ARTIFICIAL SATELLITES, Vol. 37, No. 4 – 2002

A GAUSS-MARKOV-LIKE THEOREM FOR INTEGER GNSS AMBIGUITIES

P.J.G. Teunissen
Department of Mathematical Geodesy and Positioning
Delft University of Technology
Thijsseweg 11
2629 JA Delft, The Netherlands
Fax: ++ 31 15 278 3711

ABSTRACT

Carrier phase integer ambiguity resolution is the key to high precision Global Navigation Satellite System (GNSS) positioning and navigation. It applies to a great variety of current and future models of GPS, modernized GPS and Galileo. In Teunissen (1999a, b) we introduced the class of admissible integer estimators and showed that the integer least-squares estimator is the optimal estimator within this class. In Teunissen (2002) we introduced an alternative class of ambiguity estimators. This class of integer equivariant (IE) estimators still obeys the integer remove-restore principle. In the present contribution we will determine the 'best' estimator within the IE-class. The minimum mean squared error is taken as the criterion for 'best'. As our main result we have a Gauss-Markov-like theorem which introduces a new minimum variance unbiased ambiguity estimator which is always superior to the well-known best linear unbiased ambiguity estimator (BLU) of the Gauss-Markov theorem.

Keywords: GNSS ambiguity resolution, best integer equivariant estimation, minimum variance unbiased estimation

1. INTRODUCTION

Global Navigation Satellite System (GNSS) ambiguity resolution is the process of resolving the unknown cycle ambiguities of double difference (DD) carrier phase data. Its practical importance becomes clear when one realizes the great variety of current and future GNSS models to which it applies. These models may differ greatly in complexity and diversity. They range from single-baseline models used for kinematic positioning to multi-baseline models used as a tool for studying geodynamic phenomena. The models may or may not have the relative receiver-satellite geometry included. They may also be discriminated as to whether the slave receiver(s) are stationary or in motion.. When in motion, one solves for one or more trajectories, since with the receiver-satellite geometry included, one will have new coordinate unknowns for each epoch. One may also discriminate between the models as to whether or not the differential atmospheric delays (ionosphere and troposhere) are included as unknowns. In the case of sufficiently short baselines they are usually excluded. Apart from the current Global Positioning System (GPS) models, carrier phase ambiguity resolution also applies to the future modernized GPS and the future European Galileo GNSS. An overview of GNSS models, together with their applications in surveying, navigation, geodesy and geophysics, can be found in textbooks such as [*Hofmann-Wellenhof et al.*, 2001], [*Leick*, 1995], [*Misra and Enge*, 2001], [*Parkinson and Spilker*, 1996], [*Strang and Borre*, 1997] and [*Teunissen and Kleusberg*, 1998].

In Teunissen (1999a, b) we introduced the class of admissible integer estimators and showed that the integer least-squares estimator is the optimal estimator within this class. In Teunissen (2002) we introduced an alternative class of ambiguity estimators. This class of integer equivariant (IE) estimators still obeys the integer remove-restore principle. In the present contribution we will determine the 'best' estimator within the IE-class. This new ambiguity estimator will be referred to as the best integer equivariant (BIE) estimator. The minimum mean squared error is taken as the criterion for 'best'.

It will be shown that the class of linear unbiased estimators, LU, is a subset of the class of IE-estimators, $LU \subset IE$. This automatically implies that, in the MSE-sense, the BIE-estimator always outperforms its BLU-counterpart. In addition it can be shown that the BIE-estimator is unbiased. Hence, similar to the well-known Gauss-Markov theorem, which states that the minimum variance unbiased estimator within the class of linear estimators is given by the least-squares estimator BLU, we obtain a Gauss-Markov-like theorem stating that the minimum variance unbiased estimator within the IE-class is given by the least mean-squared error estimator BIE. Both theorems hold true for any pdf the data might have. For the BLU-solution one needs to know the vc-matrix of the data up to a proportionality constant, whereas for the BIE-solution one needs to know the pdf up to a proportionality factor.

2. OPTIMAL INTEGER ESTIMATION

Let $\hat{a} \in R^n$ denote the 'float' solution of the GPS carrier phase ambiguities, with $E\{\hat{a}\} = a \in Z^n$ and $E\{.\}$ the mathematical expectation operator. The aim of GPS carrier phase ambiguity resolution is now to find ways of incorporating the integerness of the ambiguities so as to obtain a solution $\check{a}$ which outperforms $\hat{a}$. Since the ambiguities are known to be integer, it is reasonable to require $\check{a}$ to be integer as well. There are however many ways of computing an integer ambiguity vector $\check{a}$ from its real-valued counterpart $\hat{a}$. To each such method belongs a mapping $S : R^n \mapsto Z^n$ from the n-dimensional space of real numbers to the n-dimensional space of integers. Due to the discrete nature of Z^n, the map S will not be one-to-one, but instead a many-to-one map. This implies that different real-valued ambiguity vectors will be mapped to the same integer vector. One can therefore assign a subset $S_z \subset R^n$ to each integer vector $z \in Z^n$:

$$S_z = \{x \in R^n \mid z = S(x)\}, \quad z \in Z^n \tag{1}$$

The subset S_z contains all real-valued ambiguity vectors that will be mapped by S to the same integer vector $z \in Z^n$. This subset is referred to as the *pull-in region* of z. It is the region in which all ambiguity 'float' solutions are pulled to the same integer vector z.

Since the pull-in regions define the integer estimator completely, one can define classes of integer estimators by imposing various conditions on the pull-in regions. One such class is referred to as the class of admissible integer estimators.

Definition 1 (*admissible integer estimators*)
The integer estimator $\check{a} = S(\hat{a})$ is said to be ***admissible*** if its pull-in regions satisfy

$$\begin{array}{ll} (i) & \bigcup_{z \in Z^n} S_z = R^n \\ (ii) & Int(S_{z_1}) \bigcap Int(S_{z_2}) = \emptyset, \;\; \forall z_1, z_2 \in Z^n, z_1 \neq z_2 \\ (iii) & S_z = z + S_0, \;\; \forall z \in Z^n \end{array}$$

This class was introduced by Teunissen (1999a). Using the pull-in regions, one can give an explicit expression for the corresponding integer ambiguity estimator. It reads

$$\check{a} = \sum_{z \in Z^n} z s_z(\hat{a}) \tag{2}$$

with the indicator function $s_z(\hat{a}) = 1$ if $\hat{a} \in S_z$ and $s_z(\hat{a}) = 0$ otherwise. Note that the $s_z(\hat{a})$ can be interpreted as weights, since $\sum_{z \in Z^n} s_z(\hat{a}) = 1$. The integer estimator $\check{a}$ is therefore equal to a weighted sum of integer vectors with binary weights.

With the division of R^n into mutually exclusive pull-in regions, we are now in the position to consider the distribution of $\check{a}$. This distribution is of the *discrete* type and it will be denoted as $P(\check{a} = z)$. It is a probability mass function, having zero masses at nongrid points and nonzero masses at some or all grid points. If we denote the *continuous* probability density function of $\hat{a}$ as $p_{\hat{a}}(x \mid a)$, the distribution of $\check{a}$ follows as

$$P(\check{a} = z) = \int_{S_z} p_{\hat{a}}(x \mid a) dx \;, \; z \in Z^n \tag{3}$$

This distribution is of course dependent on the pull-in regions S_z and thus on the chosen integer estimator. Since various integer estimators exist which are admissible, some may be better than others. Having the problem of GNSS ambiguity resolution in mind, one is particularly interested in the estimator which maximizes the probability of correct integer estimation. This probability equals $P(\check{a} = a)$, but it will differ for different ambiguity estimators. The answer to the question which estimator maximizes the probability of correct integer estimation is given by the following theorem, due to Teunissen (1999b):

Theorem 1 (*optimal integer estimation*)
Let

$$\check{a}_{ML} = \arg \max_{a \in Z^n} p_{\hat{a}}(\hat{a} \mid a) \tag{4}$$

be admissible. Then

$$P(\check{a}_{ML} = a) \geq P(\check{a} = a) \tag{5}$$

for any admissible estimator $\check{a}$.

This result holds for an arbitrary pdf of the 'float' ambiguities $\hat{a}$. In most GNSS applications however, one assumes the data to be normally distributed. The estimator $\hat{a}$ will then be normally distributed too, with mean $a \in Z^n$ and vc-matrix $Q_{\hat{a}}$. Its pdf reads then

$$p_{\hat{a}}(x \mid a) = \frac{1}{\sqrt{\det(Q_{\hat{a}})}(2\pi)^{\frac{1}{2}n}} \exp\{-\frac{1}{2} \parallel x - a \parallel^2_{Q_{\hat{a}}}\} \tag{6}$$

with the squared weighted norm $\| \cdot \|^2_{Q_{\hat{a}}} = (.)^T Q_{\hat{a}}^{-1}(.)$. With this pdf, the optima[l] estimator becomes identical to the *integer least squares* (ILS) estimator

$$\check{a}_{ILS} = \arg\min_{a \in Z^n} \| x - a \|^2_{Q_{\hat{a}}} \tag{7}$$

The above theorem therefore gives a probabilistic justification for using the ILS estimator when the pdf is Gaussian. For GNSS ambiguity resolution it shows, that one is better off using the ILS estimator than any other admissible integer estimator.

3. BEST INTEGER EQUIVARIANT ESTIMATION

The result of the above theorem holds true for the defined class of integer estimators. One may now wonder what happens if the conditions of Definition 1 are relaxed. Would it then still be possible to find an ambiguity estimator which in some sense outperforms the 'float' solution? In order to answer this question we first start by defining a class of estimators which is larger than the class of integer estimators. It seems reasonable that the estimator should at least obey the *integer remove-restore* principle. Estimators that fulfil this condition will be called *integer equivariant* (IE), see [*Teunissen*, 2002].

Definition 2 (*integer equivariant estimators*)
The estimator $\hat{a}_{IE} = f(\hat{a})$, with $f : R^n \mapsto R^n$, is said to be *integer equivariant* if

$$f(x + z) = f(x) + z \ , \ \forall x \in R^n, z \in Z^n \tag{8}$$

It will be clear that admissibe integer estimators are also IE-estimators, but that the converse is not necessarily true.

We will now look for an IE-estimator which is 'best' in a certain sense. We will denote our *best integer equivariant* (BIE) estimator as $\hat{a}_{BIE}$ and use the mean squared error (MSE) as our criterion of 'best'. The best integer equivariant estimator will therefore be defined as

$$\hat{a}_{BIE} = \arg\min_{f \in IE} E\{\| f(\hat{a}) - a \|^2\} \tag{9}$$

in which 'IE' stands for the class of IE-estimators, $\| \cdot \|^2 = (.)^T M(.)$ and matrix M is positive semi-definite ($M \geq 0$). Note that the minimization is taken over all integer equivariant functions.

The reason for choosing the MSE-criterion is twofold. First, it is a well-known probabilistic criterion for measuring the closeness of an estimator to its target value in our case $a \in Z^n$. Second, the MSE-criterion is also often used as measure for the quality of the 'float' solution itself. The following theorem gives the solution to the above minimization problem (9).

Theorem 2 (*best integer equivariant estimation*)
Let $\hat{a}$ be the 'float' solution having $p_{\hat{a}}(x \mid a)$ as its pdf. The best integer equivariant estimator is then given as

$$\hat{a}_{BIE} = \sum_{z \in Z^n} z w_z(\hat{a}) \tag{10}$$

with the weights

$$w_z(x) = \frac{p_{\hat{a}}(x + a - z \mid a)}{\sum_{u \in Z^n} p_{\hat{a}}(x + a - u \mid a)}$$

Based on this result we are now in a position to make a number of observations. First note that the least mean squared error property of the BIE-estimator holds true for any pdf the 'float' solution might have. Secondly note that the BIE-estimator is not dependent on the choice made for matrix M. Also observe that the structure of the BIE-estimator resembles that of an admissible integer estimator, see (2). The BIE-estimator is also a weighted sum of all integer vectors in Z^n. In the present case, however, the weights are not binary. They vary between zero and one, and their values are determined by the 'float' solution and its pdf. As a consequence the BIE-estimator will be real-valued in general, instead of integer-valued. But one can also show that for the Gaussian case the BIE-solution converges to the ILS- solution when the precision of the 'float' solution improves. The two solutions will therefore not differ by much in case the probability of correct integer estimation - the ambiguity success rate - is sufficiently large.

As a final observation regarding the above theorem we claim that the following inequality must hold true:

$$E\{\| \hat{a}_{BIE} - a \|^2\} \leq E\{\| \hat{a} - a \|^2\} \tag{11}$$

The MSE of the BIE-estimator is thus always smaller than or at the most equal to the MSE of the 'float' solution. This result may come as a surprise. Afterall the 'float' solution is usually derived as the best linear unbiased (BLU) estimator, which therefore also has its MSE minimized. This apparant paradox will be explained in the next section.

4. THE BIE- AND BLU-ESTIMATORS COMPARED

4.1 Least mean squares

The 'float' solution $\hat{a}$ is usually derived from the linear model of observation equations by means of the least-squares principle with the weight matrix chosen equal to the inverse of the observational vc-matrix. The 'float' solution will then be identical to the BLU-estimator of a. In order to make this clear in our notation, we will denote the 'float' solution from now on as $\hat{a}_{BLU}$.

In order to understand why the MSE of $\hat{a}_{BIE}$ is smaller than or at the most equal to the MSE of $\hat{a}_{BLU}$, we have to consider the sets over which the minimization of the MSE takes place. In case of $\hat{a}_{BIE}$ it is the set of all integer equivariant functions, whereas in case of $\hat{a}_{BLU}$ it is the set of all linear unbiased functions. The first set will be denoted as IE and the second set as LU.

We will now show that the set of linear unbiased functions is a subset of the set of integer equivariant functions: $LU \subset IE$. Let the system of observation equations be given as $E\{y\} = Aa$, with matrix A of dimension $m \times n$, and let $F(y)$ be an IE-estimator of $a \in Z^n$. In this case integer equivariance implies

$$F(y + Aa) = F(y) + a \ , \ \forall a \in Z^n, y \in R^m \tag{12}$$

An estimator has the LU-property if it is linear and unbiased. Let Ly, with matrix L of dimension $n \times m$, be a linear estimator of a. Unbiasedness implies that $E\{Ly\} = a$ should hold for all $a \in R^n$. Hence, $LAa = a$ should hold for all $a \in R^n$. We therefore have the following characterization of an LU-estimator: $LA = I_n$. But this is equivalent to stating that

$$L(y + Aa) = Ly + a \ , \ \forall a \in R^n, y \in R^m \tag{13}$$

Upon comparing (12) with (13) it immediately follows that indeed $LU \subset IE$. Requiring integer equivariance of an estimator is therefore less restrictive than requiring the estimator to be linearly unbiased. This explains why

$$MSE(\hat{a}_{BIE}) \leq MSE(\hat{a}_{BLU}) \tag{14}$$

4.2 Minimum variance

So far we did not consider the property of unbiasedness for an IE-estimator. Let us denote the set of unbiased IE-estimators as IEU and the set of unbiased estimators as U. We already determined that $LU \subset IE$. We also have $LU \subset U$. Hence LU is a subset of the intersection of IE and U, and thus a subset of IEU: $LU \subset IEU$. From this result, combined with the fact that $IEU \subset IE$, follows that imposing the conditions of integer equivariance and unbiasedness is less restrictive than imposing the LU-conditions, but more restrictive than the IE-condition. Hence, for the MSE of a best integer equivariant unbiased estimator, denoted as $\hat{a}_{BIEU}$, we immediately have

$$MSE(\hat{a}_{BIE}) \leq MSE(\hat{a}_{BIEU}) \leq MSE(\hat{a}_{BLU}) \tag{15}$$

Since IEU-estimators and LU-estimators are unbiased by definition, minimizing their MSE is equivalent to minimizing their variance. Hence, the two estimators $\hat{a}_{BLU}$ and $\hat{a}_{BIEU}$, besides being unbiased, are also both of minimum variance within their class. It would seem, if one cherishes the properties of unbiasedness and minimum variance that then $\hat{a}_{BIEU}$ is the proper contender of $\hat{a}_{BLU}$, whereas if one goes for the least mean squared error property, the proper contender of $\hat{a}_{BLU}$ would be $\hat{a}_{BIE}$. For IE-estimation, however, this difference turns out be absent, since the two estimators $\hat{a}_{BIE}$ and $\hat{a}_{BIEU}$ can shown to be identical. This is a direct consequence of the fact that $\hat{a}_{BIE}$ is already unbiased. This can be shown by taking the expectation of (10). We therefore also have the following theorem.

Theorem 3 (*minimum variance unbiased estimation*)
The BIE-estimator is unbiased and has a better precision than the BLU-estimator:

$$\begin{aligned} (i) \quad & E\{\hat{a}_{BIE}\} = E\{\hat{a}_{BLU}\} \\ (ii) \quad & D\{\hat{a}_{BIE}\} \leq D\{\hat{a}_{BLU}\} \end{aligned} \tag{16}$$

where $D\{.\}$ denotes the dispersion operator.

As the theorem shows the BIE-estimator is already unbiased by itself. Imposing the condition of unbiasedness onto the minimization problem (9) would therefore not alter the solution. Hence $\hat{a}_{BIE} = \hat{a}_{BIEU}$.

5. SUMMARY

In this contribution we worked with a new class of estimators for the GNSS carrier phase ambiguities. This class of integer equivariant (IE) estimators is larger than the class of admissible integer estimators, but their members still obey the integer remove-restore principle. Using the mean squares error as our criterion for estimator performance, we obtained an explicit expression for the best IE-estimator by minimizing the mean squared error over the set of all integer equivariant functions.

The BIE-estimator was shown to have a similar structure as the admissible integer estimators,

$$\hat{a}_{BIE} = \sum_{z \in Z^n} z w_z(\hat{a})$$

the important difference being that the weights, as determined by $\hat{a}$ and its pdf, now vary between zero and one, instead of being zero or one. The expression given for the BIE-estimator holds true for any pdf the 'float' solution might have. In case the pdf is Gaussian, the BIE-estimator converges to the integer least-squares solution when the precision of the 'float' ambiguities is continuously improved.

Since the BIE-estimator has the smallest possible mean squared error within the class of IE-estimators and since the 'float' solution is an IE-estimator as well, the BIE-estimator outperforms the 'float' solution in terms of its mean squared error. At first instance, this result seems to be in contradiction with the fact that the 'float' solution, if derived as the best linear unbiased estimator, also has its mean squared error minimized. That this paradox is apparant but not real is due to the fact that the class of linear unbiased estimators is a subset of the class of integer equivariant estimators.

Finally it was shown that the BIE-estimator is identical to the minimum variance unbiased estimator of the IE-class. The BIE-estimator is therefore, just like the BLU-estimator, unbiased, but it outperforms the BLU-estimator in terms of precision.

6. REFERENCES

Hofmann-Wellenhof, B., H. Lichtenegger, J. Collins (2001): *Global Positioning System: Theory and Practice.* 5th edition. Springer Verlag.

Leick, A. (1995): *GPS Satellite Surveying.* 2nd edition, John Wiley, New York.

Misra, P., P. Enge (2001): *Global Positioning System: Signals, Measurements, and Performance.* Ganga-Jamuna Press, 2001.

Parkinson, B., J.J. Spilker (eds) (1996): *GPS: Theory and Applications*, Vols 1 and 2, AIAA, Washington DC.

Strang, G., K. Borre (1997): *Linear Algebra, Geodesy, and GPS*, Wellesley-Cambridge Press.

Teunissen, P.J.G., A. Kleusberg (eds) (1998): *GPS for Geodesy*, 2nd enlarged edition, Springer Verlag.

Teunissen, P.J.G. (1999a): The probability distribution of the GPS baseline for a class of integer ambiguity estimators. *Journal of Geodesy*, 73: 275-284.

Teunissen, P.J.G. (1999b): An optimality property of the integer least- squares estimator. *Journal of Geodesy*, 73: 587-593.

Teunissen, P.J.G. (2002): A new class of GNSS ambiguity estimators. *Artificial Satellites*, this issue, pages 111-119

Received: November 27, 2002,
Reviewed: December 8, 2002, by W. Pachelski,
Accepted: January 8, 2003.