

ON THE COMPUTATION OF THE BEST INTEGER EQUIVARIANT ESTIMATOR

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ABSTRACT

Carrier phase integer ambiguity resolution is the key to high precision Global Navigation Satellite System (GNSS) positioning and navigation. In this contribution we study some of the computational aspects of best integer equivariant estimation. The best integer equivariant (BIE) estimator is the optimal estimator of the class of integer equivariant estimators, which is one of the three classes of estimators for carrier phase ambiguity resolution. The two other classes are the class of integer estimators and the class of integer aperture estimators. Since the BIE-estimator can not be computed exactly, it is shown how to approximate this estimator while retaining the property of integer equivariance. It is also shown how the decorrelating Z -transformation and the integer search of the LAMBDA method can be used to speed up the computation of the BIE-estimator.

Keywords: GNSS ambiguity resolution, integer least-squares, best integer equivariant estimation

1 INTRODUCTION

Global Navigation Satellite System (GNSS) ambiguity resolution is the process of resolving the unknown cycle ambiguities of double difference (DD) carrier phase data. It is the key to fast and high-precision GNSS relative positioning. An overview of GNSS carrier phase models, together with their applications in surveying, navigation, geodesy and geophysics, can be found in textbooks such as [*Hofmann-Wellenhof et al.*, 2001], [*Leick*, 1995], [*Misra and Enge*, 2001], [*Parkinson and Spilker*, 1996], [*Strang and Borre*, 1997] and [*Teunissen and Kleusberg*, 1998].

In order to describe the problem of GNSS ambiguity resolution, we take as our point of departure the following system of linear observation equations

$$E\{y\} = Aa + Bb, \quad a \in Z^n, \quad b \in R^p \quad (1)$$

with $E\{\cdot\}$ the mathematical expectation operator, y the m -vector of observables, a the n -vector of unknown integer parameters and b the p -vector of unknown real-valued parameters. The $m \times (n + p)$ design matrix (A, B) is assumed to be of full rank.

All the linear(ized) GNSS models can in principle be cast in the above frame of observation equations. The data vector y will then usually consist of the 'observed minus computed' single- or dual- frequency double-difference (DD) phase and/or pseudorange (code) observations accumulated over all observation epochs. The entries of vector a are then the DD carrier phase ambiguities, expressed in units of cycles rather than range, while the entries of the vector b will consist of the remaining unknown parameters, such as for instance baseline components (coordinates) and possibly atmospheric delay parameters (troposphere, ionosphere).

The procedure which is usually followed for solving the GNSS model can be divided into three steps. In the *first* step one simply discards the integer constraints $a \in Z^n$ and performs a standard least-squares (LS) adjustment. As a result one obtains the LS-estimators of a and b as \hat{a} and \hat{b} , respectively. This solution is usually referred to as the 'float' solution. In the *second* step the 'float' solution \hat{a} is further adjusted so as to take the integerness of the ambiguities into account in some pre-defined way. This gives

$$\hat{a}_S = S(\hat{a}) \quad (2)$$

in which S is an n -dimensional mapping that takes the integerness of the ambiguities into account. The estimator \hat{a}_S is then used in the final and *third* step to adjust the 'float' estimator \hat{b} . As a result one obtains the so-called 'fixed' estimator of b as

$$\hat{b}_S = \hat{b} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} (\hat{a} - \hat{a}_S) \quad (3)$$

in which $Q_{\hat{a}}$ denotes the variance-covariance (vc-) matrix of \hat{a} and $Q_{\hat{b}\hat{a}}$ denotes the covariance matrix of \hat{b} and \hat{a} .

The above three-step procedure is still ambiguous in the sense that it leaves room for choosing the n -dimensional map S . Different choices for S will lead to different ambiguity estimators and thus also to different baseline estimators \hat{b}_S . One can therefore now think of constructing a family of maps S with certain desirable properties. Three such classes of ambiguity estimators are the class of integer estimators, the class of integer aperture estimators, and the class of integer equivariant estimators. These classes were introduced in, respectively, [Teunissen, 1999, 2003, 2002]. These three classes are subsets of one another. The first class, the class of integer estimators, is the most restrictive class. This is due to the fact that the outcomes of any estimator within this class are required to be integer. The integer least-squares (ILS) estimator can be shown to be the optimal estimator within this class, see [Teunissen, 1999]. It is defined as

$$\check{a}_{LS} = \arg \min_{z \in Z^n} \| \hat{a} - z \|_{Q_{\hat{a}}}^2 \quad (4)$$

and it can be shown to have the largest possible probability of correct integer estimation. In contrast to integer rounding and integer bootstrapping, an integer search is needed to compute \check{a}_{LS} . The ILS-estimator and the integer search are efficiently mechanized in the LAMBDA method [Teunissen, 1993, 1995], which is currently one of the most applied methods for GNSS carrier phase ambiguity resolution. In particular the decorrelating Z -transformation of the LAMBDA-method is responsible for speeding up the integer search.

Practical results obtained with the LAMBDA method can be found, for example, in [Boon and Ambrosius, 1997], [Boon et al., 1997], [Cox and Brading, 1999], [de Jonge and Tiberius, 1996b], [de Jonge et al., 1996], [Han, 1995], [Peng et al., 1999], [Tiberius and de Jonge, 1995].

The second class, the class of integer aperture estimators, encompasses the class of integer estimators. The integer aperture estimators are of a hybrid nature in the sense that their outcomes are either integer or noninteger. Examples of different integer aperture estimators and their properties can be found in [Teunissen, 2003a, 2004, 2005]. The most relaxed of the three classes is the class of integer equivariant estimators. These estimators are real-valued and they only obey the integer remove-restore principle. The best integer equivariant (BIE) estimator can be shown to be the optimal estimator within this relaxed class, see [Teunissen, 2003b]. Here optimality is measured by minimizing the mean squared error of the estimator. When using the BIE-estimator care should be taken in how it is computed. The purpose of the current contribution is to show how the BIE-estimator should be computed and which pitfall should be avoided.

This contribution is organized as follows. In section 2 we give a brief review of the theory of integer equivariant estimation. It includes the definition of the class of integer equivariant estimators. In section 3 we give the BIE-estimator for an arbitrary probability density function (PDF) of the ambiguity float solution. It follows from minimizing the mean squared error within the class of integer equivariant estimators. Although the BIE-estimator holds true for any probability density function the data might have, we assume in section 4 that the data are normally distributed. For this case it follows that the BIE-estimator of the baseline can be obtained in a way which is very similar to the three-step procedure of current methods of ambiguity resolution. The only difference being that the integer ambiguity estimator needs to be replaced by its BIE-counterpart. Since the BIE-estimator of the ambiguities contains an infinite sum, it can not be evaluated in an exact manner. It is shown how to approximate the BIE-estimator while retaining the property of integer equivariance. It is also shown how the decorrelating Z -transformation and the integer search of the LAMBDA method can be used to speed up the computation of the BIE-estimator.

2 INTEGER EQUIVARIANT ESTIMATION

In order to describe the class of integer equivariant (IE) estimators, we consider estimating an arbitrary linear function of the two type of unknown parameters of the GNSS model (1),

$$\theta = l_a^T a + l_b^T b, \quad l_a \in R^n, \quad l_b \in R^p \quad (5)$$

It seems reasonable that an IE estimator should at least obey the *integer remove-restore* principle, see [Teunissen, 2002]. When estimating ambiguities in case of GNSS for instance, one would like, when adding an arbitrary number of cycles to the carrier phase data, that the solution of the integer ambiguities gets shifted by the same integer amount. For the estimator of θ this would mean that adding Az to y , with arbitrary $z \in Z^n$, must result in a shift of $l_a^T z$. Likewise it seems reasonable to require of the estimator that adding $B\zeta$ to y , with arbitrary $\zeta \in R^p$, results in a shift $l_b^T \zeta$. Afterall we would not like the integer part of the estimator to become contaminated by such an addition to y . Estimators of θ that fulfil these two conditions are called *integer equivariant*. Hence, they are defined as follows

Definition 1 (*IE estimators*)

The estimator $\hat{\theta}_{IE} = f_{\theta}(y)$, with $f_{\theta} : R^m \mapsto R$, is said to be an *integer equivariant* estimator of $\theta = l_a^T a + l_b^T b$ if

$$\begin{cases} f_{\theta}(y + Az) = f_{\theta}(y) + l_a^T z, \forall y \in R^m, z \in Z^n \\ f_{\theta}(y + B\zeta) = f_{\theta}(y) + l_b^T \zeta, \forall y \in R^m, \zeta \in R^p \end{cases} \quad (6)$$

It is easy to verify that integer estimators, like integer rounding, integer bootstrapping or integer least-squares, are integer equivariant. Simply check that the above two conditions are indeed fulfilled by integer estimators. The converse, however, is not necessarily true. The class of IE-estimators is therefore a larger class than the class of integer estimators.

The class of IE-estimators is also larger than the class of linear unbiased estimators. Let $f_{\theta}^T y$, for some $f_{\theta} \in R^m$, be the linear estimator of $\theta = l_a^T a + l_b^T b$. For it to be unbiased we need, using $E\{y\} = Aa + Bb$, that $f_{\theta}^T Aa + f_{\theta}^T Bb = l_a^T a + l_b^T b, \forall a \in R^n, b \in R^p$ holds true, or that both $l_a = A^T f_{\theta}$ and $l_b = B^T f_{\theta}$ hold true. But this equivalent to stating that

$$\begin{cases} f_{\theta}^T (y + Aa) = f_{\theta}^T y + l_a^T a, \forall y \in R^m, a \in R^n \\ f_{\theta}^T (y + Bb) = f_{\theta}^T y + l_b^T b, \forall y \in R^m, b \in R^p \end{cases} \quad (7)$$

Comparing this result with (6) shows that the condition of linear unbiasedness is more restrictive than the condition of integer equivariance. Hence, the class of linear unbiased estimators is a subset of the class of integer equivariant estimators.

3 BEST INTEGER EQUIVARIANT ESTIMATION

Having defined the class of IE-estimators we will now look for an IE-estimator which is 'best' in a certain sense. We will denote our *best integer equivariant* (BIE) estimator of θ as $\hat{\theta}_{BIE}$ and use the mean squared error (MSE) as our criterion of 'best'. The best integer equivariant estimator will therefore be defined as

$$\hat{\theta}_{BIE} = \arg \min_{f_{\theta} \in IE} E\{(f_{\theta}(y) - \theta)^2\} \quad (8)$$

in which *IE* stands for the class of IE-estimators. The minimization is thus taken over all integer equivariant functions that satisfy the conditions of Definition 1.

The reason for choosing the MSE-criterion is twofold. First, it is a well-known probabilistic criterion for measuring the closeness of an estimator to its target value, in our case θ . Second, the MSE-criterion is also often used as measure for the quality of the 'float' solution itself. The following theorem gives the solution to the above minimization problem (8).

Theorem 1 (*BIE estimation*)

Let $y \in R^m$ have mean $E\{y\} = Aa + Bb$ and probability density function (PDF) $p_y(y)$, and let $\hat{\theta}_{BIE}$ be the best integer equivariant estimator of $\theta = l_a^T a + l_b^T b$. Then

$$\hat{\theta}_{BIE} = \frac{\sum_{z \in Z^n} \int_{R^p} (l_a^T z + l_b^T \beta) p_y(y + A(a - z) + B(b - \beta)) d\beta}{\sum_{z \in Z^n} \int_{R^p} p_y(y + A(a - z) + B(b - \beta)) d\beta} \quad (9)$$

Proof: see [Teunissen, 2003b].

Note that the BIE-estimator can also be written as

$$\hat{\theta}_{BIE} = l_a^T \hat{a}_{BIE} + l_b^T \hat{b}_{BIE} \quad (10)$$

where

$$\begin{cases} \hat{a}_{BIE} = \sum_{z \in Z^n} z w_z(y) & , \sum_{z \in Z^n} w_z(y) = 1 \\ \hat{b}_{BIE} = \int_{R^p} \beta w_\beta(y) d\beta & , \int_{R^p} w_\beta(y) d\beta = 1 \end{cases} \quad (11)$$

in which the weighting functions $w_z(y)$ and $w_\beta(y)$ are defined by (9). This shows that the BIE-estimator of the integer parameter vector a is a weighted sum of all integer vectors in Z^n . The weights vary between zero and one, and their values are determined by y and its PDF. As a consequence the estimator \hat{a}_{BIE} will be real-valued in general, instead of integer-valued.

The above theorem holds true for any PDF the vector of observables y might have. This is therefore a very general result indeed. A closer look at (9) reveals however, that one needs a and b , and therefore θ , in order to compute $\hat{\theta}_{BIE}$. The dependence on a and b is present in the numerator of (9) and not in its denominator. The summation over all integer vectors in Z^n and the integration over R^p makes the dependence on a and b disappear in the denominator. If the dependence of $\hat{\theta}_{BIE}$ on θ persists one would not be able to compute the BIE-estimator. Note however that this dependence disappears in case the PDF of y has the structure $p_y(y) = f(y - Aa - Bb)$. And this property is fortunately still true for a large class of probability density functions, such as the multivariate normal distribution.

A direct and important consequence of the above theorem is that the BIE-estimator is always better than or at least as good as any integer estimator as well as any linear unbiased estimator. After all the class of integer estimators and the class of linear unbiased estimators are both subsets of the class of IE-estimators. The BIE-estimator is therefore also better than the best linear unbiased (BLU) estimator. The BLU-estimator is the minimum variance estimator of the class of linear unbiased estimators and it is given by the well-known Gauss-Markov theorem. We therefore have

$$\text{MSE}(\hat{\theta}_{BIE}) \leq \text{MSE}(\hat{\theta}_{BLU}) \quad (12)$$

The two estimators $\hat{\theta}_{BIE}$ and $\hat{\theta}_{BLU}$ both minimize the mean squared error within their class. In case of the BLU-estimator this is equivalent to minimizing the variance within LU. Since the IE estimator can be shown to be unbiased as well, it follows that inequality (12) also holds true for the variances of the two type of estimators. This is summarized in the following Gauss-Markov-like theorem.

Theorem 2 (*minimum variance unbiased estimation*)

The BIE-estimator is unbiased and has a better precision than the BLU-estimator:

$$\begin{aligned} (i) \quad E\{\hat{\theta}_{BIE}\} &= E\{\hat{\theta}_{BLU}\} \\ (ii) \quad D\{\hat{\theta}_{BIE}\} &\leq D\{\hat{\theta}_{BLU}\} \end{aligned} \quad (13)$$

where $D\{\cdot\}$ denotes the dispersion operator.

Proof: see [Teunissen, 2003b].

The above result is remarkable since it shows that for a large class of PDF's of y , one can *always*, with a model like (1), improve upon the precision of the BLU-estimator while keeping the estimator unbiased. If we apply the above theorem to the problem of estimating the baseline in case of GNSS and make the comparison with the 'float' baseline estimator \hat{b} and an unbiased 'fixed' baseline estimator \check{b} , we have

$$\begin{cases} D\{\hat{b}_{BIE}\} \leq D\{\check{b}\} & \text{and } E\{\hat{b}_{BIE}\} = E\{\check{b}\} \\ D\{\hat{b}_{BIE}\} \leq D\{\hat{b}\} & \text{and } E\{\hat{b}_{BIE}\} = E\{\hat{b}\} \end{cases} \quad (14)$$

The precision of the baseline estimator \hat{b}_{BIE} is therefore always better than or at least as good as the precision of its 'float' and 'fixed' counterparts.

4 COMPUTATION OF THE BIE ESTIMATOR

4.1 THE GAUSSIAN CASE

In our discussion of the BIE-estimator and its properties we did not make a particular choice so far for the PDF of y . In many applications however, such as GNSS, it is assumed that y is normally distributed. In that case the PDF of y takes the form

$$p_y(y) = \frac{1}{(2\pi)^{\frac{m}{2}} \sqrt{\det Q_y}} \exp -\frac{1}{2} \|y - Aa - Bb\|_{Q_y}^2 \quad (15)$$

where $\| \cdot \|_{Q_y}^2 = (\cdot)^T Q_y^{-1} (\cdot)$. With this Gaussian PDF the BIE-estimator also takes on a particular shape. We have the following corollary.

Corollary (*BIE in the Gaussian case*)

Let the PDF of y be given as in (15) and let $\hat{\theta}_{BIE}$ be the best integer equivariant estimator of $\theta = l_a^T a + l_b^T b$. Then

$$\hat{\theta}_{BIE} = l_a^T \hat{a}_{BIE} + l_b^T \hat{b}_{BIE} \quad (16)$$

with

$$\begin{cases} \hat{a}_{BIE} = \sum_{z \in Z^n} z \omega_z(\hat{a}) \\ \hat{b}_{BIE} = \hat{b} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} (\hat{a} - \hat{a}_{BIE}) \end{cases} \quad (17)$$

and

$$\omega_z(\hat{a}) = \frac{\exp -\frac{1}{2} \|\hat{a} - z\|_{Q_{\hat{a}}}^2}{\sum_{z \in Z^n} \exp -\frac{1}{2} \|\hat{a} - z\|_{Q_{\hat{a}}}^2} \quad (18)$$

This result shows that in the Gaussian case we may use the three-step procedure of section 1 also for best integer equivariant estimation. Thus first the float solution is computed. Then \hat{a} is used to compute \hat{a}_{BIE} . This is then finally followed by using the ambiguity residual $\hat{a} - \hat{a}_{BIE}$ to further adjust \hat{b} so as to obtain \hat{b}_{BIE} .

4.2 THE INTEGER EQUIVARIANT APPROXIMATION

The BIE ambiguity estimator cannot be computed exactly because of the infinite sum in (17). We will now show how to approximate the BIE estimator such that the property of integer equivariance is not lost.

If the infinite sum is replaced by a sum over a finite set of integers, say Θ , one should be careful not to lose the property of integer equivariance. Special care has therefore to be taken in choosing the integer set Θ . In order to determine a finite set of integers for the approximation of the BIE ambiguity estimator without spoiling the property of integer equivariance, the finite set Θ should not be chosen as a fixed set, but instead as a set of integers that depends on the float ambiguity vector \hat{a} . In order to achieve this, we first define the ellipsoidal region:

$$E_z^\lambda = \{x \in R^n \mid \|x - z\|_{Q_{\hat{a}}}^2 < \lambda^2\}, \quad z \in Z^n \quad (19)$$

This region is centred at the integer vector $z \in Z^n$, its shape is determined by the variance matrix $Q_{\hat{a}}$, and its size is governed by the parameter λ . This set has the indicator function:

$$\delta_z^\lambda(x) = \begin{cases} 1 & \text{if } x \in E_z^\lambda \\ 0 & \text{otherwise} \end{cases} \quad (20)$$

Complementary to this ellipsoidal region, we define the integer set:

$$\Theta_x^\lambda = \{z \in Z^n \mid \|x - z\|_{Q_{\hat{a}}}^2 < \lambda^2\}, \quad x \in R^n \quad (21)$$

This set contains all integer vectors which lie within a certain distance from $x \in R^n$, where the metric of the distance is determined by $Q_{\hat{a}}$. If we now replace the summation over the whole space of integers Z^n by the summation over the integer set $\Theta_{\hat{a}}^\lambda$, we can approximate the BIE estimator as:

$$\hat{a}_{BIE}^\lambda = \sum_{z \in \Theta_{\hat{a}}^\lambda} z \frac{\exp\{-\frac{1}{2}\|\hat{a} - z\|_{Q_{\hat{a}}}^2\}}{\sum_{z \in \Theta_{\hat{a}}^\lambda} \exp\{-\frac{1}{2}\|\hat{a} - z\|_{Q_{\hat{a}}}^2\}} \quad (22)$$

Thus the integer-summation is taken over an integer set that depends on the float solution \hat{a} . When \hat{a} changes, also the integer set, over which the summation is taken, changes. We can bring the above expression for \hat{a}_{BIE}^λ into the same form as our original expression for \hat{a}_{BIE} , if we make use of the indicator function (20). This gives

$$\hat{a}_{BIE}^\lambda = \sum_{z \in Z^n} z \frac{\delta_z^\lambda(\hat{a}) \exp\{-\frac{1}{2}\|\hat{a} - z\|_{Q_{\hat{a}}}^2\}}{\sum_{z \in Z^n} \delta_z^\lambda(\hat{a}) \exp\{-\frac{1}{2}\|\hat{a} - z\|_{Q_{\hat{a}}}^2\}} \quad (23)$$

which we can write as

$$\hat{a}_{BIE}^\lambda = \sum_{z \in Z^n} z \omega_z^\lambda(\hat{a}) \quad (24)$$

with

$$\omega_z^\lambda(\hat{a}) = \frac{\delta_z^\lambda(\hat{a}) \exp\{-\frac{1}{2}\|\hat{a} - z\|_{Q_{\hat{a}}}^2\}}{\sum_{z \in Z^n} \delta_z^\lambda(\hat{a}) \exp\{-\frac{1}{2}\|\hat{a} - z\|_{Q_{\hat{a}}}^2\}} \quad (25)$$

Compare the weighting function of the approximation, $\omega_z^\lambda(\hat{a})$, with the weighting function $w_z(\hat{a})$ of the BIE estimator, cf. (18). It is now easily verified that the approximation \hat{a}_{BIE}^λ is indeed integer equivariant.

4.3 SETTING THE SIZE OF THE INTEGER SET

In order to understand the approximation involved, we first note that $\lim_{\lambda \rightarrow \infty} \hat{a}_{BIE}^\lambda = \hat{a}_{BIE}$. Thus the approximation improves when the size of the integer set increases. Since the BIE estimator is a weighted sum over all integers, the integer set Θ_a^λ in (22) should be chosen such that the weights $w_u(\hat{a})$, $\forall u \notin \Theta_a^\lambda$ are so small that $w_u(\hat{a})u \approx 0$. The difference between the BIE estimator and its approximation is:

$$\begin{aligned} \hat{a}_{BIE} - \hat{a}_{BIE}^\lambda &= \sum_{z \in Z^n} z w_z(\hat{a}) - \sum_{z \in \Theta_a^\lambda} z w_z^\lambda(\hat{a}) \\ &= \sum_{z \in \Theta_a^\lambda} z (w_z(\hat{a}) - w_z^\lambda(\hat{a})) + \sum_{z \in Z^n \setminus \{\Theta_a^\lambda\}} z w_z(\hat{a}) \end{aligned} \quad (26)$$

Note that $w_z^\lambda(\hat{a}) \geq w_z(\hat{a})$, since for their denominators we have

$$\sum_{z \in \Theta_a^\lambda} \exp\left\{-\frac{1}{2}\|\hat{a} - z\|_{Q_a}^2\right\} \leq \sum_{z \in Z^n} \exp\left\{-\frac{1}{2}\|\hat{a} - z\|_{Q_a}^2\right\}$$

Hence, the approximation error is not only the result of ignoring the last term on the right-hand side of (26), but also due to the different weights assigned to the integers in the set Θ_a^λ .

Another way to look at the approximation is to ask oneself the question for which PDF the estimator \hat{a}_{BIE}^λ would become the *exact* BIE estimator. The approximation \hat{a}_{BIE}^λ is equal to the true BIE solution when the PDF of \hat{a} would be given by the *truncated normal* distribution:

$$p_a^\lambda(x) = \frac{\delta_a^\lambda(x) \exp\left\{-\frac{1}{2}\|x - a\|_{Q_a}^2\right\}}{\int_{E_a^\lambda} \exp\left\{-\frac{1}{2}\|x - a\|_{Q_a}^2\right\} dx} \quad (27)$$

This implies that the approximation \hat{a}_{BIE}^λ is close to \hat{a}_{BIE} when the difference between the normal distribution and its truncated version (27) is small. The difference between these two distributions is small, when λ is chosen such that

$$\frac{1}{(2\pi)^{\frac{n}{2}} \sqrt{\det Q_a}} \int_{E_a^\lambda} \exp\left\{-\frac{1}{2}\|x - a\|_{Q_a}^2\right\} dx \approx 1 \quad (28)$$

Hence, in order to get a good approximation, λ can be determined from

$$P(\hat{a} \in E_a^\lambda) = P(\|\hat{a} - a\|_{Q_a}^2 < \lambda^2) = 1 - \alpha \quad (29)$$

with α set at a small value. Thus by setting α at a small value, one can determine λ from the Chi-squared distribution, since $\|\hat{a} - a\|_{Q_a}^2$ has a central χ^2 -distribution with n degrees of freedom.

4.4 CONSTRUCTING THE INTEGER SET

Now that we have set the size of the integer set, we need to identify which integer vectors reside in this set. That is, we need to identify all integer vectors z that satisfy the inequality

$$\|\hat{a} - z\|_{Q_a}^2 < \lambda^2 \quad (30)$$

For this we can make use of the integer search implemented in the LAMBDA method, see (Teunissen, 1993, 1995), (de Jonge and Tiberius, 1996). However, in order to perform the search in an efficient manner, one first will have to transform the search space such that

its shape comes close to a spherical shape. This can be achieved with the decorrelating Z -transformation of the LAMBDA method. Thus the procedure is to first determine from the variance matrix $Q_{\hat{a}}$ the decorrelating Z -transformation. Then to transform the float solution \hat{a} and its variance matrix as $\hat{z} = Z\hat{a}$ and $Q_{\hat{z}} = ZQ_{\hat{a}}Z^T$, respectively, followed by a search for all integer vectors $u \in Z^n$ satisfying

$$\|\hat{z} - u\|_{Q_{\hat{z}}}^2 < \lambda^2 \quad (31)$$

Then $z = Z^{-1}u$ is used to back-transform the integer vectors such that the integer vectors satisfying the original inequality (30) are obtained.

5 SUMMARY

In this contribution we considered the computation of the BIE-estimator. In the Gaussian case, the BIE-estimator of the baseline is given as $\hat{b}_{BIE} = \hat{b} - Q_{\hat{b}\hat{a}}Q_{\hat{a}}^{-1}(\hat{a} - \hat{a}_{BIE})$, with the BIE-estimator of the ambiguity vector given as

$$\hat{a}_{BIE} = \sum_{z \in Z^n} z \omega_z(\hat{a})$$

where

$$\omega_z(\hat{a}) = \frac{\exp\{-\frac{1}{2}\|\hat{a} - z\|_{Q_{\hat{a}}}^2\}}{\sum_{z \in Z^n} \exp\{-\frac{1}{2}\|\hat{a} - z\|_{Q_{\hat{a}}}^2\}}$$

Since the computation of the BIE-estimator of the ambiguities requires a summation over all integer vectors, no exact evaluation is possible. One is therefore forced to make use of an approximation. In defining the approximation it is of importance to retain the property of integer equivariance. It was shown that this holds true for the approximation,

$$\hat{a}_{BIE}^\lambda = \sum_{z \in Z^n} z \omega_z^\lambda(\hat{a})$$

where

$$\omega_z^\lambda(\hat{a}) = \frac{\delta_z^\lambda(\hat{a}) \exp\{-\frac{1}{2}\|\hat{a} - z\|_{Q_{\hat{a}}}^2\}}{\sum_{z \in Z^n} \delta_z^\lambda(\hat{a}) \exp\{-\frac{1}{2}\|\hat{a} - z\|_{Q_{\hat{a}}}^2\}}$$

in which $\delta_z^\lambda(x)$ is the indicator function of the ellipsoidal region

$$E_z^\lambda = \{x \in R^n \mid \|x - z\|_{Q_{\hat{a}}}^2 < \lambda^2\}.$$

It is the introduction of the indicator function which avoids the infinite sum. Instead of the infinite sum, the summation is now taken over the finite integer set

$$\Theta_{\hat{a}}^\lambda = \{z \in Z^n \mid \|\hat{a} - z\|_{Q_{\hat{a}}}^2 < \lambda^2\}.$$

In order to compute \hat{a}_{BIE}^λ , we need a good choice for λ and we need to generate the integer vectors which reside in the integer set $\Theta_{\hat{a}}^\lambda$. This set can be generated efficiently with the integer search of the LAMBDA method, provided its decorrelating Z -transformation is applied first. To obtain a proper value for λ , use is made of the probability $P(\hat{a} \in E_a^\lambda) = P(\|\hat{a} - a\|_{Q_{\hat{a}}}^2 < \lambda^2) = 1 - \alpha$, where α is set at a small value. The central Chi-square distribution with n degrees of freedom can then be used to obtain λ from α .

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