimation," we can construct a whole class of sequential integer estimators. This class is defined as follows.

Definition 6 (Sequential integer estimation). Let $\hat{a} = (\hat{a}_1, \ldots, \hat{a}_n)^T \in \mathbb{R}^n$ be the float solution. Then $\breve{a} = (\breve{a}_1, \ldots, \breve{a}_n)^T \in \mathbb{Z}^n$ is a sequential integer estimator of $\mathrm{E}(\hat{a}) = a \in \mathbb{Z}^n$ if $\breve{a}_i = \lceil \hat{a}_i + \sum_{j=1}^{i-1} r_{ij}(\hat{a}_j - \breve{a}_j) \rfloor, i = 1, \ldots, n$, or, in vector-matrix form, if

$$\breve{a} = \lceil \hat{a} + (R - I_n)(\hat{a} - \breve{a}) \rfloor \tag{37}$$

with R a unit lower triangular matrix.

By showing how the bootstrapped estimator can be computed from using the triangular factorization of the variance matrix $Q_{\hat{a}\hat{a}}$, it becomes immediately clear that the bootstrapped estimator $\breve{a}_\mathrm{B}$ is indeed a member of the class of sequential integer estimators. We have the following result (Teunissen 2007a).

Theorem 5 (Bootstrapping and the triangular decomposition). *Let $\hat{a} \in \mathbb{R}^n$ be the float solution and let the unit lower triangular decomposition of its variance matrix be given as $Q_{\hat{a}\hat{a}} = LDL^T$. The entries of L and D are then given as*

$$(L)_{ij} = \begin{cases} 0 & \text{for } 1 \le i < j \le n \\ 1 & \text{for } \quad i = j \\ \sigma_{i,j|J}\sigma_{j|J}^{-2} & \text{for } 1 \le j < i \le n \end{cases} \quad \text{and } D = \mathrm{diag}\left(\ldots, \sigma_{j|J}^2, \ldots\right) \tag{38}$$

and the bootstrapped estimator $\breve{a}_\mathrm{B} \in \mathbb{Z}^n$ of Eq. 36 can be expressed as

$$\breve{a}_\mathrm{B} = \lceil \hat{a} + (L^{-1} - I_n)(\hat{a} - \breve{a}_\mathrm{B}) \rfloor. \tag{39}$$

Comparing Eq. 39 with Eq. 37 shows that the bootstrapped estimator is indeed a sequential integer estimator.

Note that in the construction of the bootstrapped estimator, the triangular factor L of $Q_{\hat{a}\hat{a}} = LDL^T$ is used, but not the diagonal matrix D. Thus bootstrapping takes only part of the information of the variance matrix into account. Although the diagonal matrix D is not used in the bootstrapped mapping itself, it determines – as we will see below – the bootstrapped success-rate.

The Bootstrapped PMF and Success-Rate

To determine the bootstrapped PMF, we first need to determine the bootstrapped pull-in regions. The bootstrapped pull-in regions are given as

$$\mathcal{B}_z = \{x \in \mathbb{R}^n | \quad \left|c_i^T L^{-1}(x-z)\right| \leq 1/2, i = 1, \ldots, n\}, \ \forall z \in \mathbb{Z}^n, \tag{40}$$

where c_i denotes the unit vector having 1 as its ith entry and 0s otherwise.

To prove Eq. 40, let $\tilde{a} = z + L^{-1}(\hat{a} - z) = \hat{a} + (L^{-1} - I_n)(\hat{a} - z)$. Then according to Eq. 39, $\breve{a}_{\mathrm{B}} = z$ if $\lceil \tilde{a} \rfloor = z$. Thus $\breve{a}_{\mathrm{B}} = z$ if $\lceil \tilde{a} - z \rfloor = 0$, which implies that all components of $L^{-1}(\hat{a} - z)$ have to be less than or equal to $\frac{1}{2}$ in absolute value. It is easily verified that the $\mathcal{B}_z$ satisfy the conditions of Definition 2. In Xu (2006), however, it is incorrectly claimed that Teunissen's bootstrapped estimator Eq. 36 is not a genuine integer estimator.

The bootstrapped PMF follows from integrating the multivariate normal distribution over the bootstrapped pull-in regions. In contrast to the multivariate integral for integer rounding, the multivariate integral for bootstrapping can be simplified considerably. As shown by the following theorem, the bootstrapped PMF can be expressed as a product of univariate integrals (Teunissen 1998b).

Theorem 6 (Bootstrapped PMF). *Let* $\hat{a} \sim \mathrm{N}(a \in \mathbb{Z}^n, Q_{\hat{a}\hat{a}})$ *and let* $\breve{a}_{\mathrm{B}}$ *be the bootstrapped estimator of a. Then*

$$\mathrm{P}(\breve{a}_{\mathrm{B}} = z) = \prod_{i=1}^{n} \left[\Phi\left(\frac{1 - 2l_i^T(a - z)}{2\sigma_{\hat{a}_{i|I}}} \right) + \Phi\left(\frac{1 + 2l_i^T(a - z)}{2\sigma_{\hat{a}_{i|I}}} \right) - 1 \right], \forall z \in \mathbb{Z}^n, \tag{41}$$

where l_i *is the ith column vector of the unit upper triangular matrix* L^{-T}.

The bootstrapped PMF is symmetric about the mean of $\hat{a}$. This implies that the bootstrapped estimator $\breve{a}_{\mathrm{B}}$ is an unbiased estimator of $a \in \mathbb{Z}^n$.

As a direct consequence of the above theorem, we have an exact and easy-to-compute expression for the bootstrapped success-rate.

Corollary 1 (Bootstrapped success-rate). *Let* $\hat{a} \sim \mathrm{N}(a \in \mathbb{Z}^n, Q_{\hat{a}\hat{a}})$. *Then the bootstrapped success-rate is given as*

$$\mathrm{P}(\breve{a}_{\mathrm{B}} = a) = \prod_{i=1}^{n} \left[2\Phi\left(\frac{1}{2\sigma_{\hat{a}_{i|I}}} \right) - 1 \right]. \tag{42}$$

This is an important result for GNSS applications as it provides a very simple way of evaluating the bootstrapped success-rate.

The following result shows what happens to the success-rate if the computed bootstrapped estimator is based on a too optimistic or a too pessimistic description of the float precision (Teunissen 2007a).

Theorem 7 (Use of wrong weight matrix). *Let* $\hat{a} \sim \mathrm{N}(a \in \mathbb{Z}^{\mathrm{n}}, Q)$ *and let* $\breve{a}_{\mathrm{B}}^{\Sigma}$ *be the bootstrapped estimator constructed on the basis of the positive definite matrix* Σ*. Then*

$$\mathrm{P}(\breve{a}_{\mathrm{B}}^{\Sigma} = a) \leq \mathrm{P}\left(\breve{a}_{\mathrm{B}}^{Q} = a\right) \tag{43}$$

with strict inequality if the unit triangular factors of Σ *and* Q *differ.*

The success-rate will thus get smaller if a wrong weight matrix is used. This theorem has two important consequences. First, since it was shown that the bootstrapped estimator is a member of the class of sequential integer estimators, the above result directly implies that the bootstrapped estimator is the optimal estimator within this restricted class.

Second, since integer rounding is also a sequential integer estimator (i.e., take $R = I_n$ in Eq. 37), it follows that the rounding success-rate will never be larger than the bootstrapped success-rate. Thus bootstrapping is a better integer estimator than rounding.

Z-Transformations

Like rounding, bootstrapping also suffers from a lack of Z-invariance. Similarly to Eqs. 34 and 35, we have for bootstrapping,

$$\breve{z}_{\mathrm{B}} \neq Z\breve{a}_B \text{ and } \mathrm{P}(\breve{z}_{\mathrm{B}} = z) \neq \mathrm{P}(\breve{a}_{\mathrm{B}} = a) \text{ if } \hat{z} = Z\hat{a}. \tag{44}$$

Thus the success-rate of bootstrapping (and of rounding as well) changes if a different parametrization is used, say $z = Za \in \mathbb{Z}^n$ instead of $a \in \mathbb{Z}^n$. It is thus of importance, when using bootstrapping (or rounding), that a proper parametrization is used. Bootstrapping performs relatively poor, for instance, when applied to the GNSS DD ambiguities for RTK positioning. This is due to the usually high correlation between such DD ambiguities. Bootstrapping should therefore be used in combination with a decorrelating Z-transformation. For computational details on how such Z-transformations can be constructed, we refer to Teunissen (1993, 1995b), de Jonge and Tiberius (1996a), and de Jonge et al. (1996, and the references cited therein). Also see Liu et al. (1999), Grafarend (2000), Xu (2001), Joosten and Tiberius (2002), and Svendsen (2006).

Despite the fact that we have an exact and easy-to-compute formula for the bootstrapped success-rate (cf. Eq. 42), an easy-to-compute upper bound of it would still be useful if it would be Z-invariant. Such an upper bound can be constructed when use is made of the Z-invariant *Ambiguity Dilution of Precision* (ADOP) (Teunissen 2000).

Theorem 8 (Bootstrapped success-rate invariant upper bound). *Let* $\hat{a} \sim \mathrm{N}(a \in \mathbb{Z}^n, Q_{\hat{a}\hat{a}}), \hat{z} = Z\hat{a}$, *and* $\mathrm{ADOP} = \det(Q_{\hat{a}\hat{a}})^{\frac{1}{2n}}$. *Then*

$$\mathrm{P}(\breve{z}_{\mathrm{B}} = z) \leq \left[2\Phi\left(\frac{1}{2\mathrm{ADOP}}\right) - 1\right]^{n} \tag{45}$$

for any admissible Z-transformation.

Thus if the upper bound is too small, we can immediately conclude, for *any* parametrization, that bootstrapping nor rounding will be successful.

3.3 Integer Least-Squares

Mixed Integer Least-Squares

Application of the least-squares principle to the mixed integer model (3) gives the minimization problem

$$\min_{a,b} ||y - Aa - Bb||^2_{Q_{yy}}, a \in \mathbb{Z}^n, b \in \mathbb{R}^p, \tag{46}$$

where Q_{yy} is the variance matrix of y. This type of least-squares problem was first formulated in Teunissen (1993) and was coined a (mixed) integer least-squares (ILS) problem. It is a nonstandard least-squares problem due to the integer constraints $a \in \mathbb{Z}^n$.

To solve Eq. 46, we start from the orthogonal decomposition

$$||y - Aa - Bb||^2_{Q_{yy}} = ||\hat{e}||^2_{Q_{yy}} + ||\hat{a} - a||^2_{Q_{\hat{a}\hat{a}}} + \left|\left|\hat{b}(a) - b\right|\right|^2_{Q_{\hat{b}(a)\hat{b}(a)}}, \tag{47}$$

where $\hat{e} = y - A\hat{a} - B\hat{b}$, with $\hat{a}$ and $\hat{b}$ the unconstrained least-squares estimators of a and b, respectively, $\hat{b}(a) = \hat{b} - Q_{\hat{b}\hat{a}}Q_{\hat{a}\hat{a}}^{-1}(\hat{a} - a)$, and $Q_{\hat{b}(a)\hat{b}(a)} = Q_{\hat{b}\hat{b}} - Q_{\hat{b}\hat{a}}Q_{\hat{a}\hat{a}}^{-1}Q_{\hat{a}\hat{b}}$. Note that the first term on the right-hand side of Eq. (47) is constant and that the third term can be made zero for any a by setting $b = \hat{b}(a)$. Hence, the mixed integer minimizers of Eq. (46) are given as

$$\breve{a}_{\mathrm{ILS}} = \arg\min_{z \in \mathbb{Z}^n} ||\hat{a} - z||^2_{Q_{\hat{a}\hat{a}}} \text{ and } \breve{b} = \hat{b}(\breve{a}_{\mathrm{ILS}}) = \hat{b} - Q_{\hat{b}\hat{a}}Q_{\hat{a}\hat{a}}^{-1}(\hat{a} - \breve{a}_{\mathrm{ILS}}). \tag{48}$$

In contrast to rounding and bootstrapping, the ILS principle is Z-invariant. We have

$$\breve{z}_{\mathrm{ILS}} = Z\breve{a}_{\mathrm{ILS}} \text{ and } \breve{b} = \hat{b} - Q_{\hat{b}\hat{z}}Q_{\hat{z}\hat{z}}^{-1}(\hat{z} - \breve{z}_{\mathrm{ILS}}). \tag{49}$$

Application of the ILS principle to $Z\hat{a}$ gives therefore the same result as Z times the result of applying the ILS principle to $\hat{a}$. Also $\breve{b}$ is invariant for the integer reparametrization.

We have seen that the 2D pull-in regions of rounding and bootstrapping are squares and parallelograms, respectively. It follows that those of ILS are hexagons. The ILS pull-in region of $z \in \mathbb{Z}^n$ consists by definition of all those points that are closer to z than to any other integer vector in

$$\mathbb{R}^n, \mathcal{L}_z = \{x \in \mathbb{R}^n \mid ||x - z||^2_{Q_{\hat{a}\hat{a}}} \leq ||x - u||^2_{Q_{\hat{a}\hat{a}}}, \forall u \in \mathbb{Z}^n\}, z \in \mathbb{Z}^n.$$

By rewriting the inequality, we obtain a representation that more closely resembles the ones of rounding, $\mathcal{R}_z$, and bootstrapping, $\mathcal{B}_z$, (see Eqs. 32 and 40):

$$\mathcal{L}_z = \{x \in \mathbb{R}^n \mid \left|u^T Q^{-1}_{\hat{a}\hat{a}}(x - z)\right| \leq 1/2||u||^2_{Q_{\hat{a}\hat{a}}}, \forall u \in \mathbb{Z}^n\}, z \in Z^n, \tag{50}$$

This shows that the ILS pull-in regions are constructed from intersecting half-spaces. One can show that at most $2^n - 1$ pairs of such half-spaces are needed for constructing the pull-in region. It is easily verified that the ILS pull-in regions are convex, symmetric sets that satisfy the conditions of Definition 2. Note that $\mathcal{L}_z = \mathcal{R}_z$ when $Q_{\hat{a}\hat{a}}$ is diagonal.

The ILS Search

In contrast to rounding and bootstrapping, an integer search is needed to compute an ILS solution. The search space is defined as

$$\Psi_a = \{a \in \mathbb{Z}^n \mid ||\hat{a} - a||^2_{Q_{\hat{a}\hat{a}}} \leq \chi^2\}, \tag{51}$$

where χ^2 is to be chosen positive constant. This ellipsoidal search space is centred at $\hat{a}$, its elongation is governed by $Q_{\hat{a}\hat{a}}$, and its size is determined by χ^2. In the case of GNSS, the search space is usually extremely elongated due to the high correlations between the carrier-phase ambiguities. Since this extreme elongation hinders the computational efficiency of the search, the search space is first transformed to a more spherical shape by means of a decorrelating Z-transformation:

$$\Psi_z = \{z \in \mathbb{Z}^n \mid ||\hat{z} - z||^2_{Q_{\hat{z}\hat{z}}} \leq \chi^2\}, \tag{52}$$

where $\hat{z} = Z\hat{a}$ and $Q_{\hat{z}\hat{z}} = ZQ_{\hat{a}\hat{a}}Z^T$.

In order for the search to be efficient, one would like the search space to be small such that it contains not too many integer vectors. This requires the choice of a small value for χ^2, but one that still guarantees that the search space contains at least one integer vector. After all, Ψ_z has to be nonempty to guarantee that it contains $\breve{z}_{\text{ILS}}$. Since the easy-to-compute (decorrelated) bootstrapped estimator gives a good approximation to the ILS estimator (cf. Theorem 9), $\breve{z}_{\text{B}}$ is an excellent candidate for setting the size of the search space:

$$\chi^2 = ||\hat{z} - \breve{z}_{\text{B}}||^2_{Q_{\hat{z}\hat{z}}} \tag{53}$$

In this way one can work with a very small search space and still guarantee that the sought for ILS solution is contained in it. If the rounding success-rate is sufficiently high, one may also use $\breve{z}_R$ instead of $\breve{z}_B$.

For the actual search, the quadratic form $||\hat{z} - z||^2_{Q_{\hat{z}\hat{z}}}$ is first written as a sum of squares. This is achieved by using the triangular decomposition $Q_{\hat{z}\hat{z}} = LDL^T$ (cf. Theorem 5):

$$\sum_{i=1}^{n} \frac{(\hat{z}_{i|I} - z_i)^2}{\sigma^2_{i|I}} \leq \chi^2. \tag{54}$$

This sum-of-squares structure can now be used to set up the n intervals that are used for the search. These sequential intervals are given as

$$\begin{aligned}
(\hat{z}_1 - z_1)^2 &\leq \sigma_1^2 \chi^2, \\
(\hat{z}_{2|1} - z_2)^2 &\leq \sigma^2_{2|1} \left(\chi^2 - \tfrac{(\hat{z}_1 - z_1)^2}{\sigma_1^2}\right), \\
&\vdots \\
(\hat{z}_{n|(n-1),\ldots,1} - z_n)^2 &\leq \sigma^2_{n|(n-1),\ldots,1} \left(\chi^2 - \sum_{i=1}^{n-1} \tfrac{(\hat{z}_{i|I} - z_i)^2}{\sigma^2_{i|I}}\right).
\end{aligned} \tag{55}$$

To search for all integer vectors that are contained in Ψ_z, one can now proceed as follows. First, collect all integers z_1 that are contained in the first interval. Then for each of these integers, one computes the corresponding length and center point of the second interval, followed by collecting all integers z_2 that lie inside this second interval. By proceeding in this way to the last interval, one finally ends up with the set of integer vectors that lie inside Ψ_z. From this set one then picks the ILS solution as the integer vector that returns the smallest value for $||\hat{z} - z||^2_{Q_{\hat{z}\hat{z}}}$. Various refinements on this search are possible (see, e.g., Teunissen 1995b; de Jonge and Tiberius 1996a; de Jonge et al. 1996; de Jonge 1998; Chang et al. 2005).

To understand why the decorrelating Z-transformation is necessary to improve the efficiency of the search, consider the structure of the above given sequential intervals and assume that they are formulated for the DD ambiguities of a single-baseline GNSS-RTK model. The DD ambiguity sequential conditional variances will then show a large discontinuity when going from the third to the fourth ambiguity. The RTK DD ambiguities are, namely, poorly estimable, i.e., have large variances, unless already three of them are assumed known, since with three DD ambiguities known, the baseline and remaining ambiguities can be estimated with a very high precision. This discontinuity of the DD ambiguity sequential conditional variances implies that σ_1^2, $\sigma^2_{2|1}$, and $\sigma^2_{3|2,1}$ are large, while the remaining variances $\sigma^2_{i|I}$, $i = 4, \ldots, n$ are very small. Thus the first three bounds of Eq. 55 are rather loose, while those of the remaining $(n - 3)$ inequalities are very tight. As a consequence one will experience "search halting." Of many of the collected integer candidates that satisfy the first three inequalities of Eq. 55, one will not be able to find corresponding integers that satisfy the remaining inequalities. This inefficiency in the search is eliminated when using the Z-transformed ambiguities instead of the DD ambiguities. The decorrelating Z-transformation eliminates the discontinuity and, by virtue of the fact that the product of the sequential variances remains invariant (volume is preserved), also reduces the large values of the first three conditional variances. For the construction of decorrelating Z-transformations, see Teunissen (1995b, and the references cited therein).

The ILS procedure is mechanized in the GNSS LAMBDA (Least-squares AMBiguity Decorrelation Adjustment) method, which is currently one of the most applied methods for GNSS carrier-phase ambiguity resolution. For more information on the LAMBDA method, we refer to Teunissen (1993, 1995b), de Jonge and Tiberius (1996a), de Jonge et al. (1996), and Chang et al. (2005) or to the textbooks Strang and Borre (1997), Teunissen and Kleusberg (1998), Leick (2004), Misra and Enge (2006), and Hofmann-Wellenhof et al. (2008).

The LAMBDA method is in use for a variety of different applications. Examples of such applications are baseline and network positioning (Tiberius and de Jonge 1995; de Jonge and Tiberius 1996b; Boon and Ambrosius 1997; Teunissen et al. 1997; Odijk 2002; Tsai and Juang 2007), satellite formation flying (Cox and Brading 1999; Wu and Bar-Sever 2006; Buist et al. 2008), InSAR and VLBI (Kampes and Hanssen 2004; Hobiger et al. 2009), GNSS attitude determination (Park and Teunissen 2003; Dai et al. 2004; Moenikes et al. 2005; Giorgi et al. 2008), and next generation GNSS (Eissfeller et al. 2002; Wu et al. 2004; Ji et al. 2007).

The Least-Squares PMF and Success-Rate

The ILS PMF is given as

$$\mathrm{P}(\breve{a}_{\mathrm{ILS}} = z) = \int_{\mathcal{L}_z} f_{\hat{a}}(x|a)dx. \tag{56}$$

To obtain the ILS success-rate, set $z = a$. Due to the complicated geometry of the ILS pull-in regions, methods of Monte Carlo simulation are needed to evaluate the multivariate integral Eq. 56. For $\hat{a} \sim \mathrm{N}(a, Q_{\hat{a}\hat{a}})$, the ILS success-rate is given by the integral

$$\mathrm{P}(\hat{a}_{\mathrm{ILS}} = a) = \int_{\mathcal{L}_0} C \exp\left\{-\frac{1}{2}||x||^2_{Q_{\hat{a}\hat{a}}}\right\} dx. \tag{57}$$

As a consequence of Theorem 1, this success-rate is the largest of all integer estimators.

Note that a is not needed for the computation of the success-rate. Thus one may simulate as if $\hat{a}$ has a zero mean. Also note that the ILS success-rate is Z-invariant, $\mathrm{P}(\breve{z}_{\mathrm{ILS}} = Za) = \mathrm{P}(\breve{a}_{\mathrm{ILS}} = a)$. This property can be used to one's advantage when simulating. Since the simulation requires the repeated computation of an ILS solution, one is much better of doing this for a largely decorrelated $\hat{z} = Z\hat{a}$ than for the original $\hat{a}$.

The first step is to use a random generator to generate n independent samples from the univariate standard normal distribution N(0, 1), and then collect these in an

n-vector s. This vector is transformed as Gs, with G equal to the Cholesky factor of $Q_{\hat{z}\hat{z}} = GG^T$. The result is a sample Gs from $\mathrm{N}(0, Q_{\hat{z}\hat{z}})$, and this sample is used as input for the ILS estimator. If the output of this estimator equals the null vector, then it is correct; otherwise, it is incorrect. This simulation process can be repeated N number of times, and one can count how many times the null vector is obtained as a solution, say N_s times, and how often the outcome equals a nonzero integer vector, say N_f times. The approximations of the success-rate and fail-rate follow then as

$$\mathrm{P}_s \approx \frac{N_s}{N} \text{ and } \mathrm{P}_f \approx \frac{N_f}{N}. \tag{58}$$

Instead of using simulation, one may also consider using bounds on the success-rate. The following theorem gives sharp lower and upper bounds on the ILS success-rate.

Theorem 9 (ILS success-rate bounds). *Let $\hat{a} \sim \mathrm{N}(a \in \mathbb{Z}^n, Q_{\hat{a}\hat{a}})$, $\hat{z} = Z\hat{a}$, and $c_n = (\frac{n}{2}\Gamma(\frac{n}{2}))^{2/n}/\pi$, with $\Gamma(x)$ the gamma function. Then*

$$\mathrm{P}(\breve{z}_{\mathrm{B}} = z) \leq \mathrm{P}(\breve{a}_{\mathrm{ILS}} = a) \leq P\left(\chi^2(n,0) \leq \frac{c_n}{\mathrm{ADOP}^2}\right) \tag{59}$$

for any admissible Z-transformation.

The lower bound is due to Teunissen (1998a, 1999b). The upper bound was first given in Hassibi and Boyd (1998), albeit without proof. A proof is given in Teunissen (2000). The above lower bound (after decorrelation) is currently the sharpest lower bound available for the ILS success-rate. A study on the performances of the various bounds can be found in Thomsen (2000) and Verhagen (2003, 2005a,b).

4 Baseline Quality and Model Validation

4.1 Fixed and Float Baseline

The estimation of $a \in \mathbb{Z}^n$ is usually not a goal in itself. The integer constraints are usually included so as to get a better estimator for the baseline, i.e., the real-valued parameter vector $b \in \mathbb{R}^p$. To study the impact of the integer constraints, we need the distribution of the fixed baseline $\breve{b}$. It is given by the following theorem (Teunissen 1999a).

Theorem 10 (Fixed baseline PDF). *Let $\hat{a}$, $\hat{b}$ be distributed as in Eq. 4 and let $\breve{a}$ be any integer estimator of a. Then the PDF of the fixed baseline estimator, $\breve{b} = \hat{b} - Q_{\hat{b}\hat{a}}Q_{\hat{a}\hat{a}}^{-1}(\hat{a} - \breve{a})$, is given as*

$$f_{\breve{b}}(x) = \sum_{z \in \mathbb{Z}^n} f_{\hat{b}(z)}(x) \mathrm{P}(\hat{a} = z), \tag{60}$$

where $f_{\hat{b}(z)}(x)$ *is the PDF of* $\hat{b}(z) \sim \mathrm{N}(b - Q_{\hat{b}\hat{a}} Q_{\hat{a}\hat{a}}^{-1}(a - z), Q_{\hat{b}\hat{b}} - Q_{\hat{b}\hat{a}} Q_{\hat{a}\hat{a}}^{-1} Q_{\hat{a}\hat{b}})$ *and* $\mathrm{P}(\breve{a} = z)$ *is the PMF of* $\breve{a}$.

This result shows that the fixed baseline distribution is *multimodal*, it equals an infinite sum of weighted conditional baseline distributions. These conditional baseline distributions are shifted versions of one another. The size and direction of the shift are governed by $Q_{\hat{b}\hat{a}} Q_{\hat{a}\hat{a}}^{-1} z$, $z \in \mathbb{Z}^n$. Each of the conditional baseline distributions in the sum is downweighted as z gets further apart from a. The weights are the masses of $\breve{a}$'s PMF.

Knowing the fixed baseline PDF allows one to study its quality by means of the probability that $\breve{b}$ lies in a certain region $C_b \subset \mathbb{R}^p$. In general it will be difficult to evaluate such probability exactly. For practical purposes it is therefore of importance to have bounds available for the probability $\mathrm{P}(\breve{b} \in C_b)$. We will assume C_b to be convex and symmetric about $\mathrm{E}(\hat{b}) = \mathrm{E}(\breve{b}) = b$. In that case the integral of $f_{\hat{b}(z)}(x)$ over C_b reaches its maximum for $z = a$. This shows that $\mathrm{P}(\hat{b}(z = a) \in C_b)$ can be taken as upper bound. A lower bound is also easily found. Since the entries in the sum of (60) are all nonnegative, any finite sum of nonzero entries can be used to obtain a lower bound. The more nonzero entries are used in this finite sum, the sharper this lower bound becomes. As a result we have obtained the following corollary.

Corollary 2 (Fixed baseline coverage probability). *Let* $C_b \subset \mathbb{R}^p$ *be any convex set symmetric about b. Then the fixed baseline coverage probability* $\mathrm{P}(\breve{b} \in C_b)$ *can be bounded from above and below as*

$$\mathrm{P}(\hat{b}(a) \in C_b)\mathrm{P}(\breve{a} = a) \leq \mathrm{P}(\breve{b} \in C_b) \leq \mathrm{P}(\hat{b}(a) \in C_b). \tag{61}$$

Note that these bounds become tight when the success-rate approaches 1. This shows, although the probability of the conditional estimator always overestimates the probability of the fixed baseline estimator, that the two probabilities are close for large success-rates. In other words, the (unimodal) distribution of the conditional estimator is a good approximation to the (multimodal) distribution of the fixed baseline estimator, when the success-rate is sufficiently close to 1.

The tolerance value to be chosen for $P(\breve{a} = a)$ depends on the error one is willing to accept. If one accepts a *relative error* of $0 \leq [P(\hat{b}(a) \in C_b) - P(\breve{b} \in C_b)]/P(\breve{b} \in C_b) \leq 10^{-\epsilon}$, then a success-rate of $P(\breve{a} = a) = \dfrac{1}{1 + 10^{-\varepsilon}}$ is required.

4.2 Cross-Validation of Mixed Integer Model

Without the integer constraints $a \in \mathbb{Z}^n$, the mixed integer model (3) reduces to a standard linear model. For such models, validation can be executed using the standard theory of hypothesis testing. In our case, a is integer and we may consider the following null- and alternative hypothesis

$$\begin{aligned} &\mathrm{H}_0 : y \sim \mathrm{N}(Aa + Bb, Q_{yy}), \; a \in \mathbb{Z}^n, \quad b \in \mathbb{R}^p, \\ &\mathrm{H}_a : y \sim \mathrm{N}(Aa + Bb + Cc, Q_{yy}), \quad a \in \mathbb{Z}^n, \quad (b^T, c^T)^T \in \mathbb{R}^p \times \mathbb{R}^q. \end{aligned} \tag{62}$$

where the additional term Cc, with matrix C known and vector c unknown, models under the alternative hypothesis the supposed modeling errors, e.g., outliers, slips, instrumental biases, etc. (Teunissen and Kleusberg 1998, Chaps. 5 and 8).

Knowing that a is integer strengthens the model and allows one to reevaluate the model. This is the situation one will have when testing the validity of a phase-based GNSS model with integer ambiguities. Instead of validating it as a linear model, one should think of validating it as a mixed integer linear model, as was emphasized in Teunissen and Verhagen (2008). Many, however, still apply the standard theory of hypothesis testing to Eq. 62. Stating that a is known to be integer, is however not the same as stating that a is a known integer. In order to be able to validate the mixed integer model, one must be able to answer questions like: (1) What are the appropriate test statistics for testing H_0 against H_a? (2) How are these test statistics distributed under H_0 against H_a? (3) What are the appropriate acceptance and rejection regions?

Here we present a solution based on the idea of cross-validation. First note that the observation equations of Eq. 62 can always be one-to-one transformed to a set in which only a part of the transformed observables is used under H_a. For this transformed case, the C-matrix of H_a will be of the form $C = (I_q, 0)^T$, which implies that the corresponding y_0 of $y = y_0^T, y_1^{TT}$ will not contribute to the parameter solution. We may therefore use the solution of H_a to predict the extra observable(s) of H_0, namely, y_0. An unlikely prediction error is then the reason for rejecting H_0 on the basis of H_a.

To keep things simple, we assume in the following that the necessary one-to-one transformation has been applied already. This allows us to work with $y = y_0^T, y_1^{TT}$ and use y_1 to predict y_0. Let $F = F_0^T, F_1^{TT} = (A, B)$, and $x = (a^T, b^T)^T$ and let $\hat{x}(1)$ be the float estimator of x based on y_1. Then the float prediction error is given as $\hat{\epsilon}_0 = y_0 - \hat{y}_0$, with the predictor $\hat{y}_0 = F_0\hat{x}_{(1)} + Q_{y_0y_1}Q_{y_1y_1}^{-1}(y_1 - F_1\hat{x}_{(1)})$ (see, e.g., Grafarend 1976). This prediction error is distributed under H_0 as $\hat{\epsilon}_0 \sim \mathrm{N}(0, Q_{\hat{\epsilon}_0\hat{\epsilon}_0})$. To test H_0, one can check whether or not the outcome of $\hat{\epsilon}_0$ lies in the tails of $\mathrm{N}(0, Q_{\hat{\epsilon}_0\hat{\epsilon}_0})$. If it does, H_0 is rejected on the basis of H_a. This is how the tests are executed for the standard linear model. A typical example is Baarda's *data snooping* (1968) for outlier detection, where each individual observation plays on its turn the role of y_0.

For our mixed integer model, we should not use the float prediction error $\hat{\epsilon}_0 = y_0 - \hat{y}_0$, but instead the fixed prediction error $\breve{\epsilon}_0 = y_0 - \breve{y}_0$, where the predictor $\breve{y}_0$ has taken the integerness of a into account (Teunissen 2007b).

Theorem 11 (Fixed prediction error PDF). *Let $y = (y_0^T, y_1^{T\,T})$ and let $\hat{\epsilon}_0 = y_0 - \hat{y}_0$ be the float prediction error. Then for any integer estimator $\breve{a}$, the fixed prediction error is given as $\breve{\epsilon}_0 = \hat{\epsilon}_0 - Q_{\hat{\epsilon}_0\hat{a}} Q_{\hat{a}}\hat{a}^{-1}(\hat{a} - \breve{a})$ and its PDF under* H_0 *as*

$$f_{\breve{\epsilon}_0}(x) = \sum_{z\in\mathbb{Z}}^{n} f_{\hat{\epsilon}_0(z)}(x)\mathbf{P}(\breve{a} = z), \tag{63}$$

where $f_{\hat{\epsilon}_0(z)}(x)$ is the PDF of $\hat{\epsilon}_0(z) \sim \mathrm{N}(-Q_{\hat{\epsilon}_0\hat{a}} Q_{\hat{a}\hat{a}}^{-1}(a - z), Q_{\hat{\epsilon}_0\hat{\epsilon}_0} - Q_{\hat{\epsilon}_0\hat{a}} Q_{\hat{a}\hat{a}}^{-1} Q_{\hat{a}\hat{\epsilon}_0})$, and $\mathrm{P}(\breve{a} = z)$ is the PMF of $\breve{a}$.

This result can now be used to test whether or not it is likely that the outcome of $\breve{\epsilon}_0$ is a sample from $f_{\breve{\epsilon}_0}(x)$. But before one can execute the test, we need to know the acceptance and rejection regions. Determining these regions is made difficult by the multimodality of $f_{\breve{\epsilon}_0}(x)$. Let $\mathcal{A} \subset \mathbb{R}^q$ be the acceptance region with coverage probability $\mathrm{P}(\breve{\epsilon}_0 \in \mathcal{A}|\mathrm{H}_0) = 1 - \alpha$. Since we want the rejection to be rare when the underlying model is correct, the false-alarm probability α is chosen as a small value. But since there are an infinite number of subsets that can produce this false-alarm probability, we still need to determine a way of defining a proper $\mathcal{A}$. It seems reasonable to define the optimal subset as the one which has the smallest volume. In that case the probability $1 - \alpha$ would be the most concentrated. This acceptance region is given as $\mathcal{A} = \{x \in \mathbb{R}^q | f_{\breve{\epsilon}_0}(x) \leq \lambda\}$, where λ is chosen so as to satisfy the given probability constraint.

The outcome of the test leads then to rejection of H_0 if $\breve{\epsilon}_0 \notin \mathcal{A}$. Note, due to the multi-modality of $f_{\breve{\epsilon}_0}(x)$, that the acceptance region may consist of a number of disconnected regions. This is a consequence of having the integerness of a taken into account.

5 Conclusions

This chapter presented a brief review of the theory of integer inference as originally developed for GPS. The theory has however a much wider range of applicability than only GPS. It also applies directly to the next generation GNSSs, such as modernized GPS, Galileo, or Compass, as well as to VLBI, InSAR, and acoustic interferometry, to name a few examples.

The mixed integer model was introduced as the model that underlies (interferometric) measurement systems observing fractional carrier phases. Three different classes of estimators were introduced for the mixed integer model. They are the class of integer estimators (I), the class of integer aperture estimators (IA), and the class of integer equivariant estimators (IE). Each class consists of a multitude of estimators. For each class the optimal estimator was presented. As optimality criterion we used the maximization of the probability of correct integer estimation or the minimization of the mean squared error.

For the I class, we presented in addition the properties of the three most popular integer estimators. They are the estimators of integer rounding (IR), integer bootstrapping (IB), and integer least-squares (ILS). Special attention was given to computational and statistical issues. The distributional properties of the I class were given and it was shown how they impact the quality of the corresponding real-valued mixed model parameter estimators. It was also shown that the increase in strength of the mixed integer linear model over that of the standard linear model allows one to reevaluate the validation of the model. A cross-validation method for the mixed integer model was presented together with the necessary multimodal distribution.

Acknowledgements The author is a recipient of the Australian Research Council Federation Fellowship (project number FF0883188). This support is gratefully acknowledged.

References

Baarda W (1968) A testing procedure for use in geodetic networks. Publications in geodesy, vol 2, no 5. Netherlands Geodetic Commission, Delft

Betti B, Crespi M, Sanso F (1993) A geometric illustration of ambiguity resolution in GPS theory and a Bayesian approach. Manuscr Geod 18:317–330

Boon F, Ambrosius B (1997) Results of real-time applications of the LAMBDA method in GPS based aircraft landings. In: Proceedings of the KIS97, Banff, pp 339–345

Buist PJ, Teunissen PJG, Giorgi G, Verhagen S (2008) Instantaneous multi-baseline ambiguity resolution with constraints. In: Proceedings of the international symposium on GPS/GNSS, Tokyo, pp 862–871

Chang XW, Yang X, Zhou T (2005) MLAMBDA: a modified LAMBDA method for integer least-squares estimation. J Geod 79:552–565

Cox DB, Brading JDW (1999) Integration of LAMBDA ambiguity resolution with Kalman filter for relative navigation of spacecraft. In: Proceedings of the ION NTM 99, San Diego, pp 739–745

Dai L, Ling KV, Nagarajan N (2004) Real-time attitude determination for microsatellite by LAMBDA method combined with Kalman filtering. In: Proceedings of the AIAA-ICSS 2004, Montercy, vol 3118, pp 1–8

de Jonge PJ (1998) A processing strategy for the application of the GPS in networks. Publications on geodesy, vol 46. Netherlands Geodetic Commission, Delft

de Jonge PJ, Tiberius CCJM (1996a) The LAMBDA method for integer ambiguity estimation: implementation aspects. LGR-series, vol 12. Publications of the Delft Computing Centre, Delft

de Jonge PJ, Tiberius CCJM (1996b) Integer estimation with the LAMBDA method. In: Beutler G et al (eds) Proceedings of the IAG symposium, GPS trends in terrestrial, airborne and spaceborne applications, Boulder, vol 115. Springer, Berlin, pp 280–284

de Jonge PJ, Tiberius CCJM, Teunissen PJG (1996) Computational aspects of the LAMBDA method for GPS ambiguity resolution. In: Proceedings of the ION GPS-96, Kansas City, pp 935–944

Eissfeller B, Tiberius CCJM, Pany T, Biberger R, Schueler T, Heinrichs G (2002) Instantaneous ambiguity resolution for GPS/Galileo RTK positioning. J Gyroscopy Navig 38(3):71–91

Farell JA, Barth M (1999) The global positioning system and inertial navigation. McGraw-Hill, New York

Giorgi G, Teunissen PJG, Buist P (2008) A search and shrink approach for the constrained LAMBDA method: experimental results. In: Proceedings of the international GPS/GNSS symposium, Tokyo, pp 797–806
Grafarend EW (1976) Geodetic applications of stochastic processes. Phys Earth Planet Inter 12:151–179
Grafarend EW (2000) Mixed integer-real valued adjustment problems: GPS initial cycle ambiguity resolution by means of the LLL algorithm. GPS Solut 4(2):31–44
Gundlich B, Koch KR (2002) Confidence regions for GPS baselines by Bayesian statistics. J Geod 76:55–62
Gundlich B, Teunissen PJG (2004) Multiple models: fixed, switching and interacting. In: Sanso F (ed) Proceedings of the 5th Hotine-Marussi symposium. IAG symposia series, vol 127. Springer, Berlin, pp 129–136
Hassibi A, Boyd S (1998) Integer parameter estimation in linear models with applications to GPS. IEEE Trans Signal Process 46(11):2938–2952
Hobiger T, Sekido M, Koyama Y, Kondo T (2009) Integer phase ambiguity estimation in next generation very long baseline interferometry. Adv Space Res 43(1):187–192
Hofmann-Wellenhof B, Lichtenegger H, Wasle E (2008) GNSS global navigation systems; GPS, GLONASS, Galileo and More. Springer, Wien
Ji S, Chen W, Zhao C, Ding X, Chen Y (2007) Single epoch ambiguity resolution for Galileo with the CAR and LAMBDA methods. GPS Solut 11(4):259–268
Joosten P, Tiberius CCJM (2002) LAMBDA: FAQs. GPS Solut 6:109–114
Kampes BM, Hanssen RF (2004) Ambiguity resolution for permanent scatterer interferometry. IEEE Trans Geosci Remote Sens 42(11):2446–2453
Koch KR (1999) Parameter estimation and hypothesis testing in linear models, 2nd edn. Springer, Berlin
Leick A (2004) GPS satellite surveying, 3rd edn. Wiley, New York
Liu LT, Hsu HT, Zhu YZ, Ou JK (1999) A new approach to GPS ambiguity decorrelation. J Geod 73:478–490
Misra P, Enge P (2006) Global positioning system: signals, measurements, and performance, 2nd edn. Ganga-Jamuna, Lincoln
Moenikes R, Wendel J, Trommer GF (2005) A modified LAMBDA method for ambiguity resolution in the presence of position domain constraints. In: Proceedings of the ION GNSS 2005, Long Beach, pp 81–87
Odijk D (2002) Fast precise GPS positioning in the presence of ionospheric delays. Publications on geodesy, vol 52. Netherlands Geodetic Commission, Delft
Park C, Teunissen PJG (2003) A new carrier phase ambiguity estimation for GNSS attitude determination systems. In: Proceedings of the international GPS/GNSS symposium, Tokyo, pp 1–8
Parkinson B, Spilker JJ (1996) GPS: theory and applications, vols 1 and 2. AIAA, Washington, DC
Robert CP, Casella G (1999) Monte Carlo statistical methods. Springer texts in statistics. Springer, New York
Strang G, Borre K (1997) Linear algebra, geodesy, and GPS. Wellesley-Cambridge, Wellesley
Svendsen JGG (2006) Some properties of decorrelation techniques in the ambiguity space. GPS Solut 10:40–44
Teunissen PJG (1993) Least-squares estimation of the integer GPS ambiguities. In: Invited lecture, section IV theory and methodology, IAG general meeting, Beijing, Aug 1993. Also in: LGR series, no 6. Delft Geodetic Computing Centre, Delft
Teunissen PJG (1995a) The invertible GPS ambiguity transformations. Manus Geod 20:489–497
Teunissen PJG (1995b) The least-squares ambiguity decorrelation adjustment: a method for fast GPS integer ambiguity estimation. J Geod 70:65–82
Teunissen PJG (1998a) Success probability of integer GPS ambiguity rounding and bootstrapping. J Geod 72:606–612
Teunissen PJG (1998b) On the integer normal distribution of the GPS ambiguities. Artif Satell 33(2):49–65

Teunissen PJG (1999a) The probability distribution of the GPS baseline for a class of integer ambiguity estimators. J Geod 70:65–82

Teunissen PJG (1999b) An optimality property of the integer least-squares estimator. J Geod 73:587–593

Teunissen PJG (2000) ADOP based upper bounds for the bootstrapped and the least-squares ambiguity success-rates. Artif Satell 35(4):171–179

Teunissen PJG (2002) The parameter distributions of the integer GPS model. J Geod 76:41–48

Teunissen PJG (2003a) Integer aperture GNSS ambiguity resolution. Artif Satell 38(3):79–88

Teunissen PJG (2003b) Theory of integer equivariant estimation with applications to GNSS. J Geod 77:402–410

Teunissen PJG (2005) GNSS ambiguity resolution with optimally controlled failure-rate. Artif Satell 40(4):219–227

Teunissen PJG (2006) Testing theory: an introduction, 2nd edn. VSSD, Delft

Teunissen PJG (2007a) Influence of ambiguity precision on the success rate of GNSS ambiguity bootstrapping. J Geod 81:351–358

Teunissen PJG (2007b) Least-squares prediction in linear models with integer unknowns. J Geod 81:565–579

Teunissen PJG, Kleusberg A (1998) GPS for geodesy, 2nd enlarged edn. Springer, Berlin

Teunissen PJG, Verhagen S (2008) GNSS carrier phase ambiguity resolution: challenges and open problems. In: IAG symposia series, vol 133. Springer, Berlin, pp 785–792

Teunissen PJG, Verhagen S (2009) The GNSS ambiguity ratio-test revisited: a better way of using it. Surv Rev 41:138–151

Teunissen PJG, de Jonge PJ, Tiberius CCJM (1997) The least-squares ambiguity decorrelation adjustment: its performance on short GPS baselines and short observation time spans. J Geod 71:589–602

Thomsen H (2000) Evaluation of upper and lower bounds on the success probability. In: Proceedings of the ION GPS 2000, Salt Lake City, pp 183–188

Tiberius CCJM, de Jonge PJ (1995) Fast positioning using the LAMBDA method. In: Proceedings of the DSNS-95, Bergen, paper 30, p 8

Tsai YF, Juang JC (2007) Ambiguity resolution validation based on LAMBDA and eigendecomposition. In: Proceedings of the 63rd AM-ION GPS, Portland, pp 299–304

Verhagen S (2003) On the approximation of the integer least-squares success-rate: which lower or upper bound to use? J Global Position Syst 2(2):117–124

Verhagen S (2005a) The GNSS integer ambiguities: estimation and validation. Publications on geodesy, vol 58. Netherlands Geodetic Commission, Delft

Verhagen S (2005b) On the reliability of integer ambiguity resolution. Navigation 52(2):99–110

Verhagen S, Teunissen PJG (2004) PDF evaluation of the ambiguity residuals. In: Sanso F (ed) Proceedings of the 5th Hotine-Marussi symposium. IAG symposia series, vol 127. Matera, Italy, pp 120–129

Verhagen S, Teunissen PJG (2005) Performance comparison of the BIE estimator with the float and fixed GNSS ambiguity estimator. In: Sanso F (ed) A window on the future of geodesy. IAG symposia series, vol 128. Springer, Berlin, pp 428–433

Verhagen S, Teunissen PJG (2006) New global navigation satellite system ambiguity resolution method compared to existing approaches. J Guid Control Dyn 29(4):981–991

Wu SC, Bar-Sever YE (2006) Real-time sub-cm differential orbit determination of two low-earth orbiters with GPS bias-fixing. In: Proceedings of the ION GNSS 2006, Fort Worth, pp 2515–2522

Wu F, Kubo N, Yasuda A (2004) Performance evaluation of GPS augmentation using quasi zenith satellite system. IEEE Trans Aerosp Electron Syst 40(4):1249–1260

Xu P (2001) Random simulation and GPS decorrelation. J Geod 75:408–423

Xu P (2006) Voronoi cells, probabilistic bounds and hypothesis testing in mixed integer linear models. IEEE Trans Inf Theory 51(7):3122–3138