A NEW CLASS OF GNSS AMBIGUITY ESTIMATORS

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ABSTRACT

In Teunissen (1999) we introduced the class of admissible integer estimators. Members from this class are defined by their so-called pull- in regions. These pull-in regions satisfy the following three conditions. They are integer translational invariant and cover the whole ambiguity space without gaps and overlaps. Examples of such integer estimators are integer rounding, integer bootstrapping and integer least-squares. In the present contribution we will introduce a new class of GNSS ambiguity estimators. This class is referred to as the class of integer equivariant (IE) estimators since it still obeys the important integer remove-restore principle of integer estimation. It is shown that the IE- class is larger than the class of integer estimators as well as larger than the class of linear unbiased estimators. We will also give a useful representation of IE-estimators. This representation reveals the structure of IE-estimators and shows how they operate on the ambiguity 'float' solution.

Keywords: GNSS ambiguity resolution, integer equivariant estimation

1. INTRODUCTION

Ambiguity resolution applies to a great variety of GPS models currently in use. They range from single-baseline models used for kinematic positioning to multi-baseline models used as a tool for studying geodynamic phenomena. An overview of these and other GNSS models, together with their application in surveying, navigation and geodesy, can be found in textbooks such as (*Hofmann-Wellenhof et al.*, 2001), (*Leick*, 1995), (*Misra and Enge*, 2001), (*Parkinson and Spilker*, 1996), (*Strang and Borre*, 1997) and (*Teunissen and Kleusberg*, 1998). Despite the differences in application of the various GNSS models, it is important to understand that their ambiguity resolution problems are intrinsically the same. That is, the GNSS models on which ambiguity resolution is based, can all be cast in the following conceptual frame of linear(ized) observation equations

$$E\{y\} = Aa + Bb \tag{1}$$

where y is the given GNSS data vector of order m, a and b are the unknown parameter vectors respectively of order n and p, and where $E\{.\}$ denotes the mathematical expectation operator. The matrices A and B are the corresponding design matrices. The data vector y will usually consist of the 'observed minus computed' single- or dual- frequency double-difference (DD) phase and/or pseudorange (code) observations accumulated over all observation epochs. The entries of vector a are then the DD carrier phase ambiguities, expressed in units of cycles rather than range. They are known to be *integers*, $a \in Z^n$. The entries of the vector b will consist of the remaining unknown parameters, such as for instance baseline components (coordinates) and possibly atmospheric delay parameters (troposphere, ionosphere). They are known to be real- valued, $b \in R^p$.

The procedure which is usually followed for solving the GNSS model (1), can be divided into three steps [for more details we refer to e.g. (*Teunissen*, 1993) or (*de Jonge and Tiberius*, 1996)]. In the *first* step one simply disregards the integer constraints $a \in Z^n$ on the ambiguities and performs a standard adjustment. As a result one obtains the (real-valued) estimates of a and b, together with their variance-covariance matrix

$$\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix} , \begin{bmatrix} Q_{\hat{a}} & Q_{\hat{a}\hat{b}} \\ Q_{\hat{b}\hat{a}} & Q_{\hat{b}} \end{bmatrix} \qquad (2)$$

This solution is referred to as the 'float' solution. In the *second* step the 'float' ambiguity estimate $\hat{a}$ is used to compute the corresponding integer ambiguity estimate $\check{a}$. This implies that a mapping $S : R^n \mapsto Z^n$, from the n-dimensional space of reals to the n-dimensional space of integers, is introduced such that

$$\check{a} = S(\hat{a}) \qquad (3)$$

Once the integer ambiguities are computed, they are used in the *third* step finally to correct the 'float' estimate of b. As a result one obtains the 'fixed' solution

$$\check{b} = \hat{b} - Q_{\hat{b}\hat{a}}Q_{\hat{a}}^{-1}(\hat{a} - \check{a}) \qquad (4)$$

The ambiguity residual $(\hat{a} - \check{a})$ is thus used to adjust the 'float' solution so as to obtain the 'fixed' solution.

In this contribution alternatives to the choice of the map S will be considered. We first consider the class of admissible integer estimators in Sect. 2. In constructing this class, we are led by practical considerations such as: the estimator should map any 'float' solution to a unique integer solution and when the 'float' solution is perturbed by an integer amount, the integer solution should be perturbed by the same integer amount. In Sect. 3 we give three important examples of integer estimators which belong to this class of admissible estimators. They are the 'rounding' estimator, the 'bootstrapped' estimator and the integer least-squares estimator.

In Sect. 4 we introduce a new class of ambiguity estimators, which will be called integer equivariant (IE) estimators. The motivation for introducing the class of IE-estimators lies in the possible restrictive nature of the class of integer estimators. Hence, the IE-class is introduced by removing two of the three conditions of the class of integer estimators. The condition that remains is the one related to the integer remove-restore principle. And in order to be general enough the IE-class is introduced for estimating an arbitrary linear function of the two type of unknown parameters in the GNSS model, namely the integer parameters and the real-valued parameters.

We also give a useful representation of IE-estimators in Sect. 5. This representation reveals the structure of the estimators and allows one to devise one's own IE-estimator.

2. THE CLASS OF INTEGER ESTIMATIORS

There are many ways of computing an integer ambiguity vector $\check{a}$ from its real-valued counterpart $\hat{a}$. To each such method belongs a mapping $S : R^n \mapsto Z^n$ from the n-dimensional space of real numbers to the n-dimensional space of integers. Once this map has been defined, the integer ambiguity vector follows from its real-valued counterpart as $\check{a} = S(\hat{a})$. Due to the discrete nature of Z^n, the map S will not be one-to-one, but instead a many-to-one map. This implies that different real-valued ambiguity vectors may be mapped to the same integer vector. One can therefore assign a subset $S_z \subset R^n$ to each integer vector $z \in Z^n$:

$$S_z = \{x \in R^n \mid z = S(x)\}, \quad z \in Z^n \tag{5}$$

The subset S_z contains all real-valued ambiguity vectors that will be mapped by S to the same integer vector $z \in Z^n$. This subset is referred to as the *pull-in region* of z. It is the region in which all ambiguity 'float' solutions are pulled to the same 'fixed' ambiguity vector z.

Having defined the pull-in regions, we are now in a position to give an explicit expression for the corresponding integer ambiguity estimator. It reads

$$\check{a} = \sum_{z \in Z^n} z s_z(\hat{a}) \quad \text{with} \quad s_z(\hat{a}) = \left\{ \begin{array}{ll} 1 & \text{if } \hat{a} \in S_z \\ 0 & \text{otherwise} \end{array} \right. \tag{6}$$

Since the pull-in regions define the integer estimator completely, one can define a class of integer estimators by listing properties of these pull-in regions. In this section we introduce three properties of which it seems reasonable that they are possessed by the pull-in regions.

It seems reasonable to ask of the pull-in regions that their union covers the n-dimensional space completely,

$$\bigcup_{z \in Z^n} S_z = R^n \tag{7}$$

Otherwise one would have gaps, in which case not every $\hat{a} \in R^n$ can be assigned to a corresponding integer ambiguity vector.

Another property that we require of the pull-in-regions is that any two distinct regions should not have an overlap. Otherwise one could end up in a situation where a 'float' solution $\hat{a} \in R^n$ can not be assigned uniquely to a single integer vector. For the interior points of two distinct pull-in-regions we therefore require

$$S_{z_1} \bigcap S_{z_2} = \emptyset, \quad \forall z_1, z_2 \in Z^n, z_1 \neq z_2 \tag{8}$$

We allow the pull-in regions to have common boundaries however. This is permitted if we assume to have zero probability that $\hat{a}$ lies on one of the boundaries. This will be the case when the probability density function (pdf) of $\hat{a}$ is continuous.

The third and last property asked for is that the integer map S to possess the property that $S(x+z) = S(x)+z, \forall x \in R^n, z \in Z^n$. Also this property is a reasonable

one to request. It states that when the 'float' solution is moved by an integer amount z, the corresponding integer solution is moved by the same integer amount. This property allows one to use the *integer remove-restore* technique: $S(\hat{a} - z) + z = S(\hat{a})$. It therefore allows one to work with the fractional parts of the entries of $\hat{a}$, instead of with its complete entries, which may sometimes be large numbers.

The integer remove-restore property implies that $S_{z_1 + z_2} = \{x \in R^n \mid z_1 + z_2 = S(x)\} = \{x \in R^n \mid z_1 = S(x) - z_2 = S(x - z_2)\} = \{x \in R^n \mid z_1 = S(y), x = y + z_2\} = S_{z_1} + z_2, \forall z_1, z_2 \in Z^n$. Hence, it means that the pull-in regions are translated copies of one another. This third property may therefore also be stated as

$$S_z = z + S_0, \quad \forall z \in Z^n \tag{9}$$

with S_0 being the pull-in region of the origin of Z^n.

Integer ambiguity estimators that possess all three of the above stated properties form a class. This class will be referred to as the class of admissible integer ambiguity estimators. It is defined as follows:

Definition 1 (*admissible integer estimators*)
The integer estimator $\check{a} = \sum_{z \in Z^n} z s_z(\hat{a})$ is said to be *admissible* if

$$\begin{aligned}
(i) \quad & \bigcup_{z \in Z^n} S_z = R^n \\
(ii) \quad & S_{z_1} \bigcap S_{z_2} = \emptyset, \quad \forall z_1, z_2 \in Z^n, z_1 \neq z_2 \\
(iii) \quad & S_z = z + S_0, \quad \forall z \in Z^n
\end{aligned}$$

There exist various integer estimators that belong to this class. As the definition shows, one way of constructing admissible estimators is to choose a subset S_0 such that its translated copies cover R^n without gaps and overlaps. In two dimensions this can be achieved, for instance, by choosing S_0 as the unit square centred at the origin.

3. EXAMPLES OF INTEGER ESTIMATORS

In this section three different admissible integer estimators are considered. All three of them have been in use, in one way or another, for GNSS ambiguity resolution. They are the 'rounding' estimator, the 'bootstrapped' estimator and the least-squares estimator.

Integer rounding
The simplest way to obtain an integer vector from the real-valued 'float' solution is to round each of the entries of $\hat{a}$ to its nearest integer. The corresponding integer estimator reads therefore

$$\check{a}_R = ([\hat{a}_1], \ldots, [\hat{a}_n])^T \tag{10}$$

where '[.]' denotes rounding to the nearest integer. This estimator is clearly admissible. The first two conditions of the definition are satisfied, since - apart from ties in rounding - any 'float' solution $\hat{a} \in R^n$ gets mapped to a unique integer vector. The third condition is also satisfied since rounding admits the integer remove- restore technique, that is, $[x - z] + z = [x], \forall x \in R, z \in Z$.

Since componentwise rounding implies that each real-valued ambiguity estimate $\hat{a}_i, i = 1, \ldots, n$, is mapped to its nearest integer, the absolute value of the difference

between the two is at most $\frac{1}{2}$. The pull-in regions $S_{R,z}$ that belong to this integer estimator are therefore given as

$$S_{R,z} = \cap_{i=1}^{n} \{x \in R^n \mid \mid x_i - z_i \mid \leq \frac{1}{2}\}, \ \forall z \in Z^n \tag{11}$$

They are n-dimensional cubes, centred at the $z \in Z^n$, all having sides of length one.

Integer bootstrapping

Another relatively simple integer ambiguity estimator is the bootstrapped estimator. The bootstrapped estimator can be seen as a generalization of the previous estimator. It still makes use of integer rounding, but it also takes some of the correlation between the ambiguities into account. The bootstrapped estimator follows from a sequential conditional least- squares adjustment and it is computed as follows. If n ambiguities are available, one starts with the first ambiguity $\hat{a}_1$, and rounds its value to the nearest integer. Having obtained the integer value of this first ambiguity, the real-valued estimates of all remaining ambiguities are then corrected by virtue of their correlation with the first ambiguity. Then the second, but now corrected, real-valued ambiguity estimate is rounded to its nearest integer. Having obtained the integer value of the second ambiguity, the real-valued estimates of all remaining $n-2$ ambiguities are then again corrected, but now by virtue of their correlation with the second ambiguity. This process is continued until all ambiguities are considered. The components of the bootstrapped estimator $\check{a}_B$ are given as

$$
\begin{aligned}
\check{a}_{B,1} &= [\hat{a}_1] \\
\check{a}_{B,2} &= [\hat{a}_{2|1}] &= [\hat{a}_2 - \sigma_{\hat{a}_2 \hat{a}_1} \sigma_{\hat{a}_1}^{-2} (\hat{a}_1 - \check{a}_{B,1})] \\
&\vdots \\
\check{a}_{B,n} &= [\hat{a}_{n|N}] &= [\hat{a}_n - \sum_{i=1}^{n-1} \sigma_{\hat{a}_n \hat{a}_{i|I}} \sigma_{\hat{a}_{i|I}}^{-2} (\hat{a}_{i|I} - \check{a}_{B,i})]
\end{aligned}
\tag{12}
$$

where the shorthand notation $\hat{a}_{i|I}$ stands for the ith least-squares ambiguity obtained through a conditioning on the previous $I = \{1, \ldots, (i-1)\}$ sequentially rounded ambiguities.

Also the bootstrapped estimator is admissible. The first two conditions of the definition are satisfied, since - apart from ties in rounding - any 'float' solution gets mapped to a unique integer ambiguity vector. Also the integer remove-restore technique applies. To see this, let $\check{a}'_B$ be the bootstrapped estimator which corresponds with $\hat{a}' = \hat{a} - z$. It follows then from (12) that $\check{a}_B = \check{a}'_B + z$.

The real-valued sequential conditional least-squares solution can be obtained by means of the triangular decomposition of the ambiguity variance-covariance matrix. Let the LDU-decomposition of the variance-covariance matrix be given as $Q_{\hat{a}} = LDL^T$, with L a unit lower triangular matrix and D a diagonal matrix. Then $(\hat{a} - z) = L(\hat{a}^c - z)$, where $\hat{a}^c$ denotes the conditional least-squares solution obtained from a sequential conditioning on the entries of z. The variance-covariance matrix of $\hat{a}^c$ is given by the diagonal matrix D. This shows, when a componentwise rounding is applied to $\hat{a}^c$, that z is the integer solution of the bootstrapped method. Thus $\check{a}_B$ satisfies $[L^{-1}(\hat{a} - \check{a}_B)] = 0$. Hence, if c_i denotes the ith canonical unit vector having a 1 as its ith entry, the pull-in regions $S_{B,z}$ that belong to the bootstrapped estimator follow as

$$S_{B,z} = \cap_{i=1}^{n} \{x \in R^n \mid \mid c_i^T L^{-1}(x - z) \mid \leq \frac{1}{2}\}, \ \forall z \in Z^n \tag{13}$$

Note that these subsets reduce to the ones of (11) when L becomes diagonal. This is the case when the ambiguity variance-covariance matrix is diagonal. In that case the two integer estimators $\breve{a}_R$ and $\breve{a}_B$ are identical.

Integer least-squares

The integer least-squares estimator is defined as

$$\breve{a}_{ILS} = \arg \min_{z \in Z^n} \| \hat{a} - z \|^2_{Q_{\hat{a}}} \tag{14}$$

where $\| \cdot \|^2_{Q_{\hat{a}}} = (.)^T Q_{\hat{a}}^{-1}(.)$. This ambiguity estimator was introduced for the first time in (*Teunissen*, 1993). Also this estimator is admissible. Apart from boundary ties, it produces a unique integer vector for any 'float' solution $\hat{a} \in R^n$. And since $\breve{a}_{ILS} = \arg\min_{z \in Z^n} \| \hat{a} - u - z \|^2_{Q_{\hat{a}}} +u$ holds true for any integer u, also the integer remove-restore technique applies.

It follows from (14) that the 'float' solutions $\hat{a} \in R^n$ which are mapped to the same integer vector $\breve{a}_{ILS}$ are those that lie closer to this integer vector than to any other integer vector $z \in Z^n$. This shows that the least-squares pull-in regions $S_{ILS,z}$ consist of intersecting half-spaces, each one of which is bounded by the plane orthogonal to $(c - z), c \in Z^n$ and passing through the mid-point $\frac{1}{2}(z + c)$. Here, orthogonality is taken with respect to the metric as defined by the ambiguity variance-covariance matrix. Since $\hat{a}$ lies in one of these half-spaces when the length of the orthogonal projection of $(\hat{a} - z)$ onto $(c - z)$ is less than or equal to half the distance between c and z, it follows that

$$S_{ILS,z} = \cap_{c \in Z^n} \{ x \in R^n \mid \; | \, w_c(x) \, | \leq \frac{1}{2} \| c \|_{Q_{\hat{a}}} \} , \; \forall z \in Z^n \tag{15}$$

with

$$w_c(x) = \frac{c^T Q_{\hat{a}}^{-1}(x - z)}{\sqrt{c^T Q_{\hat{a}}^{-1} c}}$$

Note that $(c-z)$ has been replaced by c in (15). This is permitted since the intersection is taken with respect to all $c \in Z^n$.

In our comparison of $\breve{a}_R$ and $\breve{a}_B$, we noted that the two estimators became identical in case the unit triangular matrix L reduced to the identity matrix. The same holds true in case of $\breve{a}_{ILS}$. Hence, all three estimators become identical in case the ambiguity variance-covariance matrix is diagonal. This condition can be relaxed however, when comparing $\breve{a}_B$ with $\breve{a}_{ILS}$. These two estimators will already become identical when all matrix entries of L are integer.

4. THE CLASS OF INTEGER EQUIVARIANT ESTIMATORS

We will now introduce a new class of estimators which is larger than the previous defined class of integer estimators. And in order to be general enough, we consider estimating an arbitrary linear function of the two type of unknown parameters of the GNSS model (1),

$$\theta = l_a^T a + l_b^T b \; , \; l_a \in R^n \; , \; l_b \in R^p \tag{16}$$

Thus if $l_b = 0$, then linear functions of the ambiguities are estimated, while if $l_a = 0$, then linear functions of the baseline components are estimated. Linear functions of both the ambiguities and baseline components, e.g. the carrier phases, are estimated when $l_a \neq 0$ and $l_b \neq 0$.

It seems reasonable that the estimator should at least obey the *integer remove-restore* principle. When estimating ambiguities in case of GNSS for instance, one would like, when adding an arbitrary number of cycles to the carrier phase data, that the solution of the integer ambiguities gets shifted by the same integer amount. For the estimator of θ this would mean that adding Az to y, with arbitrary $z \in Z^n$, must result in a shift of $l_a^T z$. Likewise it seems reasonable to require of the estimator that adding $B\zeta$ to y, with arbitrary $\zeta \in R^p$, results in a shift $l_b^T \zeta$. Afterall we would not like the integer part of the estimator to become contaminated by such an addition to y. Estimators of θ that fulfil these two conditions will be called *integer equivariant* (IE). Hence, they are defined as follows.

Definition 2 (*integer equivariant estimators*)
The estimator $\hat{\theta}_{IE} = f_\theta(y)$, with $f_\theta : R^m \mapsto R$, is said to be an *integer equivariant* estimator of $\theta = l_a^T a + l_b^T b$ if

$$\begin{cases} f_\theta(y + Az) & = & f_\theta(y) + l_a^T z \ , \ \forall y \in R^m, z \in Z^n \\ f_\theta(y + B\zeta) & = & f_\theta(y) + l_b^T \zeta \ , \ \forall y \in R^m, \zeta \in R^p \end{cases} \tag{17}$$

It is not difficult to verify that the integer estimators of the previous section are integer equivariant. Simply check that the above two conditions are indeed fulfilled by the estimator $\breve{\theta} = l_a^T \breve{a} + l_b^T \breve{b}$. The converse, however, is not necessarily true. The class of IE-estimators is therefore indeed a larger class.

We will now show that the class of IE-estimators is also larger than the class of linear unbiased estimators. Let $f_\theta^T y$, for some $f_\theta \in R^m$, be the linear estimator of $\theta = l_a^T a + l_b^T b$. For it to be unbiased we need, using $E\{y\} = Aa + Bb$, that $f_\theta^T Aa + f_\theta^T Bb = l_a^T a + l_b^T b$, $\forall a \in R^n, b \in R^p$ holds true, or that both $l_a = A^T f_\theta$ and $l_b = B^T f_\theta$ hold true. But this equivalent to stating that

$$\begin{cases} f_\theta^T(y + Aa) & = & f_\theta^T y + l_a^T a \ , \ \forall y \in R^m, a \in R^n \\ f_\theta^T(y + Bb) & = & f_\theta^T y + l_b^T b \ , \ \forall y \in R^m, b \in R^p \end{cases} \tag{18}$$

Comparing this result with (17) shows that the condition of linear unbiasedness is more restrictive than the condition of integer equivariance. Hence, the class of linear unbiased estimators is a subset of the class of integer equivariant estimators. This result also automatically implies that IE-estimators exist which are unbiased. Thus, if we denote the class of IE-estimators as IE, the class of unbiased estimators as U, the class of unbiased IE-estimators as IEU, the class of unbiased integer estimators as IU, and the class of linear unbiased estimators as LU, we may summarize their relationships as: $IEU = IE \cap U \neq \emptyset$, $LU \subset IEU$ and $IU \subset IEU$.

5. REPRESENTATION OF IE-ESTIMATORS

In order to get a better understanding of how IE-estimators operate, it would be useful to have a representation that reveals their structure. One such representation is given in the following lemma.

Lemma (*IE-representation*)
Let $\hat{\theta}_{IE} = f_\theta(y)$ be the IE-estimator of $\theta = l_a^T a + l_b^T b$, let $y = A\alpha + B\beta + C\gamma$, with the $m \times (m - n - p)$ matrix C chosen such that (A, B, C) is invertible, and let $g_\theta(\alpha, \beta, \gamma) = f_\theta(A\alpha + B\beta + C\gamma)$. Then functions $h_\theta : R^n \times R^{m-n-p} \mapsto R$ exist such that

$$g_\theta(\alpha, \beta, \gamma) = l_a^T \alpha + l_b^T \beta + h_\theta(\alpha, \gamma) \tag{19}$$

with $h_\theta(\alpha + z, \gamma) = h_\theta(\alpha, \gamma)$ for all $z \in Z^n$.

Proof: We will start with the 'if part': If $g_\theta(\alpha, \beta, \gamma) = l_a^T \alpha + l_b^T \beta + h_\theta(\alpha, \gamma)$ with h_θ periodic in its first slot, then clearly $g_\theta(\alpha + z, \beta + \zeta, \gamma) = g_\theta(\alpha, \beta, \gamma) + l_a^T z + l_b^T \zeta$, $\forall z \in Z^n$, $\zeta \in R^p$. Therefore, since $f_\theta(A\alpha + B\beta + C\gamma) = g_\theta(\alpha, \beta, \gamma)$ and $y = A\alpha + B\beta + C\gamma$ with (A, B, C) invertible,

$$f_\theta(y + Az + B\zeta) = f_\theta(y) + l_a^T z + l_b^T \zeta \ , \ \forall y \in R^m , z \in Z^n , \zeta \in R^p \qquad (20)$$

which is equivalent to the properties of Definition 2.

For the 'only if part' we have: If (20) holds true, the h_θ function defined as $h_\theta(\alpha, \gamma) = f_\theta(A\alpha + B\beta + C\gamma) - l_a^T \alpha - l_b^T \beta$ with (A, B, C) invertible, will be periodic in its first slot and hence $g_\theta(\alpha, \beta, \gamma) = f_\theta(A\alpha + B\beta + C\gamma)$ can be written as $g_\theta(\alpha, \beta, \gamma) = l_a^T \alpha + l_b^T \beta + h_\theta(\alpha, \gamma)$. *End of proof*

Note that the lemma now easily allows one to design one's own IE-estimator. Also note that when devising one's own IE-estimator, there are essentially two types of degrees of freedom involved. The choice of the matrix C and the choice of the function h_θ.

Here are some examples of IE-estimators obtained for specific choices of C and h_θ:

Example 1: For arbitrary C and $h_\theta = 0$ we get

$$\hat{\theta}_{IE} = l_a^T \alpha + l_b^T \beta$$

Note that this is a linear unbiased estimator of θ for any choice of C. Hence, matrix C governs the choice of these linear unbiased estimators.

Example 2: For $h_\theta = 0$ and C chosen such that $C^T Q_y^{-1}(A, B) = 0$ we get the least-squares estimator

$$\hat{\theta}_{IE} = l_a^T \hat{a} + l_b^T \hat{b}$$

in which we recognize the 'float' solution of the ambiguities and the baseline.

Example 3: For $h_\theta(\alpha, \gamma) = -(l_a^T + l_b^T Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1})(\alpha - S(\alpha))$ and C chosen such that $C^T Q_y^{-1}(A, B) = 0$ we get the estimator

$$\hat{\theta}_{IE} = l_a^T \breve{a} + l_b^T \breve{b}$$

in which we recognize the integer solution of the ambiguities and the corresponding 'fixed' solution for the baseline.

Example 4: For $h_\theta(\alpha, \gamma) = -(l_a^T + l_b^T Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1})(\alpha - S(\alpha))$, $S(\alpha) = \arg\min_{z \in Z^n}(\alpha - z)^T Q_{\hat{a}}^{-1}(\alpha - z)$ and C chosen such that $C^T Q_y^{-1}(A, B) = 0$ we get the integer least-squares estimator

$$\hat{\theta}_{IE} = l_a^T \breve{a}_{ILS} + l_b^T \breve{b}_{ILS}$$

Example 5: For $h_\theta(\alpha, \gamma) = -(l_a^T + l_b^T Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1})(\alpha - \sum_{z \in Z^n} z w(\alpha - z))$, with $\sum_{z \in Z^n} w(\alpha - z) = 1, \forall \alpha \in R^n$, and C chosen such that $C^T Q_y^{-1}(A, B) = 0$ we get the estimator

$$\hat{\theta}_{IE} = l_a^T \hat{a}_{IE} + l_b^T \hat{b}_{IE}$$

with

$$\begin{cases} \hat{a}_{IE} &= \sum_{z \in Z^n} z w(\hat{a} - z) \\ \hat{b}_{IE} &= \hat{b} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1}(\hat{a} - \hat{a}_{IE}) \end{cases}$$

The above are just a few examples of IE-estimators. Note that the ambiguity estimator of the last example resembles the integer ambiguity estimator (6). The important difference between these two estimators is however the range of values taken by the weights. In case of the integer estimator (6) the weights $s_z(\hat{a})$ are binary, whereas in case of $\hat{a}_{IE}$ the weights $w(\hat{a} - z)$ may vary between zero and one.

Since the class of IE-estimators includes all integer estimators as well as all linear unbiased estimators, estimators which are optimal in the IE-class will automatically outperform their integer counterparts as well as their linear unbiased counterparts. The search for such an optimal IE-estimator will therefore be taken up in a future study.

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