

GNSS Ambiguity Resolution for Attitude Determination: Theory and Method

P.J.G. Teunissen
Delft University of Technology
Delft Institute of Earth Observation and Space Systems (DEOS)
The Netherlands

P.J.G.Teunissen@tudelft.nl

BIOGRAPHY

Peter Teunissen is full professor of Earth Observation and Navigation at the Delft University of Technology and Head of the Mathematical Geodesy and Positioning (MGP) group of the Delft Institute of Earth Observation and Space Systems (DEOS). He is the inventor of the LAMBDA method and has 25 years of GNSS research experience. His current research focuses on investigating the capabilities of the next generation GNSS for relative navigation and attitude determination.

ABSTRACT

In this contribution we give a brief review of the integer least-squares theory for GNSS attitude determination, together with a description of the key elements that make up the LAMBDA method for resolving the baseline constrained integer ambiguities.

1 INTRODUCTION

Global Navigation Satellite System (GNSS) ambiguity resolution is the process of resolving the unknown cycle ambiguities of the carrier phase data as integers. The sole purpose of ambiguity resolution is to use the integer ambiguity constraints as a means of improving significantly on the precision of the remaining model parameters.

In this contribution we consider the problem of ambiguity resolution for GNSS attitude determination. Attitude determination based on GNSS is a rich field of current studies, with a wide variety of challenging (terrestrial, sea, air and space) applications, see e.g. [1-18].

In the present contribution we give a brief review of the integer least-squares theory for GNSS attitude determination as it has been developed in [18], [22] and [26]. This review includes a description of the various elements that make up the LAMBDA method for resolving the baseline constrained integer ambiguities. Although our theory and method are applicable to the multi-baseline GNSS attitude determination problem, we restrict attention in

the present contribution to the single baseline case (two antennas) and therefore only consider the determination of heading and elevation (or yaw and pitch). The corresponding model is referred to as the GNSS compass model.

This contribution is organised as follows. In Section 2, we present the integer least-squares (ILS) solution of the standard GNSS model. This solution can be used for precise attitude determination provided the underlying GNSS model has sufficient strength. Since this is true in the multi-frequency case, GNSS attitude determination does not really pose a challenge if multiple frequencies are observed. In that case already the standard GNSS model has sufficient strength, thus implying that the standard LAMBDA method can be used for ambiguity resolution. The real ambiguity resolution challenge for GNSS attitude determination occurs when only single frequency data of a single epoch is used. In that case an additional strengthening of the underlying GNSS model is needed. In the GNSS compass model, this additional strengthening is found in assuming the length of the baseline known. Instead of using the baseline length as a hard constraint, we will use it as a weighted constraint, thus adding additional flexibility to our approach. The weighted constrained ILS solution is presented in Section 3. If the weight is set to zero, the solution reduces to the standard ILS solution and if the weight goes to infinity, one obtains the so-called quadratically constrained ILS solution.

In Section 4, we focus on the different issues that come up when solving the weighted constrained ILS problem. We describe the key elements of the constrained LAMBDA method and show how bounding functions combined with a search and shrink approach allows one to find the sought for integer solution in an efficient manner.

We also present an approximate method that is based on a quadratic approximation of the nonlinear objective function at the constrained float solution. In addition to being useful for setting the size of the search space for the

constrained LAMBDA method, this approximate method may also be used in its own right if the length of the baseline is not too short. This has the advantage that then the standard LAMBDA method may be used again. The contribution is concluded with a summary.

2 THE STANDARD GNSS MODEL

2.1 Heading and Elevation

In principle all the GNSS baseline models can be cast in the following frame of linearized observation equations,

$$E(y) = Aa + Bb, \quad D(y) = Q_{yy} \quad (1)$$

where y is the given GNSS data vector of order m , and a and b are the unknown parameter vectors of order n and p respectively. $E(\cdot)$ and $D(\cdot)$ denote the *expectation* and *dispersion* operator, and A and B are the given design matrices that link the data vector to the unknown parameters. Matrix A contains the carrier wavelengths and the geometry matrix B contains the receiver-satellite unit line-of-sight vectors. The variance matrix of y is given by the positive definite matrix Q_{yy} . The data vector y will usually consist of the 'observed minus computed' single- or multi-frequency double-difference (DD) phase and/or pseudorange (code) observations accumulated over all observation epochs. The entries of vector a are then the DD carrier phase ambiguities, expressed in units of cycles rather than range. They are known to be *integers*, $a \in \mathbb{Z}^n$. The entries of the vector b will consist of the remaining unknown parameters, such as baseline components (coordinates) and possibly atmospheric delay parameters (troposphere, ionosphere). They are known to be real-valued, $b \in \mathbb{R}^p$. Vectors a and b are referred to as the *ambiguity* vector and *baseline* vector, respectively.

Since we consider the GNSS-Compass application in the present contribution, we restrict attention to the case of satellite tracking with two near-by antennas. The short distance between the two antennas implies that we may neglect the (differential) atmospheric delays. Thus $p = 3$ and $b = (b_1, b_2, b_3)^T \in \mathbb{R}^3$ consists then only of the three coordinates of the baseline vector between the two antennas. If the baseline vector is parametrized with respect to the local North-East-Up frame, the heading H and elevation E can be computed from the baseline coordinates b_1 , b_2 and b_3 as

$$H = \arctan \frac{b_2}{b_1} \quad \text{and} \quad E = \arctan \frac{b_3}{\sqrt{b_1^2 + b_2^2}} \quad (2)$$

To obtain the most precise estimates of heading and elevation, use needs to be made of the very precise carrier phase data. The inclusion of the carrier phase data into the model accounts for the presence of the unknown integer ambiguity vector a in (1).

2.2 Integer Least Squares

We apply the least-squares estimation principle to model (1) to solve for the unknown parameter vectors a and b .

This gives the minimization problem

$$\min_{a,b} \|y - Aa - Bb\|_{Q_{yy}}^2, \quad a \in \mathbb{Z}^n, b \in \mathbb{R}^3 \quad (3)$$

with the square of the weighted norm defined as $\|\cdot\|_{Q_{yy}}^2 = (\cdot)^T Q_{yy}^{-1} (\cdot)$. The minimization problem (3), first introduced in [19], is subject to the integer constraint $a \in \mathbb{Z}^n$ and has therefore been coined an *Integer Least Squares* (ILS) problem by the author.

According to [19], the objective function of (3) can be decomposed as

$$\|y - Aa - Bb\|_{Q_{yy}}^2 = \|\hat{e}\|_{Q_{yy}}^2 + \|\hat{a} - a\|_{Q_{\hat{a}\hat{a}}}^2 + \|\hat{b}(a) - b\|_{Q_{\hat{b}(a)\hat{b}(a)}}^2 \quad (4)$$

where $\hat{e} = y - A\hat{a} - B\hat{b}$ is the least-squares residual vector, \hat{a} and \hat{b} are the solutions of the normal equations

$$\begin{bmatrix} A^T Q_{yy}^{-1} A & A^T Q_{yy}^{-1} B \\ B^T Q_{yy}^{-1} A & B^T Q_{yy}^{-1} B \end{bmatrix} \begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix} = \begin{bmatrix} A^T Q_{yy}^{-1} y \\ B^T Q_{yy}^{-1} y \end{bmatrix} \quad (5)$$

and

$$\hat{b}(a) = (B^T Q_{yy}^{-1} B)^{-1} B^T Q_{yy}^{-1} (y - Aa) \quad (6)$$

is the conditional baseline solution (conditioned on assuming a known). The real-valued least-squares solutions \hat{a} and \hat{b} are referred to as the *float* solutions of model (1). The variance matrix of the float ambiguity vector \hat{a} is the inverse of the b -reduced normal matrix of (5) and it is given as $Q_{\hat{a}\hat{a}} = (\bar{A}^T Q_{yy}^{-1} \bar{A})^{-1}$, with $\bar{A} = P_B^\perp A$ and $P_B^\perp = I_m - B(B^T Q_{yy}^{-1} B)^{-1} B^T Q_{yy}^{-1}$. The variance matrix of the conditional baseline vector $\hat{b}(a)$ is given as $Q_{\hat{b}(a)\hat{b}(a)} = (B^T Q_{yy}^{-1} B)^{-1}$.

From the orthogonal decomposition (4) it is clear that the third term on the right side can be made zero for any $a \in \mathbb{Z}^n$. The solution to the ILS minimization problem (3) follows therefore as

$$\check{a} = \arg \min_{a \in \mathbb{Z}^n} \|\hat{a} - a\|_{Q_{\hat{a}\hat{a}}}^2 \quad \text{and} \quad \check{b} = \hat{b}(\check{a}) \quad (7)$$

These solutions are referred to as the *fixed* solutions of model (1). The computation of \check{a} involves a search for the integer vector that is closest to \hat{a} in the metric of $Q_{\hat{a}\hat{a}}$. It can be computed efficiently with the LAMBDA method [20], [21]. Once \check{a} has been computed, the baseline solution \check{b} follows from substituting \check{a} for a in (6).

For $\check{b} = \hat{b}(\check{a})$, and thus heading \check{H} and elevation \check{E} , to take full advantage of the very high precision of the carrier phase data, the uncertainty in \check{a} needs to be as small as possible. This implies that the probability of correct integer ambiguity estimation, the so-called ambiguity *success rate*, needs to be close enough to 1.

For ambiguity resolution to be successful, the underlying GNSS model needs to have sufficient strength. Clearly,

the strength of the GNSS model improves and therefore the success rate gets larger when the number of satellites tracked gets larger, when the measurement precision improves, when the number of measurement epochs increases, or when the number of frequencies used gets larger. It can be shown that successful instantaneous ambiguity resolution and therefore precise epoch-by-epoch heading and elevation determination is possible using the standard LAMBDA method, when two or more frequencies are observed. This is not possible however, for the single-frequency case, as is shown in table 1. This table shows typical values of *single-epoch*, *single-frequency*, *short-baseline*, *ILS success rates* for different measurement precision and a varying number of satellites. Since single-frequency, single-epoch ambiguity resolution is not possible for the GNSS model (1), (unless the number of tracked satellites is larger than 8 and the code precision is better than 5 cm), a further strengthening of the model is needed.

3 BASELINE CONSTRAINED GNSS MODEL

3.1 Constrained Integer Least Squares

If we may assume that the two antennas are firmly attached to the rigid body of the moving platform, the constant length of the baseline vector may be determined a priori. In that case we can strengthen the GNSS model (1) by including the additional constraint $\|b\| = l$, with l the given baseline length (note: to denote the unweighted norm, we write $\|\cdot\|$ instead of $\|\cdot\|_I$). Our least-squares minimization problem becomes then

$$\min_{a,b} \|y - Bb - Aa\|_{Q_{yy}}^2, \quad a \in \mathbb{Z}^n, b \in \mathbb{R}^3, \|b\|^2 = l^2 \quad (8)$$

This least-squares problem has been coined a *Quadratically Constrained Integer Least-Squares* (QC-ILS) problem in [22]. It was introduced for the first time in [9], see also [15]. If we make use of decomposition (4), it follows that

$$\begin{aligned} & \min_{a \in \mathbb{Z}^n, b \in \mathbb{R}^3, \|b\|=l} \|y - Aa - Bb\|_{Q_{yy}}^2 = \|\hat{e}\|_{Q_{yy}}^2 + \\ & + \min_{a \in \mathbb{Z}^n} \left(\|\hat{a} - a\|_{Q_{\hat{a}\hat{a}}}^2 + \min_{b \in \mathbb{R}^3, \|b\|=l} \|\hat{b}(a) - b\|_{Q_{\hat{b}(a)\hat{b}(a)}}^2 \right) \end{aligned} \quad (9)$$

Note that now the third term on the right hand side does not vanish anymore. This is due to the presence of the baseline length constraint. We denote the minimizer of the third term as

$$\check{b}(a) = \arg \min_{b \in \mathbb{R}^3, \|b\|=l} \|\hat{b}(a) - b\|_{Q_{\hat{b}(a)\hat{b}(a)}}^2 \quad (10)$$

It is the vector on the sphere of radius l that has smallest distance to $\hat{b}(a)$, where distance is measured with respect to the metric as defined by the variance matrix $Q_{\hat{b}(a)\hat{b}(a)}$. Recall that $\hat{b}(a)$ is the conditional baseline solution (conditioned on assuming a known). The solution $\check{b}(a)$, having length $\|\check{b}(a)\| = l$, is therefore the baseline-length

constrained, conditional baseline solution. It can geometrically be depicted as the point where the ellipsoid $E = \{b \in \mathbb{R}^3 \mid \|\hat{b}(a) - b\|_{Q_{\hat{b}(a)\hat{b}(a)}}^2 = \text{constant}\}$ just touches the sphere $S_l = \{b \in \mathbb{R}^3 \mid \|b\| = l\}$, see Figure 1. This solution can be computed by means of an eigenvalue decomposition, see [18].

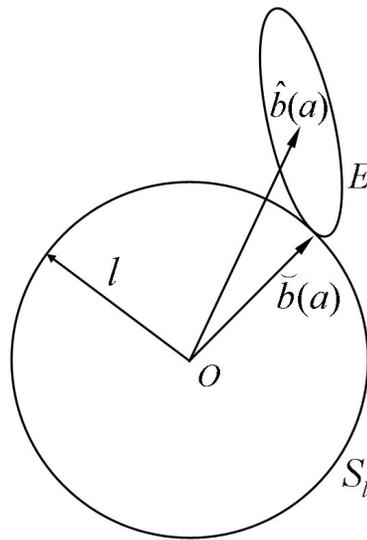


Fig. 1: The conditional baseline solution $\check{b}(a)$ is point of contact of ellipsoid $E = \{b \in \mathbb{R}^3 \mid \|\hat{b}(a) - b\|_{Q_{\hat{b}(a)\hat{b}(a)}}^2 = \text{constant}\}$ and sphere $S_l = \{b \in \mathbb{R}^3 \mid \|b\| = l\}$.

With (10), the minimizers \check{a} and \check{b} of the QC-ILS problem (9) follow as

$$\begin{aligned} \check{a} &= \arg \min_{a \in \mathbb{Z}^n} \left(\|\hat{a} - a\|_{Q_{\hat{a}\hat{a}}}^2 + \|\hat{b}(a) - \check{b}(a)\|_{Q_{\hat{b}(a)\hat{b}(a)}}^2 \right) \\ \check{b} &= \check{b}(\check{a}) \end{aligned} \quad (11)$$

Compare this solution with the unconstrained solution (7). In the unconstrained case, \check{a} is the integer vector closest to \hat{a} in the metric of $Q_{\hat{a}\hat{a}}$. This is not true anymore in the constrained case. In the constrained case a second term is added to the objective function. This second term measures the distance, in the metric of $Q_{\hat{b}(a)\hat{b}(a)}$, between $\hat{b}(a)$ and the sphere S_l . Thus potential candidates $a \in \mathbb{Z}^n$ are now not only downweighted if they are further away from \hat{a} , but also if their corresponding conditional baseline $\hat{b}(a)$ is further apart from the sphere S_l .

3.2 Weighted Constrained Integer Least Squares

Instead of using the baseline length as a *hard* constraint, it could also be used as a *weighted* constraint. This could be viewed as a more realistic approach, as the baseline length l is after all the result of an a priori measurement. If we consider the baseline length as an observable, we

| σ_ϕ [mm] | 3 | | |
|--------------------|------|------|------|
| σ_p [cm] | 30 | 15 | 5 |
| N=5 | 3.3 | 19.1 | 86.7 |
| N=6 | 24.8 | 66.7 | 96.9 |
| N=7 | 50.2 | 79.7 | 99.5 |
| N=8 | 86.2 | 94.5 | 99.9 |

Table 1: Single-frequency, single-epoch, short-baseline, ILS success rates (%) for different measurement precision (undifferenced σ_ϕ and σ_p) and different number of satellites (N).

need to extend model (1) with the observation equation

$$E(l) = \|b\|, \quad D(l) = \sigma_l^2 \quad (12)$$

Application of the least-squares principle to (1) and (12) gives

$$\min_{a \in \mathbb{Z}^n, b \in \mathbb{R}^3} \left\{ \|y - Aa - Bb\|_{Q_{yy}}^2 + \sigma_l^{-2} (l - \|b\|)^2 \right\} \quad (13)$$

This formulation is clearly more general than the previous ones. For $\sigma_l^2 \rightarrow \infty$, it reduces to the ILS-problem and for $\sigma_l^2 \rightarrow 0$ it reduces to the QC-ILS problem. Through the choice of σ_l^2 in (13), one can thus weigh the contribution of the baseline length constraint to the objective function. This offers the flexibility to allow the baseline length to differ from l , as will be the case for a nonrigid antenna configuration (e.g. nonrigid wing or fuselage of aircraft). The weighted constrained integer least-squares problem (13) was introduced for the first time in [22].

If we make use of decomposition (4), the minimizers \check{a} and \check{b} of (13) follow as

$$\check{a} = \arg \min_{a \in \mathbb{Z}^n} (\|\hat{a} - a\|_{Q_{aa}}^2 + G(a)) \quad \text{and} \quad \check{b} = \check{b}(\check{a}) \quad (14)$$

where

$$\begin{aligned} \check{b}(a) &= \arg \min_{b \in \mathbb{R}^3} H(a, b) \\ G(a) &= H(a, \check{b}(a)) \\ H(a, b) &= \left(\|\hat{b}(a) - b\|_{Q_{\hat{b}(a)\hat{b}(a)}}^2 + \sigma_l^{-2} (l - \|b\|)^2 \right) \end{aligned} \quad (15)$$

Compare this solution with the ILS solution (7) and the QC-ILS solution (11). Although we used the same notation \check{a} , \check{b} and $\check{b}(a)$ as before, this should not be any reason for confusion.

3.3 The Baseline by Nonlinear Least-Squares

Although the resolution of the integer ambiguities is the main topic of this contribution, it is useful to make a few remarks about the different approaches that can be used for computing the conditional baseline $\check{b}(a)$. Thus, next to the computation of the integer ambiguity minimizer \check{a} , one also needs to solve the baseline minimization problem

(see (15))

$$\check{b}(a) = \arg \min_{b \in \mathbb{R}^3} \left(\|\hat{b}(a) - b\|_{Q_{\hat{b}(a)\hat{b}(a)}}^2 + \sigma_l^{-2} (l - \|b\|)^2 \right) \quad (16)$$

In fact, this minimization problem needs to be solved every time the function $F(a) = \|\hat{a} - a\|_{Q_{aa}}^2 + G(a)$ is evaluated for some a , see (14) and (15).

The conditional baseline $\check{b}(a)$ is the least-squares solution of the nonlinear model

$$\begin{aligned} E \begin{bmatrix} \hat{b}(a) \\ l \end{bmatrix} &= \begin{bmatrix} b \\ \|b\| \end{bmatrix}, \\ D \begin{bmatrix} \hat{b}(a) \\ l \end{bmatrix} &= \begin{bmatrix} Q_{\hat{b}(a)\hat{b}(a)} & 0 \\ 0 & \sigma_l^2 \end{bmatrix} \end{aligned} \quad (17)$$

Hence, one can try to compute $\hat{b}(a)$ by using standard methods for solving nonlinear least-squares problems.

These are, for instance, the iterative descent methods, such as the *steepest descent* method, the *Gauss-Newton* method, or *Newton's* method. These methods have good convergence properties, provided the model is moderately nonlinear, the data are moderately inconsistent, and the initial value is close enough to the solution [23]. In case of the Gauss-Newton method, for instance, local convergence is guaranteed if

$$|k_n| \|\hat{r}\|_{Q_{yy}} \leq 1 \quad (18)$$

where $|k_n|$ is the in absolute value largest normal curvature of the nonlinear manifold (17) and \hat{r} is the corresponding least-squares residual vector.

The curvature of the nonlinear manifold (17) is in a large part determined by the length of the baseline. The smaller the baseline, the more curved the manifold becomes and the more difficult it will be for the standard iterative descent methods to converge to the required solution. Hence, the standard iterative descent methods can be used for large baselines but not for very small baselines. Thus if the baselines are too small, one will need to resort to alternative methods, such as, for instance, the globally convergent method as described in [18].

4 THE CONSTRAINED LAMBDA METHOD

In this section we describe the various elements of the constrained LAMBDA method. As we will see, the strength of the underlying model and the length of the baseline

determine the approach that can be taken to numerically solve the integer minimization problem (14).

We start by pointing out some of the challenges of computing \check{a} . Then we briefly describe two approaches for which the standard LAMBDA method can be used, the global and the local ellipsoidal search. For the most challenging cases, such as the single-epoch, single-frequency, short-baseline scenario, we describe how the various elements of the constrained LAMBDA method work.

4.1 The Challenges

In order to determine the integer ambiguities, we need to solve the *ambiguity* minimization problem

$$\check{a} = \arg \min_{a \in \mathbb{Z}^n} F(a) \quad \text{with} \quad F(a) = \|\hat{a} - a\|_{Q_{\hat{a}\hat{a}}}^2 + G(a) \quad (19)$$

In principle the integer minimizer \check{a} can be computed by means of an exhaustive search in the search space

$$\Omega(\chi^2) = \{a \in \mathbb{Z}^n \mid F(a) \leq \chi^2\} \quad (20)$$

First one sets the size of the search space by choosing a value for χ^2 . This value should be such that the search space is nonempty. Then one collects all integer vectors that lie inside $\Omega(\chi^2)$ and from this set one selects the integer vector that returns the smallest value for $F(a)$. This integer vector will then be the solution sought, i.e. the integer minimizer of $F(a)$. Hence, the global search for \check{a} has the following components:

1. Set size: choose small χ^2 such that $\Omega(\chi^2)$ nonempty.
2. Enumerate: find all integer vectors inside $\Omega(\chi^2)$.
3. Minimize: select \check{a} such that $F(\check{a}) \leq F(a)$ for all $a \in \Omega(\chi^2)$.

Each of these three components poses challenges if one wants to compute (19) in an efficient and timely manner. In order to guarantee that $\Omega(\chi^2)$ is nonempty, we choose $\chi^2 = F(z)$ for some $z \in \mathbb{Z}^n$. A proper choice of z is one that returns a small enough value for χ^2 . This requires that one can choose z to be close to \check{a} . This is not too difficult if the underlying GNSS model has sufficient strength. Rounding or bootstrapping the float solution would then give an integer solution that is already close to the solution sought. For models that lack sufficient strength, an alternative method of choice needs to be made. We will come back to this issue in Section 4.3.

Also the enumeration step of the above scheme is non-trivial. The presence of the additional term $G(a)$ in $F(a)$ makes that the geometry of the search space $\Omega(\chi^2)$ is not ellipsoidal anymore. The more complicated geometry of the search space makes the enumeration more difficult. To simplify the enumeration, it would help if we could work with a simpler search space such as, for instance, an

ellipsoidal search space. We will come back to this issue in the next two sections.

Finally, in order to find \check{a} , we need to evaluate $F(a)$ for all possible candidates. The evaluation of $F(a)$ involves the nontrivial computation of the conditional baseline $\check{b}(a)$ (see Section 3.3). Hence, if possible, one would like to avoid the need of having to evaluate $F(a)$ many times.

4.2 A Global Ellipsoidal Search

The complication of having to enumerate a search space with a somewhat complicated geometry can be remedied if one can work with an ellipsoidal search space instead. In order to ensure that the global minimizer is included in the enumeration, one needs to work with an ellipsoidal search space that encompasses $\Omega(\chi^2)$. The steps for computing \check{a} are then as follows:

1. Set size: choose small χ^2 such that $\Omega(\chi^2)$ nonempty.
2. Enumerate: find all integer vectors inside the larger ellipsoidal search space (see Figure 2)
$$\Omega_0(\chi^2) = \{a \in \mathbb{Z}^n \mid \|\hat{a} - a\|_{Q_{\hat{a}\hat{a}}}^2 \leq \chi^2\} \supset \Omega(\chi^2)$$

This can be done very efficiently with the LAMBDA method.
3. Minimize: select \check{a} such that $F(\check{a}) \leq F(a)$ for all $a \in \Omega_0(\chi^2)$.

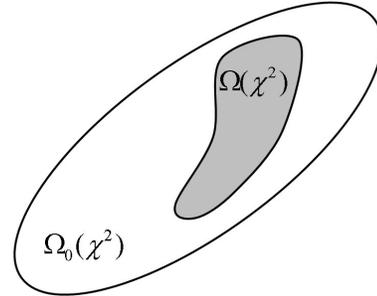


Fig. 2: Visualization of $\Omega(\chi^2) \subset \Omega_0(\chi^2)$.

Clearly this exhaustive search is rigorous, simple and rather straightforward to apply. This search is therefore an attractive method for finding \check{a} , provided it can be performed in a timely manner. The search becomes inefficient though if the search space contains too many integer vectors. Both the enumeration and minimization will then contribute to a slow down of the computational process. The larger the search space, the more integer vectors need to be enumerated and the more often the function $F(a)$ needs to be evaluated.

A too large search space $\Omega_0(\chi^2)$ can be avoided if one is able to compute a small enough value for χ^2 . This is possible if the underlying unconstrained GNSS model has sufficient strength. This is the case, for example, with

short-baseline multi-frequency models. For such models, bootstrapping (or rounding) the float solution usually already gives an integer close to the sought for integer solution. Such a bootstrapped (or rounded) integer can then be used to compute χ^2 . Thus for GNSS models that have sufficient strength, the above LAMBDA-based search is still attractive. This is generally not the case however for models that are based on single-frequency, single-epoch data. Such models are too weak to permit such a straightforward approach as described above. Hence, for these challenging cases we present alternative methods.

4.3 A Local Ellipsoidal Search

Instead of trying to solve $\tilde{a} = \arg \min_{a \in \mathbb{Z}^n} F(a)$ rigorously, one can also try to solve it in an approximate sense, i.e. by minimizing an approximation to the function $F(a)$. The standard LAMBDA method can then again be applied if we make use of a *quadratic* approximation. This idea was first introduced in [18], see also [22]. We therefore approximate $F(a)$ in a quadratic sense using the 'best' float solution available. The 'best' float solution is the *constrained* float solution, which is defined as the real-valued least-squares solution of the baseline constrained GNSS model (1) and (12). It can be expected to be better than the unconstrained float solution \hat{a} , since it incorporates the baseline length information. We denote the constrained float ambiguity solution as \bar{a} . Since $\partial_a F(\bar{a}) = 0$, the quadratic approximation of $F(a)$ at \bar{a} is given as

$$F(a) \approx F(\bar{a}) + \frac{1}{2}(\bar{a} - a)^T \partial_{aa}^2 F(\bar{a})(\bar{a} - a) \quad (21)$$

The Hessian matrix of (21) can be shown to read as (see [22])

$$\frac{1}{2} \partial_{aa}^2 F(\bar{a}) =$$

$$A^T Q_{yy}^{-1} A - A^T Q_{yy}^{-1} B [B^T Q_{yy}^{-1} B + \sigma_l^{-2} X]^{-1} B^T Q_{yy}^{-1} A \quad (22)$$

with

$$X = P_{\bar{b}} + \left(1 - \frac{l}{\|\bar{b}\|}\right) P_{\bar{b}}^\perp \quad (23)$$

and where $P_{\bar{b}} = \bar{b}(\bar{b}^T \bar{b})^{-1} \bar{b}^T$, $P_{\bar{b}}^\perp = I - P_{\bar{b}}$ and $\bar{b} = \arg \min_b H(\bar{a}, b)$.

Note, if matrix X is set to zero, that (22) reduces to $Q_{\hat{a}\hat{a}}^{-1}$, the inverse of the variance matrix of the unconstrained float ambiguity solution. Hence, the presence of matrix X in (22) is due to the baseline length constraint.

Also note that \bar{b} is the constrained float solution of the baseline. Hence, since σ_l^2 will usually be very small, the 'residual' $1 - l/\|\bar{b}\|$ can be expected to be small as well. In fact, in the limit $\sigma_l^2 \rightarrow 0$ we would have $\|\bar{b}\| = l$. If we neglect $1 - l/\|\bar{b}\|$, then $X = P_{\bar{b}}$ and (22) reduces to the matrix one would get when working with the *linearized* model

$$E \begin{bmatrix} y \\ l \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & \bar{b}^T / \|\bar{b}\| \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \quad (24)$$

This is the model one gets when $\|b\|$ in (12) is replaced by its first order approximation $\|b\| \approx \|\bar{b}\| + (\bar{b}/\|\bar{b}\|)^T (b - \bar{b}) = \bar{b}^T b / \|\bar{b}\|$.

Depending on how well the above quadratic approximation works, it may be used to compute an integer solution that is used either as a replacement of \tilde{a} or as a way of setting the size of the search space. For instance, one may use (21) to compute an approximate integer solution as

$$\tilde{a}' = \arg \min_{a \in \mathbb{Z}^n} \|\bar{a} - a\|_Q^2 \quad \text{with} \quad Q^{-1} = \frac{1}{2} \partial_{aa}^2 F(\bar{a}) \quad (25)$$

Either this solution is used in its own right and is thus used as a replacement of \tilde{a} . This is allowed if one can show that \tilde{a} and \tilde{a}' have comparable success rates. Or, alternatively, one uses \tilde{a}' , assuming that it is close to \tilde{a} , to set the size of the search space through $\chi^2 = F(\tilde{a}')$.

Instead of applying the ILS-principle to the quadratic approximation, one may also think of using bootstrapping or rounding. In case of bootstrapping, one uses both \bar{a} and the Hessian $\partial_{aa}^2 F(\bar{a})$, with the latter matrix playing the role of the weight matrix, whereas in case of rounding, one simply rounds the entries of \bar{a} to their nearest integer. Whether the above quadratic approximation can be used as a basis for computing a useful integer solution depends to a large extent on the length of the baseline. The longer the baseline, the better the quadratic approximation.

Hence, if the baseline is long enough, one can perform successful, single-frequency, single-epoch ambiguity resolution with the standard LAMBDA method, provided it is not based on the unconstrained model, but instead on the above quadratic approximation that follows from the constrained model (see Table 2).

4.4 Upper/Lower Bounding of Search Space

The quadratic approximation (21) avoids the complexity of having to compute $\tilde{b}(a)$ during the search. But it does of course not guarantee a global integer minimizer as result. In order to approximate $F(a)$ and still be able to guarantee that we can find the global minimizer, we introduce functions that are easy to evaluate and that bound $F(a)$ from below and from above:

$$F_1(a) \leq F(a) \leq F_2(a) \quad (26)$$

With these two bounding functions correspond the two search spaces

$$\begin{aligned} \Omega_1(\chi^2) &= \{a \in \mathbb{Z}^n \mid F_1(a) \leq \chi^2\} \\ \Omega_2(\chi^2) &= \{a \in \mathbb{Z}^n \mid F_2(a) \leq \chi^2\} \end{aligned} \quad (27)$$

They bound $\Omega(\chi^2)$ as

$$\Omega_2(\chi^2) \subset \Omega(\chi^2) \subset \Omega_1(\chi^2) \quad (28)$$

Note that the set ordering is the reverse of the function ordering. Also note that the sizes of the three sets are defined by the same χ^2 . Thus by varying χ^2 , all three sets

| σ_ϕ [mm] | 3 | | |
|--------------------|------|------|------|
| σ_p [cm] | 30 | 15 | 5 |
| $\ b\ = 0.5$ m | 8.2 | 18.2 | 71.8 |
| $\ b\ = 1$ m | 8.5 | 23.6 | 87.6 |
| $\ b\ = 2$ m | 11.5 | 38.1 | 97.1 |
| $\ b\ = 5$ m | 21.4 | 63.7 | 99.8 |
| $\ b\ = 20$ m | 53.5 | 94.3 | 99.9 |
| $\ b\ = 50$ m | 77.5 | 99.3 | 99.9 |

Table 2: Single-frequency, single-epoch, ILS success rates (%) based on a quadratic approximation of $F(a)$ for different measurement precision (undifferenced σ_ϕ and σ_p) and for different baseline lengths ($\|b\|$). Number of satellites was taken as $N = 6$.

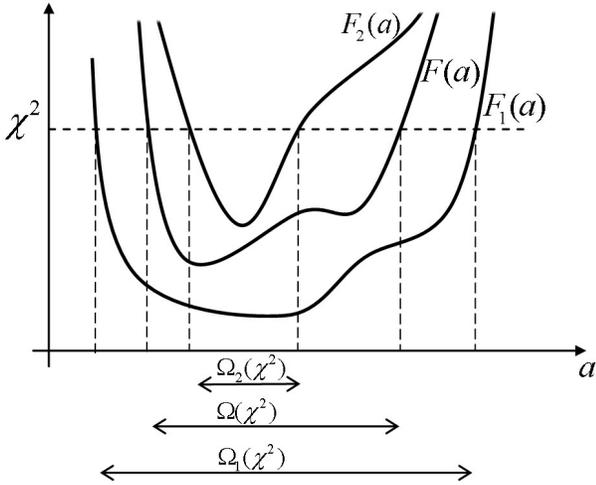


Fig. 3: Visualization of (26) and (28).

change in size, but the set ordering (28) remains intact (see Figure 3).

Now recall that the global minimizer \tilde{a} is found by evaluating $F(a)$ for all vectors in $\Omega(\chi^2)$, followed by selecting the vector that returns the smallest function value. Hence, one would like $\Omega(\chi^2)$ to be small and nonempty. The nonemptiness of $\Omega(\chi^2)$ guarantees that it contains \tilde{a} , and the smallness helps avoiding a multitude of function evaluations $F(a)$. These two requirements are met if $\Omega_2(\chi^2) \subset \Omega(\chi^2)$ is nonempty and $\Omega_1(\chi^2) \supset \Omega(\chi^2)$ is small.

To obtain a small $\Omega_1(\chi^2)$ while ensuring that $\Omega_2(\chi^2)$ is nonempty, we compute the smallest χ^2 for which $\Omega_2(\chi^2)$ is nonempty. Hence, we determine χ^2 as the integer minimizer of $F_2(a)$. This integer minimizer is determined by means of a *search and shrink* strategy. Starting with a certain initial χ_0^2 (using the method of Section 4.3), we search for an integer vector in the space Ω_2 :

$$\Omega_2(\chi_0^2) = \{a \in \mathbb{Z}^n \mid F_2(a) \leq \chi_0^2\} \subset \Omega(\chi_0^2) \quad (29)$$

As soon as such an integer vector is found, say \tilde{a} , the space is shrunk to the value $\tilde{\chi}^2 = F_2(\tilde{a}) < \chi_0^2$ and the

search continues in this smaller set. In this way the search proceeds rather quickly toward the integer minimizer of $F_2(a)$, which we denote as \tilde{a}_2 . This integer vector is not necessarily the minimizer of $F(a)$, but it is known to lie inside the set

$$\Omega(\chi_1^2) \subset \Omega_1(\chi_1^2) = \{a \in \mathbb{Z}^m \mid F_1(a) \leq \chi_1^2\} \quad (30)$$

with $\chi_1^2 = F_1(\tilde{a}_2)$. The sought for minimizer \tilde{a} is then found as the integer vector of $\Omega_1(\chi_1^2)$ that returns the smallest value of $F(a)$.

Next to the search and shrink strategy, the constrained LAMBDA method also has the option of the *search and expansion* strategy as introduced in [9], see also [10], [15]. This strategy was also shown to have excellent numerical performance.

4.5 The Bounding Functions

In principle, different choices for the bounding functions can be made. For instance, it is possible to choose $F_1(a)$ and $F_2(a)$ as quadratic forms so that $\Omega_1(\chi^2)$ and $\Omega_2(\chi^2)$ of (28) become ellipsoidal regions. With the search and shrink strategy, this gives the possibility to speed up the global ellipsoidal search strategy of Section 4.2.

The goal of working with the bounding sets is that we end up with a set $\Omega_1(\chi^2) \supset \Omega(\chi^2)$ that is small and nonempty. This can be achieved better if we work with bounding functions that have similar properties as $F(a)$ and therefore also produce similar search spaces.

Recall that

$$F(a) = \|\hat{a} - a\|_{Q_{\hat{a}\hat{a}}}^2 + \min_{b \in \mathbb{R}^3} H(a, b) \quad (31)$$

with

$$H(a, b) = \|\hat{b}(a) - b\|_Q^2 + \sigma_l^{-2} (l - \|b\|)^2 \quad (32)$$

for $Q = Q_{\hat{b}(a)\hat{b}(a)}$. We can bound $F(a)$ from below and from above with similar functions, if replace $Q = Q_{\hat{b}(a)\hat{b}(a)}$ by $Q = \frac{1}{\lambda} I_3$, where λ is the smallest and largest eigenvalue of $Q_{\hat{b}(a)\hat{b}(a)}$, respectively. We have the following Lemma.

Lemma [22]: If $Q = \frac{1}{\lambda}I$ and $\hat{b}(a) \neq 0$, then

$$\check{b}(a) = \arg \min_{b \in \mathbb{R}^3} H(a, b) = \frac{l + \sigma_l^2 \lambda \|\hat{b}(a)\|}{1 + \sigma_l^2 \lambda} \frac{\hat{b}(a)}{\|\hat{b}(a)\|} \quad (33)$$

and

$$\min_{b \in \mathbb{R}^3} H(a, b) = \frac{\lambda}{1 + \sigma_l^2 \lambda} \left(l - \|\hat{b}(a)\| \right)^2 \quad (34)$$

If $\hat{b}(a) = 0$, then the minimizer is not unique, but the minimum still is.

With the help of this lemma it follows that $F(a)$ is bounded from below and from above by the functions

$$\begin{aligned} F_1(a) &= \|\hat{a} - a\|_{Q_a}^2 + \frac{\lambda_{\min}}{1 + \sigma_l^2 \lambda_{\min}} \left(l - \|\hat{b}(a)\| \right)^2 \\ F_2(a) &= \|\hat{a} - a\|_{Q_a}^2 + \frac{\lambda_{\max}}{1 + \sigma_l^2 \lambda_{\max}} \left(l - \|\hat{b}(a)\| \right)^2 \end{aligned} \quad (35)$$

Note that these functions are indeed easy to evaluate. They avoid solving the baseline minimization problem (16).

It is our experience, that with this chosen set of bounding functions, the final number of integer vectors in the reduced search space $\Omega_1(\chi_1^2)$, $\chi_1^2 = F_1(\hat{a}_2)$, is very small and often simply equal to one, see [25].

5 SUMMARY

In this contribution we introduced the GNSS-Compass model as

$$\mathbf{E} \begin{bmatrix} y \\ l \end{bmatrix} = \begin{bmatrix} Aa + Bb \\ \|b\| \end{bmatrix}, \mathbf{D} \begin{bmatrix} y \\ l \end{bmatrix} = \begin{bmatrix} Q_{yy} & 0 \\ 0 & \sigma_l^2 \end{bmatrix} \quad (36)$$

with $a \in \mathbb{Z}^n$, $b \in \mathbb{R}^3$. It is the standard GNSS-baseline model extended with an extra observation equation that acts as a weighted baseline length constraint. Application of the least-squares principle gives the minimization problem

$$\min_{a \in \mathbb{Z}^n, b \in \mathbb{R}^3} \left\{ \|y - Aa - Bb\|_{Q_{yy}}^2 + \sigma_l^{-2} (l - \|b\|)^2 \right\} \quad (37)$$

For $\sigma_l^2 \rightarrow \infty$, it reduces to the ILS-problem (3) and for $\sigma_l^2 \rightarrow 0$ it reduces to the QC-ILS problem (8), see [9], [15], [18], [19], [20], [21], [24], [25]. Through the choice of σ_l^2 , one can thus weigh the contribution of the baseline length constraint to the objective function.

It was shown that the integer estimation part of (37) reduces to the computation of

$$\check{a} = \arg \min_{a \in \mathbb{Z}^n} F(a) \quad (38)$$

with $F(a) = \{\|\hat{a} - a\|_{Q_{\hat{a}\hat{a}}}^2 + G(a)\}$ and where $G(a) = \min_{b \in \mathbb{R}^3} \{\|\hat{b}(a) - b\|_{Q_{\hat{b}(a)\hat{b}(a)}}^2 + \sigma_l^{-2} (l - \|b\|)^2\}$.

The inclusion of the baseline length constraint increases the strength of the GNSS model and in particular enables

one to obtain higher ambiguity success rates. However, the inclusion of the constraint also introduces an additional curvature which results in a more complex ambiguity resolution process, in particular in case of very short baselines. Whether or not the increase in success rate is really needed and whether or not one really needs to take the complete curvature into account, all depends on the strength of the unconstrained GNSS model ($\sigma_l^{-2} = 0$) and on the length of the baseline.

For the precise determination of GNSS-compass information, we therefore discriminated between the following three classes of problems:

- Class I: If already the unconstrained GNSS model has sufficient strength, one case base the fixed baseline computation on the standard ILS estimator

$$\check{a} = \arg \min_{a \in \mathbb{Z}^n} \|\hat{a} - a\|_{Q_{\hat{a}\hat{a}}}^2 \quad (39)$$

In this case no baseline length constraint is needed ($\sigma_l^{-2} = 0$) and the standard LAMBDA method can be applied. This situation is applicable in the multi-frequency case.

- Class II: Since the nonlinearity of $F(a)$ gets smaller for longer baselines, one may use a quadratic approximation if the baseline is not too short. The integer ambiguity vector is then computed as

$$\check{a} = \arg \min_{a \in \mathbb{Z}^n} \|\bar{a} - a\|_{(\partial_{aa}^2 F(\bar{a}))^{-1}}^2 \quad (40)$$

with $\partial_{aa}^2 F(\bar{a})$ the Hessian matrix evaluated at the constrained float ambiguity vector \bar{a} . As with the problems of the previous class, this class permits the application of the standard LAMBDA method.

- Class III: The above approximate ambiguity solution will not benefit sufficiently from the baseline length constraint if the length is small. Hence, in that case one will have to compute the integer ambiguity vector rigorously as

$$\check{a} = \arg \min_{a \in \mathbb{Z}^n} \{\|\hat{a} - a\|_{Q_{\hat{a}\hat{a}}}^2 + G(a)\} \quad (41)$$

This requires the use of the constrained LAMBDA method.

A typical example from the last class is the single-frequency, single-epoch, short-baseline ambiguity resolution problem. The fact that in this case only single-frequency data on a single epoch is available makes the unconstrained GNSS model too weak for successful ambiguity resolution. Hence, the additional strength due to the baseline length constraint is then really needed. However, the quadratic approximation is not usable if the baseline length is too short. Thus in order to resolve this most challenging ambiguity resolution problem we need to apply the

baseline *constrained LAMBDA* method. For this method we described the following steps: (i) set initial size of $\Omega_2(\chi^2)$, by choosing $\chi^2 = F_2(z)$, where z follows from using the quadratic approximation; (ii) apply the search and shrink strategy to Ω_2 to get $\hat{a}_2 = \arg \min_{a \in \mathbb{Z}^n} F_2(a)$; (iii) set $\chi^2 = F_1(\hat{a}_2)$, enumerate $\Omega_1(\chi^2)$ and select integer minimizer of $F(a)$. As our experience shows, the method allows for a very fast, single-frequency, single-epoch ambiguity resolution of the GNSS compass model, see [25].

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