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GNSS AMBIGUITY RESOLUTION WITH OPTIMALLY CONTROLLED FAILURE-RATE

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ABSTRACT. GNSS carrier phase ambiguity resolution is the key to fast and high- precision satellite positioning and navigation. It applies to a great variety of current and future models of GPS, modernized GPS and Galileo. In [*Teunissen*, 2003] we described the general principle of integer aperture (IA) ambiguity estimation. In the present contribution we will derive the optimal IA-estimator. The optimal IA-estimator is defined as the estimator which has the largest possible success rate given a user-defined fail rate. Hence, it is the optimal estimator for which the fail rate can be controlled.

Keywords: GNSS ambiguity resolution, integer least-squares, optimal aperture estimation

1 INTRODUCTION

In [*Teunissen*, 2003] we introduced the class of integer aperture (IA) estimators for carrier phase ambiguity resolution. This class allows one to design ambiguity estimators such that the ambiguity resolution process will have a user-defined fail-rate. Examples of such estimators have been given in [Teunissen, 2003a, 2005]. In this contribution we will derive the optimal IA-estimator. The optimal IA-estimator is defined as the estimator which has the largest possible success rate given a user-defined fail rate. We first discuss in section 2 integer estimation in relation to integer aperture estimation. The differences and similarities between the two type of estimators are discussed, and it is shown how their performances can be measured. In section 3 we derive the optimal IA estimator. This is done for an arbitary distribution of the float ambiguities. The derivation is presented in two steps. In the first step, only the aperture space is assumed unknown, but not the pull-in region. The result obtained is optimal for those cases where one already has chosen the integer estimator. In the second step, we assume both the aperture space and the pull-in region unknown. The resulting IA-estimator is the one which has the largest possible success rate given a fixed fail rate. We also give the optimal IA-estimator in case the float ambiguities are normally distributed. In this case, it is again the integer least-squares estimator which plays a prominant role.

2 INTEGER ESTIMATION AND INTEGER APERTURE ESTIMATION

2.1 THE CLASSES OF I- AND IA-ESTIMATORS

In order to understand the principle of integer aperture (IA) estimation properly, we first consider the principle of integer (I) estimation. Let $\hat{a} \in R^n$ be the unbiased real-valued float estimator of the GNSS ambiguity vector. Thus $E(\hat{a}) = a$, where $E(.)$ denotes the mathematical expectation operator and $a \in Z^n$ denotes the unknown integer ambiguity vector. In order to estimate a as an integer vector, a mapping $S : R^n \mapsto Z^n$ is introduced such that $\check{a} = S(\hat{a})$ is the integer estimator of a. The mapping S will be a many-to-one map due to the discrete nature of Z^n. This implies that different real-valued vectors will be mapped by S to one and the same integer vector. One can therefore assign a subset $S_z \subset R^n$ to each integer vector $z \in Z^n$:

$$S_z = \{x \in R^n \mid z = S(x)\}, \quad z \in Z^n \tag{1}$$

This subset is referred to as the *pull-in region* of z. It is the region in which all vectors are pulled to the same integer vector z. One can now define a class of integer estimators by imposing certain conditions on the pull-in regions. For instance, it seems reasonable that the pull-in regions cover R^n without gaps and overlaps. Furthermore, it is reasonable to require the pull-in regions to be translationally invariant. This implies that when the float solution $\hat{a}$ is perturbed by $z \in Z^n$, the corresponding integer solution is perturbed by the same amount. This property allows one to apply the *integer remove-restore* technique: $S(\hat{a}-z)+z = S(\hat{a})$. It therefore allows one to work with the fractional parts of the entries of $\hat{a}$, instead of with its complete entries.

The pull-in regions are translationally invariant and cover R^n without gaps and overlaps, if they satisfy the following three conditions:

$$\begin{array}{ll} (i) & \bigcup_{z \in Z^n} S_z = R^n \\ (ii) & Int(S_{z_1}) \bigcap Int(S_{z_2}) = \emptyset, \ \forall z_1, z_2 \in Z^n, z_1 \neq z_2 \\ (iii) & S_z = z + S_0, \ \forall z \in Z^n \end{array} \tag{2}$$

Using these pull-in regions, one can now give an explicit definition of I-estimators. It reads

Definition 1 (*Integer estimators*)
Integer estimators are defined as

$$\check{a} = \sum_{z \in Z^n} z s_z(\hat{a}) \ \text{ with } \ s_z(\hat{a}) = \begin{cases} 1 & \text{if} \ \ \hat{a} \in S_z \\ 0 & \text{if} \ \ \hat{a} \notin S_z \end{cases} \tag{3}$$

with the pull-in regions S_z satisfying the three conditions of (2)

Note that the $s_z(\hat{a})$ can be interpreted as weights, since $\sum_{z \in Z^n} s_z(\hat{a}) = 1$. Any integer estimator $\check{a}$ is therefore equal to a weighted sum of integer vectors with binary weights. Examples of integer estimators are, for instance, those obtained by integer rounding, integer bootstrapping or integer least-squares.

The outcome of an I-estimator is always integer. It may happen, however, that one is not willing to accept the integer outcome. This will be the case when one is doubtful about the correctness of the integer outcome. The class of integer aperture (IA) estimators has been introduced so as to relax the condition that the outcome of the ambiguity estimator should always be an integer. The pull-in regions of IA-estimators are therefore allowed to have gaps, thus making it possible that their outcomes could be equal to the float solution as well. Thus instead of R^n, now a subset $\Omega \subset R^n$ is taken as the region for which $\hat{a}$ is mapped to an integer if $\hat{a} \in \Omega$. It seems reasonable to ask of the region Ω that it has the property that if $\hat{a} \in \Omega$ then also $\hat{a} + z \in \Omega$, for all $z \in Z^n$. If this property would not hold, then float solutions could be mapped to integers whereas their fractional parts would not. We thus require Ω to be translationally invariant with respect to an arbitrary integer vector: $\Omega + z = \Omega$, for all $z \in Z^n$. Knowing Ω is however not sufficient for defining our IA-estimator. Ω only determines whether or not the float solution is mapped to an integer, but it does not tell us yet to which integer the float solution is mapped. We therefore define

$$\Omega_z = \Omega \cap S_z \ , \ \forall z \in Z^n \tag{4}$$

where S_z is a pull-in region satisfying the three conditions of (2). Then

$$
\begin{array}{ll}
(i) & \bigcup_z \Omega_z = \bigcup_z (\Omega \bigcap S_z) = \Omega \bigcap (\bigcup_z S_z) = \Omega \bigcap R^n = \Omega \\
(ii) & \Omega_{z_1} \bigcap \Omega_{z_2} = (\Omega \bigcap \Omega_{z_1}) \bigcap (\Omega \bigcap \Omega_{z_2}) = \Omega \bigcap (S_{z_1} \bigcap S_{z_2}) = \emptyset, \ \forall z_1, z_2 \in Z^n, z_1 \neq z_2 \\
(iii) & \Omega_0 + z = (\Omega \bigcap S_0) + z = (\Omega + z) \bigcap (S_0 + z) = \Omega \bigcap S_z = \Omega_z, \ \forall z \in Z^n
\end{array}
$$

This shows that the subsets $\Omega_z \subset S_z$ satisfy similar conditions as are satisfied by S_z, be it that R^n has now been replaced by $\Omega \subset R^n$. Hence, the mapping of the IA-estimator can now be defined as follows. The IA-estimator maps the float solution $\hat{a}$ to the integer vector z when $\hat{a} \in \Omega_z$ and it maps the float solution to itself when $\hat{a} \notin \Omega$. The class of IA-estimators can therefore be defined as follows.

Definition 2 (*Integer aperture estimators*)
Integer aperture estimators are defined as

$$\hat{a}_{IA} = \hat{a} + \sum_{z \in Z^n} (z - \hat{a}) \omega_z(\hat{a}) \tag{5}$$

with $\omega_z(x)$ the indicator function of $\Omega_z = \Omega \cap S_z$ and $\Omega \subset R^n$ translationally invariant.

Note, since the indicator functions $s_z(x)$ of the pull-in regions S_z sum up to unity, $\sum_{z \in Z^n} s_z(x) = 1$, that any I-estimator can be written as

$$\check{a} = \hat{a} + \sum_{z \in Z^n} (z - \hat{a}) s_z(x) \tag{6}$$

This shows that the class of I-estimators is a subset of the class of IA-estimators. A comparison shows that the difference between the two type of estimators lies in their binary weights, $s_z(x)$ versus $\omega_z(x)$. Since the $s_z(x)$ sum up to unity for all $x \in R^n$, the outcome of an I-estimator will always be integer. This is not true for an IA-estimator, since the binary weights $\omega_z(x)$ do not sum up to unity for all $x \in R^n$. The IA-estimator is therefore an hybrid estimator having as outcome either the real-valued float solution $\hat{a}$ or an integer solution. The IA-estimator returns the float solution if $\hat{a} \notin \Omega$ and it will be equal to z when $\hat{a} \in \Omega_z$. Note, since Ω is the collection of all $\Omega_z = \Omega_0 + z$, that the

IA-estimator is completely determined once Ω_0 is known. Thus $\Omega_0 \subset S_0$ plays the same role for the IA-estimators as S_0 does for the I-estimators. By changing the size and shape of Ω_0 one changes the outcome of the IA-estimator. The subset Ω_0 can therefore be seen as an adjustable pull-in region with two limiting cases The limiting case in which Ω_0 is empty and the limiting case when Ω_0 equals S_0. In the first case the IA-estimator becomes identical to the float solution $\hat{a}$, and in the second case the IA-estimator becomes identical to an I-estimator. The subset Ω_0 therefore determines the *aperture* of the pull-in region.

2.2 EVALUATION OF I- AND IA-ESTIMATORS

For the evaluation of the I- and IA-estimators we need the distributions of $\check{a}$ and $\bar{a}_{IA}$, respectively. For the I-estimator this distribution is of the discrete type and it will be denoted as $P(\check{a} = z)$. This distribution is obtained from integrating the probability density function (PDF) of $\hat{a}$, $f_{\hat{a}}(x)$, over the pull-in regions S_z,

$$P(\check{a} = z) = \int_{S_z} f_{\hat{a}}(x)dx \ , \ z \in Z^n \tag{7}$$

This distribution is of course dependent on S_z and thus on the chosen integer estimator. Since various integer estimators exist which are admissible, some may be better than others. Having the problem of GNSS ambiguity resolution in mind, one is particularly interested in the estimator which maximizes the probability of correct integer estimation, the *success-rate* P_S. This probability equals $P_S = P(\check{a} = a)$, but it will differ for different ambiguity estimators. The answer to the question which estimator maximizes the probability of correct integer estimation is given by the following theorem.

Theorem 1 (*Optimal integer estimation*)
Let $f_{\hat{a}}(x \mid a)$ be the PDF of the float solution $\hat{a}$ and let

$$\check{a}_{ML} = \arg\max_{a \in Z^n} f_{\hat{a}}(\hat{a} \mid a) \tag{8}$$

be an integer estimator. Then

$$P(\check{a}_{ML} = a) \geq P(\check{a} = a) \tag{9}$$

for any arbitrary integer estimator $\check{a}$.

Proof: see [*Teunissen*, 1999]

Note that we have denoted the PDF of $\hat{a}$ for the occasion as $f_{\hat{a}}(x \mid a)$ instead of as $f_{\hat{a}}(x)$. This has been done to explicitly show the dependence of the PDF on the true but unknown ambiguity vector $a \in Z^n$. In this contribution we will make a limited use of this notation. We will use the notation $f_{\hat{a}}(x \mid a)$ only when it is really needed to show the dependence on a explicitly.

The above theorem holds true for an arbitrary PDF of the float ambiguity vector $\hat{a}$. In most GNSS applications however, one assumes the data to be normally distributed. The estimator $\hat{a}$ will then be normally distributed too, with mean $a \in Z^n$ and variance-covariance (vc-) matrix $Q_{\hat{a}}$, $\hat{a} \sim N(a, Q_{\hat{a}})$. In this case the optimal estimator becomes identical to the integer least-squares (ILS) estimator

$$\check{a}_{LS} = \arg\min_{a \in Z^n} \| x - a \|^2_{Q_{\hat{a}}} \tag{10}$$

The above theorem therefore gives a probabilistic justification for using the ILS estimator when the PDF is Gaussian For GNSS ambiguity resolution it shows, that one is better off using the ILS estimator than any other admissible integer estimator. In contrast to integer rounding and integer bootstrapping, an integer search is needed to compute $\check{a}_{LS}$. The ILS procedure is mechanized in the LAMBDA method, [Teunissen, 1993, 1995]. Note that the ILS pull-in region of $z \in Z^n$, is the set described by all $x \in R^n$ satisfying

$$z = \arg\min_{u \in Z^n} \| x - u \|^2_{Q_{\hat{a}}} \tag{11}$$

Hence, it consists of all those points which are closer to z than to any other integer vector in R^n, $S_{LS,z} = \{x \in R^n \mid \| x - z \|^2_{Q_{\hat{a}}} \leq \| x - u \|^2_{Q_{\hat{a}}}, \forall u \in Z^n\}$.

Since the outcome of an I-estimator is always integer, the outcome is either correct or wrong Hence, for an I-estimator, the probability of a correct integer outcome (success), $P_S = P(\check{a} = a)$, and the probability of an incorrect integer outcome (failure), $P_F = \sum_{z \in Z^n \setminus \{a\}} P(\check{a} = z)$, add up to one: $P_S + P_F = 1$ This, however, will not be the case for an IA-estimator. In case of an IA-estimator, we have to take three different possible outcomes into account They are

$$\hat{a}_{IA} = \begin{cases} a \in Z^n & \text{(correct integer)} \\ z \in Z^n \setminus \{a\} & \text{(incorrect integer)} \\ \hat{a} \in R^n \setminus Z^n & \text{(no integer)} \end{cases} \tag{12}$$

The respective probabilities are therefore given as

$$\begin{cases} P_S &= P(\hat{a}_{IA} = a) &= \int_{\Omega_a} f_{\hat{a}_{IA}}(x)dx &= \int_{\Omega_a} f_{\hat{a}}(x)dx \quad \text{(success)} \\ P_F &= \sum_{z \neq a} P(\hat{a}_{IA} = z) &= \sum_{z \neq a} \int_{\Omega_z} f_{\hat{a}_{IA}}(x)dx &= \sum_{z \neq a} \int_{\Omega_z} f_{\hat{a}}(x)dx \quad \text{(failure)} \\ P_U &= P(\hat{a}_{IA} = \hat{a}) &= 1 - \int_{\Omega} f_{\hat{a}_{IA}}(x)dx &= 1 - P_S - P_F \quad \text{(undecided)} \end{cases} \tag{13}$$

All three probabilities are completely governed by $f_{\hat{a}}(x)$, the PDF of the float solution, and by Ω_0, the aperture pull-in region which uniquely defines the IA-estimator. Hence, one can proceed with the evaluation of IA-estimators once this information is available.

3 IA-ESTIMATION WITH OPTIMALLY CONTROLLED FAILURE-RATE

Since the outcome of an I-estimator is always integer and therefore $P_U = 0$, the fail-rate of an I-estimator equals one minus its success-rate. Thus although the optimal I-estimator has the largest possible success-rate, one can not excercise any control over its fail-rate. That is, the fail-rate of an I-estimator is determined completely by the strength, or the lack of strength for that matter, of the underlying mathematical model It can not be fixed a priori independently of the model This situation changes however in case of IA-estimation. Due to the fact that IA-estimators allow one to excercise control over the aperture of the pull-in region, it also gives one the possibility to excercise control over the fail-rate. The idea is therefore to constrain the fail-rate to a user-defined fixed value and then to find the size and shape of the pull-in region which maximizes the success-rate Hence, the optimization problem which needs to be solved is a constrained maximization problem. It reads

$$\max_{\Omega_0} P_S \text{ subject to given } P_F \tag{14}$$

Recall that the aperture pull-in region Ω_0 is governed by the choice made for the translationally invariant aperture space Ω and by the choice made for the pull-in region S_0, $\Omega_0 = \Omega \cap S_0$. This implies that we can think of three different maximization problems. One where we keep S_0 fixed and vary Ω. Another where we keep Ω fixed and vary S_0. And a third where we vary both Ω and S_0. Note, however, that the second maximization problem is not of much use. Since the sum $P_S + P_F$ is independent of S_0, it follows, when P_F and Ω are given, that P_S can not be varied by varying S_0. Hence, this leaves us with two different maximization problems. We will now consider them separately.

3.1 OPTIMAL IA-ESTIMATION FOR A GIVEN INTEGER ESTIMATOR

In this section we will solve the maximization problem

$$\max_{\Omega} P_S \text{ subject to given } S_0 \text{ and } P_F \tag{15}$$

Since the objective function and one of the constraints are both integrals, one may think of the well-known Neyman-Pearson lemma as a way to find the solution of (15). This lemma reads as follows.

Neyman-Pearson Lemma
Let $f(x)$ and $g(x)$ be integrable functions over R^n. Then the region

$$\hat{\Omega} = \{x \in R^n \mid f(x) \geq \lambda g(x)\} \ , \ \lambda \in R \tag{16}$$

solves the constrained maximization problem

$$\max_{\Omega \subset R^n} \int_{\Omega} f(x)dx \quad \text{subject to} \quad \int_{\Omega} g(x)dx = \text{constant} \tag{17}$$

if λ is chosen so as to satisfy the integral constraint.

Proof: See e.g. (*Rao*, 1973, p. 446)

Unfortunately, however, one can not make a direct use of the Neyman-Pearson lemma, since in our case the region of integration Ω has to satisfy the additional constraint that it is translationally invariant. A modification of the Neyman-Pearson lemma is therefore needed in order to suit our purposes. This modification is given in the following lemma.

Lemma 1 (*modified Neyman-Pearson lemma*)
Let $f(x)$ and $g(x)$ be integrable functions over R^n. Then the region

$$\hat{\Omega} = \{x \in R^n \mid \sum_{z \in Z^n} f(x+z) \geq \lambda \sum_{z \in Z^n} g(x+z)\} \ , \ \lambda \in R \tag{18}$$

solves the constrained maximization problem

$$\max_{\Omega \subset R^n} \int_{\Omega} f(x)dx \tag{19}$$

subject to the two constraints

$$\int_{\Omega} g(x)dx = \text{constant} \ \text{ and } \ \Omega = \Omega + z \ , \ \forall z \in Z^n$$

if λ is chosen so as to satisfy the integral constraint.

Proof: It is clear that $\hat{\Omega}$ satisfies both constraints. Now let Ω be any region that also satisfies both constraints. Then

$$\begin{array}{rcl} \int_{\hat{\Omega}} f(x)dx - \int_{\Omega} f(x)dx & = & \sum_{z\in Z^n} \int_{\hat{\Omega}\cap S_z} f(x)dx - \sum_{z\in Z^n} \int_{\Omega\cap S_z} f(x)dx \\ & = & \int_{\hat{\Omega}\cap S_0} \sum_{z\in Z^n} f(x+z)dx - \int_{\Omega\cap S_0} \sum_{z\in Z^n} f(x+z)dx \\ & = & \int_{(\hat{\Omega}-\hat{\Omega}\cap\Omega)\cap S_0} \sum_{z\in Z^n} f(x+z)dx - \int_{(\Omega-\hat{\Omega}\cap\Omega)\cap S_0} \sum_{z\in Z^n} f(x+z)dx \\ & \geq & \lambda \int_{(\hat{\Omega}-\hat{\Omega}\cap\Omega)\cap S_0} \sum_{z\in Z^n} g(x+z)dx - \lambda \int_{(\Omega-\hat{\Omega}\cap\Omega)\cap S_0} \sum_{z\in Z^n} g(x+z)dx \\ & = & \lambda \int_{\hat{\Omega}-\hat{\Omega}\cap\Omega} g(x)dx - \lambda \int_{\Omega-\hat{\Omega}\cap\Omega} g(x)dx = 0 \end{array}$$

The first equality follows from using the property that the pull-in regions S_z cover R^n without gaps and overlap, while the second equality follows from using the property of translational invariance. In the third equality the common part of the two integrals has been canceled. The inequality follows from using the definition of $\hat{\Omega}$, while the last equality follows from again using the property of translational invariance and the fact that both $\hat{\Omega}$ and Ω satisfy the same integral constraint. *End of proof.*

In order to apply Lemma 1 for solving (15), first note that (15) is equivalent to $\max_{\Omega} (P_S + P_F)$ subject to given P_F and S_0. With $P_S + P_F = \int_{\Omega} f_{\hat{a}}(x)dx$ and $P_F = \int_{\Omega\backslash\Omega_a} f_{\hat{a}}(x)dx = \int_{\Omega} f_{\hat{a}}(x)\,(1 - s_a(x))\,dx$, where $s_a(x)$ denotes the indicator function of S_a, we obtain the equivalent optimization problem

$$\max_{\Omega} \int_{\Omega} f_{\hat{a}}(x)dx \ \text{ subject to } \left\{ \begin{array}{l} P_F = \int_{\Omega} f_{\hat{a}}(x)\,(1 - s_a(x))\,dx \\ \Omega = \Omega + z\ ,\ \forall z \in Z^n \end{array} \right. \tag{20}$$

This formulation is now suitable for an application of the above lemma, where $f_{\hat{a}}(x)\,(1 - s_a(x))$ plays the role of the function $g(x)$. As a result we get $\Omega = \{x \in R^n \mid \sum_{z\in Z^n} f_{\hat{a}}(x+z) \geq \lambda \sum_{z\in Z^n} f_{\hat{a}}(x+z)\,(1 - s_a(x+z))\}$ and therefore, with $\Omega_0 = \Omega \cap S_0$, $\Omega_0 = \{x \in S_0 \mid \sum_{z\in Z^n} f_{\hat{a}}(x+z) \leq \frac{\lambda}{\lambda-1} f_{\hat{a}}(x+a)\}$. We therefore have the following theorem

Theorem 2 (*optimal IA-estimation for a given I-estimator*)
Let $f_{\hat{a}}(x)$ be the PDF of the float solution $\hat{a}$, and let P_S and P_F be respectively the success-rate and the fail-rate of the IA-estimator. Then the solution to

$$\max_{\Omega_0=\Omega\cap S_0} P_S \ \text{ subject to given } P_F \text{ and } S_0 \tag{21}$$

is given as

$$\Omega_0 = \{x \in S_0 \mid \sum_{z\in Z^n} f_{\hat{a}}(x+z) \leq \lambda_{FF} f_{\hat{a}}(x+a)\} \tag{22}$$

with λ_{FF} chosen so as to satisfy the a priori fixed fail-rate P_F.

This theorem is applicable to the situation where we have already decided which I-estimator to use (recall that S_0 defines the I-estimator). This could, for instance, be integer rounding, integer bootstrapping or integer least-squares. Given the I-estimator chosen, the theorem shows how we need to choose the aperture pull-in region in order to have the largest possible success rate for a user-defined given fail rate.

3.2 OPTIMAL IA-ESTIMATION

So far the pull-in region S_0 has been chosen arbitrarily. We therefore still need to find the best choice for S_0. Since $P_S + P_F$ is independent of S_0, but P_S and P_F are not, the best choice for S_0 is the pull-in region that maximizes P_S. This leads to the answer as to what the optimal IA-estimator is.

Theorem 3 (*optimal IA-estimation*)
Let $f_{\hat{a}}(x)$ be the PDF of the float solution $\hat{a}$, and let P_S and P_F be respectively the success-rate and the fail-rate of the IA-estimator. Then the solution to

$$\max_{\Omega_0=\Omega\cap S_0} P_S \text{ subject to given } P_F \tag{23}$$

is given by

$$\begin{cases} (i)\ \ \Omega_0 = \{x \in S_0 \mid \sum_{z\in Z^n} f_{\hat{a}}(x+z) \leq \lambda_{FF} f_{\hat{a}}(x+a)\} \\ (ii)\ \ \text{with } S_0 = \{x \in R^n \mid 0 = \arg\max_{z\in R^n} f_{\hat{a}}(x \mid z)\} \end{cases} \tag{24}$$

and λ_{FF} chosen so as to satisfy the a priori fixed fail-rate P_F.

Proof We have already proven (24*i*) for an arbitrary given pull-in region in Theorem 2. What remains to be shown is that the choice (24*ii*) maximizes the success rate. The success rate is given as $P_F = \int_{\Omega\cap S_0} f_{\hat{a}}(x+a \mid a)dx$. From Theorem 1 it follows that S_0 of (24*ii*) satisfies $\int_{S_0} f_{\hat{a}}(x+a \mid a)dx \geq \int_{S_0'} f_{\hat{a}}(x+a \mid a)dx$ for any arbitrary pull-in region S_0' satisfying the conditions of (2). We therefore also have $\int_{\Omega\cap S_0} f_{\hat{a}}(x+a \mid a)dx \geq \int_{\Omega\cap S_0'} f_{\hat{a}}(x+a \mid a)dx$. *End of proof*

The above result shows that the optimal IA-estimator reduces to the optimal I-estimator in case the volume of the aperture pull-in region reaches is maximum volume of one

3.3 THE GAUSSIAN CASE

The above results have been given for an arbitrary PDF of the float solution. In most GNSS applications, however, one assume the float solution to be Gaussian distributed, $\hat{a} \sim N(a, Q_{\hat{a}})$. In that case the PDF is given as

$$f_{\hat{a}}(x \mid a) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det Q_{\hat{a}}}} \exp\{-\frac{1}{2} \parallel x-a \parallel^2_{Q_{\hat{a}}}\} \tag{25}$$

The corresponding optimal aperture pull-in region is then given as

$$\Omega_0 = \left\{ x \in S_0 \mid \sum_{z\in Z^n\backslash\{0\}} \exp\{-\frac{1}{2} \parallel x-z \parallel^2_{Q_{\hat{a}}}\} \leq (\lambda_{FF}-1)\exp\{-\frac{1}{2} \parallel x \parallel^2_{Q_{\hat{a}}}\} \right\} \tag{26}$$

in which S_0 is the ILS pull-in region. Note that the contribution of the exponentials in the sum will get smaller the more peaked the PDF of the float solution is. The aperture pull-in region Ω_0 will therefore get larger the more peaked the PDF is. This is also what one would expect.

The computational steps in computing the optimal IA-estimator are now as follows. First we compute the ILS-solution from the float solution, $\check{a}_{LS} = \arg\min_{z\in Z^n} \parallel \hat{a} - z \parallel^2_{Q_{\hat{a}}}$. Then we form the ambiguity residual vector $\check{\epsilon}_{LS} = \hat{a} - \check{a}_{LS}$ and check whether $\check{\epsilon}_{LS} \in \Omega_0$. If this is the case, then the outcome of the optimal IA-estimator is $\check{a}_{LS}$, otherwise the outcome is $\hat{a}$. For the purpose of computational efficiency it is advisable to compute $\check{a}_{LS}$ with the LAMBDA method and use the LAMBDA-transformed ambiguities also for the evaluation of $\check{\epsilon}_{LS} \in \Omega_0$.

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