STATISTICAL GNSS CARRIER PHASE AMBIGUITY RESOLUTION: A REVIEW

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Biography
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Abstract
Global Navigation Satellite System carrier phase ambiguity resolution is the key to high precision positioning and navigation. In this contribution a brief review is given of the probabilistic theory of integer carrier phase ambiguity estimation. Various ambiguity estimators are discussed. Among them are the estimators of integer rounding, integer bootstrapping, integer least-squares and the Bayesian solution. We also discuss the various relationships that exist between these estimators.

1. INTRODUCTION

Global Navigation Satellite System (GNSS) ambiguity resolution is the process of resolving the unknown cycle ambiguities of double difference (DD) carrier phase data as integers. The sole purpose of ambiguity resolution is to use the integer ambiguity constraints as a means of improving significantly on the precision of the remaining model parameters, such as baseline coordinates and/or atmospheric (troposphere, ionosphere) delays.

Ambiguity resolution applies to a great variety of current and future GNSS models. These models may differ greatly in complexity and diversity. They range from single-baseline models used for kinematic positioning to multi-baseline models used as a tool for studying geodynamic phenomena. The models may or may not have the relative receiver-satellite geometry included. They may also be discriminated as to whether the slave receiver(s) are stationary or in motion. When in motion, one solves for one or more trajectories, since with the receiver-satellite geometry included, one will have new coordinate unknowns for each epoch. One may also discriminate between the models as to whether or not the differential atmospheric delays (ionosphere and troposphere) are included as unknowns. In the case of sufficiently short baselines they are usually excluded.

Apart from the current Global Positioning System (GPS) models, carrier phase ambiguity resolution also applies to the future modernized GPS and the future European Galileo GNSS. An overview of GNSS models, together with their applications in surveying, navigation, geodesy and geophysics, can be found in textbooks such as [*Hofmann-Wellenhof et al.*, 1997], [*Leick*, 1995], [*Parkinson and Spilker*, 1996], [*Strang and Borre*, 1997] and [*Teunissen and Kleusberg*, 1998].

In this contribution we review the probabilistic theory for integer carrier phase ambiguity estimation. It is the key to high precision GNSS positioning and navigation. This contribution is organized as follows. In section 2 we introduce a general class of integer ambiguity estimators, determine their probability mass functions and show how their variability affect the uncertainty in the computed GNSS baselines. This theory is worked out in sections 3 and 4 for two of the most important integer ambiguity estimators. In section 3 we discuss the properties of integer bootstrapping and in section 4 those of integer least-squares. In the final section, section 5, we discuss the Bayesian solution to carrier phase ambiguity resolution. Although the Bayesian approach has not yet find a wide-spread use in any of the GNSS applications, the basic concepts involved are of interest in their own right. Where possible, the various ambiguity estimation principles are compared.

2. INTEGER AMBIGUITY RESOLUTION

2.1. The GNSS model

As our point of departure we will take the following system of linear(ized) observation equations

$$y = Aa + Bb + e \tag{1}$$

where y is the given GNSS data vector of order m, a and b are the unknown parameter vectors respectively of order n and p, and where e is the noise vector. In principle all the GNSS models can be cast in this frame of observation equations. The data vector y will usually consist of the 'observed minus computed' single- or dual- frequency double-difference (DD) phase and/or pseudorange (code) observations accumulated over all observation epochs. The entries of vector a are then the DD carrier phase ambiguities, expressed in units of cycles rather than range. They are known to be *integers*, $a \in Z^n$. The entries of the vector b will consist of the remaining unknown parameters, such as for instance baseline components (coordinates) and possibly atmospheric delay parameters (troposphere, ionosphere). They are known to be real- valued, $b \in R^p$.

The procedure which is usually followed for solving the GNSS model (1), can be divided into three steps. In the *first* step one simply disregards the integer constraints $a \in Z^n$ on the ambiguities and performs a standard least-squares adjustment. As a result one obtains the (real-valued) estimates of a and b, together with their variance-covariance (vc-) matrix

$$\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix}, \begin{bmatrix} Q_{\hat{a}} & Q_{\hat{a}\hat{b}} \\ Q_{\hat{b}\hat{a}} & Q_{\hat{b}} \end{bmatrix} \tag{2}$$

This solution is referred to as the 'float' solution. In the *second* step the 'float' ambiguity estimate $\hat{a}$ is used to compute the corresponding integer ambiguity estimate $\check{a}$. This implies that a mapping $S : R^n \mapsto Z^n$, from the n-dimensional space of reals to the n-dimensional space of integers, is introduced such that

$$\check{a} = S(\hat{a}) \tag{3}$$

Once the integer ambiguities are computed, they are used in the *third* step to finally correct the 'float' estimate of b. As a result one obtains the 'fixed' solution

$$\check{b} = \hat{b} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} (\hat{a} - \check{a}) \tag{4}$$

In the present review we will primarily focus our attention on the probabilistic properties of (3) and (4).

2.2. Admissible integer estimation

There are many ways of computing an integer ambiguity vector $\check{a}$ from its real-valued counterpart $\hat{a}$. To each such method belongs a mapping $S : R^n \mapsto Z^n$ from the n-dimensional space of real numbers to the n-dimensional space of integers. Due to the discrete nature of Z^n, the map S will not be one-to-one, but instead a many-to-one map. This implies that different real-valued ambiguity vectors will be mapped to the same integer vector. One can therefore assign a subset $S_z \subset R^n$ to each integer vector $z \in Z^n$:

$$S_z = \{x \in R^n \mid z = S(x)\}, \quad z \in Z^n \tag{5}$$

The subset S_z contains all real-valued ambiguity vectors that will be mapped by S to the same integer vector $z \in Z^n$. This subset is referred to as the *pull-in region* of z [*Jonkman*, 1998]. It is the region in which all ambiguity 'float' solutions are pulled to the same 'fixed' ambiguity vector z. Using the pull-in regions, one can give an explicit expression for the corresponding integer ambiguity estimator. It reads

$$\check{a} = \sum_{z \in Z^n} z s_z(\hat{a}) \tag{6}$$

with the indicator function

$$s_z(\hat{a}) = \begin{cases} 1 & \text{if } \hat{a} \in S_z \\ 0 & \text{otherwise} \end{cases}$$

Since the pull-in regions define the integer estimator completely, one can define classes of integer estimators by imposing various conditions on the pull-in regions. One such class is referred to as the class of admissible integer estimators. These integer estimators are defined as follows.

Definition 1
The integer estimator $\check{a} = \sum_{z \in Z^n} z s_z(\hat{a})$ is said to be *admissible* if

$$\begin{array}{ll} (i) & \bigcup_{z \in Z^n} S_z = R^n \\ (ii) & Int(S_{z_1}) \bigcap Int(S_{z_2}) = \emptyset, \;\; \forall z_1 \neq z_2 \in Z^n \\ (iii) & S_z = z + S_0, \;\; \forall z \in Z^n \end{array}$$

This definition is motivated as follows. Each one of the above three conditions describe a property of which it seems reasonable that it is possessed by an arbitrary integer ambiguity estimator. The first condition states that the pull-in regions should not leave any gaps and the second that they should not overlap. The absence of gaps is needed in order to be able to map any 'float' solution $\hat{a} \in R^n$ to Z^n, while the absence of overlaps is needed to guarantee that the 'float' solution is mapped to just one integer vector. Note that we allow the pull-in regions to have common boundaries. This is permitted if we assume to have zero probability that $\hat{a}$ lies on one of the boundaries. This will be the case when the probability density function (pdf) of $\hat{a}$ is continuous.

The third and last condition follows from the requirement that $S(x+z) = S(x) + z, \forall x \in R^n, z \in Z^n$. Also this condition is a reasonable one to ask for. It states that when the 'float' solution is perturbed by $z \in Z^n$, the corresponding integer solution is perturbed by the same amount. This property allows one to apply the *integer remove-restore* technique: $S(\hat{a} - z) + z = S(\hat{a})$. It therefore allows one to work with the fractional parts of the entries of $\hat{a}$, instead of with its complete entries.

With the division of R^n into mutually exclusive pull-in regions, we are now in the position to consider the distribution of $\check{a}$. This distribution is of the *discrete* type and it will be denoted as $P(\check{a} = z)$. It is a probability mass function, having zero masses at nongrid points and nonzero masses at some or all grid points. If we denote the *continuous* probability density function of $\hat{a}$ as $p_{\hat{a}}(x)$, the distribution of $\check{a}$ follows as

$$P(\check{a} = z) = \int_{S_z} p_{\hat{a}}(x) dx \, , \, z \in Z^n \tag{7}$$

This expression holds for any distribution the 'float' ambiguities $\hat{a}$ might have. In most GNSS applications however, one assumes the vector of observables y to be normally distributed. The estimator $\hat{a}$ is therefore normally distributed too, with mean $a \in Z^n$ and vc-matrix $Q_{\hat{a}}$. Its probability density function reads

$$p_{\hat{a}}(x) = \frac{1}{\sqrt{\det(Q_{\hat{a}})}(2\pi)^{\frac{1}{2}n}} \exp\{-\frac{1}{2} \parallel x - a \parallel^2_{Q_{\hat{a}}}\} \tag{8}$$

with the squared weighted norm $\parallel . \parallel^2_{Q_{\hat{a}}} = (.)^T Q_{\hat{a}}^{-1} (.)$. Note that $P(\check{a} = a)$ equals the probability of *correct* integer ambiguity estimation. It describes the expected success rate of GNSS ambiguity resolution.

2.3. The baseline solution

We are now in the position to determine the pdf of the 'fixed' baseline estimator (4). In order to determine this pdf, one needs to propagate the uncertainty of the 'float' solution, $\hat{a}$ and $\hat{b}$, as well as the uncertainty of the integer solution $\check{a}$ through (4). Should one neglect the random character of the integer solution and therefore consider the ambiguity vector $\check{a}$ as deterministic and equal to, say, z, then the pdf of $\check{b}$ would equal the conditional baseline distribution

$$p_{\hat{b}|\hat{a}}(x \mid z) = \frac{\exp\{-\frac{1}{2} \parallel x - b(z) \parallel^2_{Q_{\hat{b}|\hat{a}}}\}}{\sqrt{\det Q_{\hat{b}|\hat{a}}}(2\pi)^{\frac{1}{2}p}} \tag{9}$$

with conditional mean $b(z) = b - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} (a - z)$, conditional variance matrix $Q_{\hat{b}|\hat{a}} = Q_{\hat{b}} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} Q_{\hat{a}\hat{b}}$ and $\parallel . \parallel^2_{Q_{\hat{b}|\hat{a}}} = (.)^T Q_{\hat{b}|\hat{a}}^{-1} (.)$. However, since $\check{a}$ is random and not deterministic, the conditional baseline distribution will give a too optimistic description of the quality of the 'fixed' baseline. To get a correct description of the 'fixed' baseline's pdf, the integer ambiguity's pmf needs to be considered. As the following theorem shows this results in a baseline distribution, which generally will be multi-modal.

Theorem 1 (*Pdf of the 'fixed' baseline*)
Let the 'float' solution, $\hat{a}$ and $\hat{b}$, be normally distributed with mean $a \in Z^n$ and mean $b \in R^p$, and vc-matrix (2), let $\check{a}$ be an admissible integer estimator and let the 'fixed' baseline $\check{b}$ be given as in (4). The pdf of $\check{b}$ reads then

$$p_{\check{b}}(x) = \sum_{z \in Z^n} p_{\hat{b}|\hat{a}}(x \mid z) P(\check{a} = z) \tag{10}$$

Note that, although the model (1) is linear and the observables normally distributed, the distribution of the 'fixed' baseline is not normal, but multi-modal. As the theorem shows, the 'fixed' baseline distribution equals an infinite sum of weighted conditional baseline distributions. These conditional baseline distributions $p_{\hat{b}|\hat{a}}(x \mid z)$ are shifted versions of one another. The size and direction of the shift is governed by $Q_{\hat{b}\hat{a}}Q_{\hat{a}}^{-1}z$, $z \in Z^n$. Each of the conditional baseline distributions in the infinite sum is downweighted. These weights are given by the probability masses of the distribution of the integer bootstrapped ambiguity estimator $\check{a}$. This shows that the dependence of the 'fixed' baseline distribution on the choice of integer estimator is only felt through the weights $P(\check{a} = z)$.

2.4. On the quality of the 'fixed' baseline

In order to describe the quality of the 'fixed' baseline, one would like to know how close one can expect the baseline estimate $\check{b}$ to be to the unknown, but true baseline value b. As a measure of confidence, we take

$$P(\check{b} \in R) = \int_R p_{\check{b}}(x)dx \text{ with } R \subset R^p \tag{11}$$

But in order to evaluate this integral, we first need to make a choice about the shape and location of the subset R. Since it is common practice in GNSS positioning to use the vc-matrix of the conditional baseline estimator as a measure of precision for the 'fixed' baseline, the vc-matrix $Q_{\hat{b}|\hat{a}}$ will be used to define the shape of the confidence region. For its location, we choose the confidence region to be centered at b. After all, we would like to know by how much the baseline estimate $\check{b}$ can be expected to differ from the true, but unknown baseline value b. That is, one would like (11) to be a measure of the baseline's probability of concentration about b.

With these choices on shape and location, the region R takes the form

$$R = \{x \in R^p \mid (x-b)^T Q_{\hat{b}|\hat{a}}^{-1}(x-b) \leq \beta^2\} \tag{12}$$

The size of the region can be varied by varying β. The following theorem shows how the baseline's probability of concentration (11) can be evaluated as a weighted sum of probabilities of noncentral Chi-square distributions.

Theorem 2 (*The 'fixed' baseline's probability of concentration*)
Let $\check{b}$ be the 'fixed' baseline estimator, let R be defined as in (12), and let $\chi^2(p, \lambda_z)$ denote the noncentral Chi-square distribution with p degrees of freedom and noncentrality parameter λ_z. Then

$$P(\check{b} \in R) = \sum_{z \in Z^n} P(\chi^2(p, \lambda_z) \leq \beta^2) P(\check{a} = z) \tag{13}$$

with

$$\lambda_z = \| \nabla\check{b}_z \|^2_{Q_{\hat{b}|\hat{a}}} \text{ and } \nabla\check{b}_z = Q_{\hat{b}\hat{a}}Q_{\hat{a}}^{-1}(z-a)$$

This result shows that the probability of the 'fixed' baseline lying inside the ellipsoidal region R centered at b equals an infinite sum of probability products. If one considers the two probabilities of these products separately, two effects are observed. First the probabilistic effect of shifting the conditional baseline estimator away from b and secondly the probabilistic effect of the peakedness or nonpeakedness of the ambiguity pmf. The second effect is related to the expected performance of ambiguity resolution, while the first effect has to do with the sensitivity of the baseline for changes in the values of the integer ambiguities. This effect is measured by the noncentrality parameter λ_z. Since the tail of a noncentral Chi-square distribution becomes heavier when the noncentrality parameter increases, while the degrees of freedom remain fixed, $P(\chi^2(p, \lambda_z) \leq \beta^2)$ gets smaller when λ_z gets larger.

The two probabilities in the product reach their maximum values when $z = a$. The following corollary shows how these two maxima can be used to lower bound and to upper bound the probability $P(\check{b} \in R)$. Such bounds are of importance for practical purposes, since it is difficult in general to evaluate (13) exactly.

Corollary 1 (*Lower and upper bounds*)
Let $\check{b}$ be the 'fixed' baseline estimator and let R be defined as in (12). Then

$$P(\hat{b}_{|\hat{a}=a} \in R)P(\check{a} = a) \leq P(\check{b} \in R) \leq P(\hat{b}_{|\hat{a}=a} \in R) \tag{14}$$

with

$$P(\hat{b}_{|\hat{a}=a} \in R) = P(\chi^2(p, 0) \leq \beta^2)$$

Note that the two bounds relate the probability of the 'fixed' baseline estimator to that of the conditional estimator and the bootstrapped succes rate. The above bounds become tight when the ambiguity success rate approaches one. This shows, although the probability of the conditional estimator always overestimates the probability of the 'fixed' baseline estimator, that the two probabilities are close for large values of the success rate. This implies that in case of GNSS ambiguity resolution, one should first evaluate the success rate $P(\check{a} = a)$ and make sure that its value is close enough to one, before making any inferences on the basis of the distribution of the conditional baseline estimator. In other words, the (unimodal) distribution of the conditional estimator is a good approximation to the (multimodal) distribution of the bootstrapped baseline estimator, when the success rate is sufficiently close to one.

3. INTEGER BOOTSTRAPPING

3.1. The bootstrapped estimator

The distributional results presented so far hold for any admissible ambiguity estimator. The simplest way to obtain an integer vector from the real-valued 'float' solution is to round each of the entries of $\hat{a}$ to its nearest integer. The corresponding integer estimator reads therefore

$$\check{a}_R = ([\hat{a}_1], \ldots, [\hat{a}_n])^T \tag{15}$$

where '[.]' denotes rounding to the nearest integer. The pull-in region of this integer estimator equals the multivariate version of the unit-square.

Another relatively simple integer ambiguity estimator is the bootstrapped estimator. The bootstrapped estimator can be seen as a generalization of the previous estimator. It still makes use of integer rounding, but it also takes some of the correlation between the ambiguities into account. The bootstrapped estimator follows from a sequential conditional least- squares adjustment and it is computed as follows. If n ambiguities are available, one starts with the first ambiguity $\hat{a}_1$, and rounds its value to the nearest integer. Having obtained the integer value of this first ambiguity, the real-valued estimates of all remaining ambiguities are then corrected by virtue of their correlation with the first ambiguity. Then the second, but now corrected, real-valued ambiguity estimate is rounded to its nearest integer. Having obtained the integer value of the second ambiguity, the real-valued estimates of all remaining $n-2$ ambiguities are then again corrected, but now by virtue of their correlation with the second ambiguity. This process is continued until all ambiguities are considered. We thus have the following definition.

Definition 2 (*Integer bootstrapping*)
Let $\hat{a} = (\hat{a}_1, \ldots, \hat{a}_n)^T \in R^n$ be the ambiguity 'float' solution and let $\check{a}_B = (\check{a}_{B,1}, \ldots, \check{a}_{B,n})^T \in Z^n$ denote the corresponding integer bootstrapped solution. The entries of the bootstrapped ambiguity estimator are then defined as

$$\begin{array}{rcl} \check{a}_{B,1} & = & [\hat{a}_1] \\ \check{a}_{B,2} & = & [\hat{a}_{2|1}] = [\hat{a}_2 - \sigma_{21}\sigma_1^{-2}(\hat{a}_1 - \check{a}_{B,1})] \\ & \vdots & \\ \check{a}_{B,n} & = & [\hat{a}_{n|N}] = \\ & & [\hat{a}_n - \sum_{j=1}^{n-1}\sigma_{n,j|J}\sigma_{j|J}^{-2}(\hat{a}_{j|J} - \check{a}_{B,j})] \end{array} \tag{16}$$

where $'[.]'$ denotes the operation of rounding to the nearest integer, and $\sigma_{i,j|J}$ denotes the covariance between $\hat{a}_i$ and $\hat{a}_{j|J}$, and $\sigma^2_{j|J}$ is the variance of $\hat{a}_{j|J}$. The shorthand notation $\hat{a}_{i|I}$ stands for the ith least-squares ambiguity obtained through a conditioning on the previous $I = \{1, \ldots, (i-1)\}$ sequentially rounded ambiguities.

Note that the bootstrapped estimator is not unique. Changing the order in which the ambiguities appear in vector $\hat{a}$ will already produce a different bootstrapped estimator. Although the principle of bootstrapping remains the same, every choice of ambiguity parametrization has its own bootstrapped estimator.

3.2. The bootstrapped pull-in regions

The pull-in regions for rounding are unit-cubes centred at integer grid points. For bootstrapping the shape of the pull-in regions will depend on the vc-matrix of the ambiguities. They will coincide with the unit- cubes only in case the vc-matrix is a diagonal matrix. Bootstrapping reduces namely to rounding in the absence of any correlation between the ambiguities. The following theorem gives a description of the bootstrapped pull-in regions in the general case.

Theorem 3 (*Bootstrapped pull-in regions*)
The pull-in regions of the bootstrapped ambiguity estimator $\check{a}_B = (\check{a}_{B,1}, \ldots, \check{a}_{B,n})^T \in Z^n$ are given as

$$S_{B,z} = \{x \in R^n \mid | c_i^T L^{-1}(x-z) | \leq \frac{1}{2} \, , \, i = 1-n\} \tag{17}$$

$\forall z \in Z^n$ where L denotes the unique unit lower triangular matrix of the ambiguity vc-matrix' decomposition $Q_{\hat{a}} = LDL^T$ and c_i denotes the ith canonical unit vector having a 1 as its ith entry and zeros otherwise.

That the bootstrapped estimator is indeed admissible, can now be seen as follows. The first two conditions of Definition 1 are easily verified using the definition of the bootstrapped estimator. Since every real-valued vector $\hat{a}$ will be mapped by the bootstrapped estimator to an integer vector, the pull-in regions $S_{B,z}$ cover R^n without any gaps. There is also no overlap between the pull-in regions, since - apart from boundary ties - any real-valued vector $\hat{a}$ is mapped to not more than one integer vector. To verify the last condition of Definition 1, we make use of (17). From

$$\begin{array}{l} S_{B,z} = \\ \{x \in R^n \mid | c_i^T L^{-1}(x-z) | \leq \frac{1}{2} \, , \, i = 1,n\} = \\ \{x \in R^n \mid | c_i^T L^{-1} y | \leq \frac{1}{2} \, , \, x = y + z \, , \, i = 1,n\} = \\ S_{B,0} + z \end{array}$$

it follows that all bootstrapped pull-in regions are translated copies of $S_{B,0}$. All pull-in regions have therefore the same shape and the same volume. Their volumes all equal 1. This can be shown by transforming $S_{B,0}$ to the unit cube centered at the origin. Consider the linear transformation $y = L^{-1}x$. Then

$$L^{-1}(S_{B,0}) = \{y \in R^n \mid | c_i^T y | \leq \frac{1}{2} \, , \, i = 1, \ldots, n\}$$

equals the unit cube centered at the origin. Since the determinant of the unit lower triangular matrix L^{-1} equals one and since the volume of the unit cube equals one, it follows that the volume of $S_{B,0}$ must equal one as well. To infer the shape of the bootstrapped pull-in region, we consider the two-dimensional case first. Let the lower triangular matrix L be given as

$$L = \begin{bmatrix} 1 & 0 \\ l & 1 \end{bmatrix}$$

Then

$$\begin{array}{rcl} S_{B,0} & = & \{x \in R^2 \mid | c_i^T L^{-1} x | \leq \frac{1}{2} \, , \, i = 1,2\} \\ & = & \{x \in R^2 \mid | x_1 | \leq \frac{1}{2}, | x_2 - l x_1 | \leq \frac{1}{2}\} \end{array}$$

which shows that the two-dimensional pull-in region equals a parallellogram. Its region is bounded by the two vertical lines $x_1 = 1/2$ and $x_1 = -1/2$, and the two parallel slopes $x_2 = lx_1 + 1/2$ and $x_2 = lx_1 - 1/2$. The direction of the slope is governed by $l = \sigma_{21}\sigma_1^{-2}$. Hence, in the absence of correlation between the two ambiguities, the parallellogram reduces to the unit square. In higher dimensions the above construction of the pull-in region can be continued. In three dimensions for instance, the intersection of the pull-in region with the x_1x_2-plane remains a parallellogram, while along the third axis the pull-in region becomes bounded by two parallel planes.

3.3. The bootstrapped pmf

Since the integer bootstrapped estimator is defined as $\check{a}_B = z \Longleftrightarrow \hat{a} \in S_{B,z}$, it follows that $P(\check{a}_B = z) = P(\hat{a} \in S_{B,z})$. The pmf of $\check{a}_B$ follows therefore as

$$P(\check{a}_B = z) = \int_{S_{B,z}} p_{\hat{a}}(x)dx \ , \ z \in Z^n \tag{18}$$

Hence, the probability that $\check{a}_B$ coincides with z is given by the integral of the pdf $p_{\hat{a}}(x)$ over the bootstrapped pull-in region $S_{B,z} \subset R^n$. The above expression holds for any distribution the 'float' ambiguities $\hat{a}$ might have. In most GNSS applications however, one usually assumes the vector of observables y to be normally distributed. For that case the following theorem gives an exact expression of the bootstrapped pmf.

Theorem 4 (*The integer bootstrapped pmf*)
Let $\hat{a}$ be distributed as $\mathrm{N}(a, Q_{\hat{a}})$, $a \in Z^n$, and let $\check{a}_B$ be the corresponding integer bootstrapped estimator. Then

$$\begin{aligned} P(\check{a}_B = z) &= \textstyle\prod_{i=1}^{n} [\Phi(\frac{1-2l_i^T(a-z)}{2\sigma_{\hat{a}_{i|I}}}) \\ &+ \Phi(\frac{1+2l_i^T(a-z)}{2\sigma_{\hat{a}_{i|I}}}) - 1] \ , \ z \in Z^n \end{aligned} \tag{19}$$

with

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}v^2\} dv$$

and with l_i the ith column vector of the unit lower triangular matrix L^{-T} and $\sigma^2_{\hat{a}_{i|I}}$ the variance of the ith least- squares ambiguity obtained through a conditioning on the previous $I = \{1, \ldots, (i-1)\}$ ambiguities.

The bootstrapped pmf equals a product of univariate pmf's and is therefore easy to compute. Note that the bootstrapped pmf is completely governed by the ambiguity vc-matrix $Q_{\hat{a}}$. The pmf follows once the triangular factor L and the diagonal matrix D of the decomposition $Q_{\hat{a}} = LDL^T$ are given. The above result also shows that the bootstrapped pmf is symmetric about the mean of $\hat{a}$. This implies that the bootstrapped estimator $\check{a}_B$ is an unbiased estimator of $a \in Z^n$. Since the 'float' solutions, $\hat{a}$ and $\hat{b}$, are unbiased too, it follows from taking the expectation of (4) that the bootstrapped baseline is also unbiased.

For the purpose of predicting the success of ambiguity resolution, the probability of correct integer estimation is of particular interest. For the bootstrapped estimator this success rate is given in the following corollary.

Corollary 2 (*The bootstrapped success rate*)
The bootstrapped probability of correct integer estimation (the success rate) is given as

$$P(\check{a}_B = a) = \prod_{i=1}^{n} [2\Phi(\frac{1}{2\sigma_{\hat{a}_{i|I}}}) - 1] \tag{20}$$

The method of integer bootstrapping is easy to implement and it does not need, as opposed to the method of integer least-squares (see next section), an integer search for computing the sought for integer solution. However, as it was mentioned earlier, the outcome of bootstrapping depends on the chosen ambiguity parametrization. Bootstrapping of DD ambiguities, for instance, will produce an integer solution which generally differs from the integer solution obtained from bootstrapping of reparametrized ambiguities. Since this dependency also holds true for the bootstrapped pmf, one still has some important degrees of freedom left for improving (20) or for sharpening the lower bound of (14).

In order to improve the bootstrapped success rate, one should work with decorrelated ambiguities instead of with the original ambiguities. The method of bootstrapping performs relatively poor, for instance, when applied to the DD ambiguities. This is due to the usually high correlation between the DD ambiguities. Bootstrapping should therefore only be used in combination with the decorrelating Z-transformation of the LAMBDA method [*Teunissen*, 1993, 1995]. This transformation decorrelates the ambiguities further than the best reordering would achieve and thereby reduces the values of the sequential conditional variances. By reducing the values of the sequential conditional variances, the bootstrapped success rate gets enlarged.

It may however happen that it is simply not possible to resolve the complete vector of ambiguities with sufficient probability. As an alternative of resolving the complete vector of ambiguities, one might then consider resolving only a subset of the ambiguities. The idea of partial ambiguity resolution is based on the fact that the success rate will generally increase when fewer integer constraints are imposed. However, in order to apply partial ambiguity resolution, one first will have to determine which subset of ambiguities to choose. It will be clear that this decision should be based on the precision of the 'float' ambiguities. The more precise the ambiguities, the larger the ambiguity success rate. It is at this point where the decorrelation step of the LAMBDA method and the bootstrapping principle can be applied. Once the transformed and decorrelated ambiguity vc-matrix is obtained, the construction of the subset proceeds in a sequential fashion. One first starts with the most precise ambiguity, say $\hat{z}_1$, and computes its success rate $P(\check{a}_1 = z_1)$. If this success rate is large enough, one continues and determines the most precise pair of ambiguities, say $(\hat{z}_1, \hat{z}_2)$. If their success rate is still large enough, one continues again by trying to extend the set. This procedure continues until one reaches a point where the corresponding success rate becomes unacceptably small. When this point is reached, one can expect that the previously identified ambiguities can be resolved successfully.

Once the subset for partial ambiguity resolution has been identified, one still needs to determine what this will do to improve the baseline estimator. After all, being able to successfully resolve the ambiguities does not necessarily mean that the 'fixed' solution is significantly better than the 'float' solution. The theory presented in the previous sections provide the necessary tools for performing such an evaluation.

4. INTEGER LEAST-SQUARES

4.1. The ILS estimator

In this section we review some integer least-squares' theory for solving the GNSS model (1). When using the least-squares principle, the GNSS model can be solved by means of the minimization problem

$$\min_{a,b} \| y - Aa - Bb \|^2_{Q_y} \ \ , a \in Z^n \ , b \in R^p \tag{21}$$

with Q_y the vc-matrix of the GNSS observables. This type of least-squares problem was first introduced in [*Teunissen*, 1993] and has been coined with the term '*integer least-squares*'. It is a nonstandard least-squares problem due to the integer constraints $a \in Z^n$.

The solution of (21) is consistent with the three solution steps of section 1. This can be seen as follows. It follows from the orthogonal decomposition

$$\begin{array}{rcl} \| y-Aa-Bb \|^2_{Q_y} & = & \| \hat{e} \|^2_{Q_y} + \| \hat{a}-a \|^2_{Q_{\hat{a}}} \\ & + & \| \hat{b}(a)-b \|^2_{Q_{\hat{b}|\hat{a}}} \end{array} \quad (22)$$

with $\hat{e} = y - A\hat{a} - B\hat{b}$ and $\hat{b}(a) = \hat{b} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1}(\hat{a}-a)$, that the sought for minimum is obtained when the second term on the right-hand side is minimized for $a \in Z^n$ and the last term is set to zero. The integer least-squares (ILS) estimator of the ambiguities is therefore defined as follows.

Definition 3 (*Integer least-squares*)
Let $\hat{a} = (\hat{a}_1, \ldots, \hat{a}_n)^T \in R^n$ be the ambiguity 'float' solution and let $\check{a}_{LS} \in Z^n$ denote the corresponding integer least-squares solution. Then

$$\check{a}_{LS} = \arg\min_{z \in Z^n} \| \hat{a} - z \|^2_{Q_{\hat{a}}} \quad (23)$$

In contrast to integer rounding and integer bootstrapping, an integer search is needed to compute $\check{a}_{LS}$. Although we will refrain from discussing the computational intricacies of ILS estimation, the conceptual steps of the computational procedure will be described briefly. The ILS procedure is mechanized in the GNSS LAMBDA (Least-squares AMBiguity Decorrelation Adjustment) method, which is currently one of the most applied methods for GNSS carrier phase ambiguity resolution. For more information on the LAMBDA method, we refer to e.g. [*Teunissen*, 1993], [*Teunissen*, 1995] and [*de Jonge and Tiberius*, 1996a] or to the textbooks [*Hofmann-Wellenhof*, 1997], [*Strang and Borre*, 1997], [*Teunissen and Kleusberg*, 1998]. Practical results obtained with it can be found, for example, in [*Boon and Ambrosius*, 1997], [*Boon et al.*, 1997], [*Cox and Brading*, 1999], [*de Jonge and Tiberius*, 1996b], [*de Jonge et al.*, 1996], [*Han*, 1995], [*Jonkman*, 1998], [*Peng et al.*, 1999], [*Tiberius and de Jonge*, 1995], [*Tiberius et al.*, 1997].

The main steps as implemented in the LAMBDA method are as follows. One starts by defining the ambiguity search space

$$\Omega_a = \{ a \in Z^n \mid (\hat{a}-a)^T Q_{\hat{a}}^{-1} (\hat{a}-a) \leq \chi^2 \} \quad (24)$$

with χ^2 a to be chosen positive constant. The boundary of this search space is ellipsoidal. It is centred at $\hat{a}$, its shape is governed by the vc-matrix $Q_{\hat{a}}$ and its size is determined by χ^2. In case of GNSS, the search space is usually extremely elongated, due to the high correlations between the ambiguities. Since this extreme elongation usually hinders the computational efficiency of the search, the search space is first transformed to a more spherical shape,

$$\Omega_z = \{ z \in Z^n \mid (\hat{z}-z)^T Q_{\hat{z}}^{-1} (\hat{z}-z) \leq \chi^2 \} \quad (25)$$

using the admissible ambiguity transformations $\hat{z} = Z^T \hat{a}$, $Q_{\hat{z}} = Z^T Q_{\hat{a}} Z$. Ambiguity transformations Z are said to be admissible when both Z and its inverse Z^{-1} have integer entries. Such matrices preserve the integer nature of the ambiguities. In order for the transformed search space to become more spherical, the volume-preserving Z-transformation is constructed as a transformation that decorrelates the ambiguities as much as possible. Using the triangular decomposition of $Q_{\hat{z}}$, the left-hand side of the quadratic inequality in (25) is then written as a sum-of-squares:

$$\sum_{i=1}^{n} \frac{(\hat{z}_{i|I} - z_i)^2}{\sigma^2_{i|I}} \leq \chi^2 \quad (26)$$

On the left-hand side one recognizes the conditional least-squares estimator $\hat{z}_{i|I}$, which follows when the conditioning takes place on the integers $z_1, z_2, \ldots, z_{i-1}$. Using the sum-of-squares structure, one can finally set up the n intervals which are used for the search. These sequential intervals are given as

$$\begin{array}{rcl} (\hat{z}_1 - z_1)^2 & \leq & \sigma_1^2 \chi^2 \\ (\hat{z}_{2|1} - z_2)^2 & \leq & \sigma^2_{2|1} \left(\chi^2 - \frac{(\hat{z}_1 - z_1)^2}{\sigma_1^2} \right) \\ & \vdots & \end{array} \quad (27)$$

In order for the search to be efficient, one not only would like the vc- matrix $Q_{\hat{z}}$ to be as close as possible to a diagonal matrix, but also that the search space does not contain too many integer grid points. This requires the choice of a small value for χ^2, but one that still guarantees that the search space contains at least one integer grid point. Since the bootstrapped estimator is so easy to compute and at the same time gives a good approximation to the ILS estimator (see section 4.4), the bootstrapped solution is an excellent candidate for setting the size of the ambiguity search space. Following the decorrelation step $\hat{z} = Z^T \hat{a}$, the LAMBDA-method therefore uses, as one of its options, the bootstrapped solution $\check{z}_B$ for setting the size of the ambiguity search space as

$$\chi^2 = (\hat{z} - \check{z}_B)^T Q_{\hat{z}}^{-1} (\hat{z} - \check{z}_B) \quad (28)$$

In this way one can work with a very small search space and still guarantee that the sought for integer least-squares solution is contained in it.

4.2. The ILS pull-in region

The pull-in regions of integer rounding are unit cubes, while those of integer bootstrapping are multivariate versions of parallellograms. To determine the ILS pull-in regions we need to know the set of 'float' solutions $\hat{a} \in R^n$ that are mapped to the same integer vector $z \in Z^n$. This set is described by all $x \in R^n$ that satify $z = \arg\min_{u \in Z^n} \| x-u \|^2_{Q_{\hat{a}}}$. The ILS pull-in-region that belongs to the integer vector z follows therefore as

$$S_{LS,z} = \{ x \in R^n \mid \| x-z \|^2_{Q_{\hat{a}}} \leq \| x-u \|^2_{Q_{\hat{a}}}, \forall u \in Z^n \} \quad (29)$$

It consists of all those points which are closer to z than to any other integer point in R^n. The metric used for measuring these distances is determined by the vc-matrix $Q_{\hat{a}}$. Based on (29), one can give a representation of the ILS pull-in regions that resembles the representation of the bootstrapped pull-in regions. This representation reads as follows.

Theorem 5 (*ILS pull-in regions*)
The pull-in regions of the ILS ambiguity estimator $\check{a}_{LS} \in Z^n$ are given as

$$\begin{array}{l} S_{LS,z} = \\ \cap_{c_i \in Z^n} \{ x \in R^n \mid \mid c_i^T Q_{\hat{a}}^{-1} (x-z) \mid \leq \frac{1}{2} \| c_i \|^2_{Q_{\hat{a}}} \}, \\ \forall z \in Z^n \end{array} \quad (30)$$

This shows that the ILS pull-in regions are constructed from intersecting half-spaces. One can also show that at most $2^n - 1$ pairs of such half spaces are needed for constructing the pull-in region.

The ILS pull-in regions are convex, symmetric sets of volume 1, which satisfy the conditions of Definition 1. The ILS estimator is therefore admissible. The ILS pull-in regions are hexagons in the two-dimensional case.

4.3. Maximizing the success-rate

Although various integer estimators exist which are admissible, some may be better than others. Having the problem of GNSS ambiguity resolution in mind, one is particularly interested in the estimator which maximizes the probability of correct integer estimation. This probability equals $P(\check{a} = a)$, but it will differ for different ambiguity estimators. The following theorem shows that the ILS estimator maximizes the probability of correct integer estimation.

Theorem 6 (*ILS is optimal*)
Let the pdf of the 'float' solution $\hat{a}$ be given as

$$p_a(x) = \sqrt{\det(Q_{\hat{a}}^{-1})}G(\| x - a \|^2_{Q_{\hat{a}}}) \tag{31}$$

where $G : R \mapsto [0, \infty)$ is decreasing and $Q_{\hat{a}}$ is positive-definite. Then

$$P(\check{a}_{LS} = a) \geq P(\check{a} = a) \tag{32}$$

for any admissible estimator $\check{a}$.

This theorem gives a probabilistic justification for using the ILS estimator. For GNSS ambiguity resolution it shows, that one is better off using the ILS estimator than any other admissible integer estimator. The family of distributions defined in (31), is known as the family of elliptically contoured distributions. Several important distributions belong to this family. The multivariate normal distribution can be shown to be a member of this family by choosing $G(x) = (2\pi)^{-\frac{n}{2}} \exp -\frac{1}{2}x, x \in R$. Another member is the multivariate t-distribution.

As a direct consequence of the above theorem we have the following corollary.

Corollary 3 (*The effect of the weight matrix*)
Let Σ be any positive-definite matrix of order n and define

$$\check{a}_\Sigma = \arg\min_{z \in Z^n} \| \hat{a} - z \|^2_\Sigma \tag{33}$$

Then $\check{a}_\Sigma$ is admissible and

$$P(\check{a}_{LS} = a) \geq P(\check{a}_\Sigma = a) \tag{34}$$

In order to prove the corollary, we only need to show that $\check{a}_\Sigma$ is admissible. Once this has been established, the stated result (34) follows from theorem 6. The admissibility can be shown as follows. The first two conditions of Definition 1 are satisfied, since the ILS-map produces - apart from boundary ties - a unique integer vector for any 'float' solution $\hat{a} \in R^n$. And since $\check{a}_\Sigma = \arg\min_{z \in Z^n} \| \hat{a} - u - z \|^2_\Sigma + u$ holds true for any integer $u \in Z^n$, also the integer remove-restore technique applies.

As the corollary shows, a proper choice of the data weight matrix is also of importance for ambiguity resolution. The choice of weights is optimal when the weight matrix equals the inverse of the ambiguity vc-matrix. A too optimistic precision description or a too pessimistic precision description, will both result in a less than optimal ambiguity success rate. In the case of GNSS, the observation equations (the functional model) are sufficiently known and well documented. However, the same can not yet be said of the vc-matrix of the GNSS data. In the many GNSS textbooks available, we will usually find only a few comments, if any, on this vc-matrix. Examples of studies that have been reported in the literature are: [*Euler and Goad*, 1991], [*Gerdan*, 1995], [*Gianniou*, 1996], and [*Jin and de Jong*, 1996], who studied the elevation dependence of the observation variances; [*Jonkman*, 1998] and [*Tiberius*, 1998], who considered time correlation and cross correlation; and [*Schaffrin and Bock*, 1988], [*Bock*, 1998] and [*Teunissen*, 1998a], who considered the inclusion of stochastic ionospheric constraints.

4.4. Bounding the ILS success-rate

A very useful application of theorem 6 is that it shows how one can *lower-bound* the ILS probability of correct integer estimation. This is particularly useful since the ILS success rate is usually difficult to compute. This is due to the rather complicated geometry of the ILS pull-in region. The bootstrapped success-rate is a good candidate for the ILS success-rates' lower-bound. The bootstrapped success-rate is easy to compute and it becomes a sharp lower-bound when applied to the decorrelated ambiguities $\hat{z} = Z^T\hat{a}$. In fact, at present, the bootstrapped success-rate is the sharpest available lower-bound of the ILS success-rate.

Apart from having a lower-bound, it is also useful to have an upper-bound available. For obtaining an upper-bound one can make use of the *geometric mean* of the ambiguity conditional variances. This geometric mean is referred to as the Ambiguity Dilution of Precision (ADOP) and it is given as

$$\text{ADOP} = \sqrt{\det Q_{\hat{a}}}^{\frac{1}{n}} \quad (\text{cycles}) \tag{35}$$

Note that this scalar measure of the ambiguity precision is invariant for the admissible volume preserving ambiguity transformations. With the ADOP one can obtain an upper-bound by making use of the fact that the probability content of the ILS pull-in region $S_{LS,a}$ would be maximal if its shape would coincide with that of the ambiguity search space, while its volume would still be constrained to 1. We have the following bounds for the ILS success-rate.

Theorem 7 (*Bounds on the ILS success-rate*)
The ILS success-rate $P(\check{a}_{LS} = a)$ is bounded from below and from above as

$$\begin{array}{l} P(\check{z}_B = z) \leq P(\check{a}_{LS} = a) \leq P\left(\chi^2(n,0) \leq \frac{c_n}{\text{ADOP}^2}\right) \\ \text{with } \; c_n = (\frac{n}{2}\Gamma(\frac{n}{2}))^{2/n}/\pi \end{array} \tag{36}$$

5. A BAYESIAN APPROACH

5.1. The Bayes estimate

The Bayesian approach to GNSS carrier phase ambiguity resolution starts from a set of assumptions which differs fundamentally from the one used in the previous sections, see e.g.[*Betti et al.*, 1993], [*Gundlich and Koch*, 2001]. In the Bayesian approach, not only the vector of observables, y, is assumed to be random, but the vector of unknown parameters, a and b, as well. Although the Bayesian approach has not yet find a wide-spread use in any of the GNSS applications, the basic concepts involved are of interest in their own right, also in their comparison with the nonBayesian theory of the previous sections.

Let us for the moment take the two type of parameter vectors a and b together in one vector $x = (a^T, b^T)^T$. If both y and x are random, we have according to Bayes' theorem

$$p(x \mid y) = \frac{p(y \mid x)p(x)}{p(y)} \tag{37}$$

Thus the posterior density $p(x \mid y)$ is proportional to the product of the likelihood function $p(y \mid x)$ and the prior density $p(x)$. Given the data vector y, that is, given the observations, the posterior density gives a complete description of the probabilistic properties of x. The idea of the Bayesian approach is therefore to use the posterior density for parameter estimation.

In the Bayesian approach to ambiguity resolution it is the so-called *Bayes estimate* which is used as the solution for the ambiguities and baseline. This estimate is defined as follows.

Definition 4 (*The Bayes estimate*)
The Bayes estimate $\hat{x}_{Bayes}$ of the random parameter vector x is defined as the conditional mean

$$\hat{x}_{Bayes} = \mathrm{E}\{x \mid y\} = \int x p(x \mid y) dx \tag{38}$$

This definition can be motivated as follows. In order to find a 'good' estimate $\hat{x}$ of the parameter vector x, we would like to determine a function of the data, say $\hat{x} = \hat{x}(y)$, which in a certain sense is close to x. Let $L(x, \hat{x}(y))$ be our measure of discrepancy, or our measure of loss, between x and $\hat{x}$. It then seems reasonable to take $\hat{x}$ as the solution which minimizes this discrepancy on the average. This amounts to solving the minimization problem

$$\min_{\hat{x}} \mathrm{E}\{L(x, \hat{x}(y)) \mid y\} = \min_{\hat{x}} \int L(x, \hat{x}) p(x \mid y) dx \tag{39}$$

This minimization problem is particularly easy to solve in case the loss function equals the quadratic form, $L(x, \hat{x}) = \| x - \hat{x} \|_Q^2$, with matrix Q being positive definite. From the decomposition

$$\begin{array}{l} \mathrm{E}\{L(x, \hat{x}(y)) \mid y\} = \\ \int \| x - \hat{x} \|_Q^2 \; p(x \mid y) dx = \\ \int \| x - \mathrm{E}\{x \mid y\} \|_Q^2 \; p(x \mid y) dx + \| \mathrm{E}\{x \mid y\} - \hat{x} \|_Q^2 \end{array}$$

it directly follows that the posterior expected loss is minimized when $\hat{x}$ is taken equal to the Bayes estimate. When the Bayes estimate is substituted into the loss function, the expected loss equals $\mathrm{E}\{L(x, \hat{x}_{Bayes})\} = \mathrm{trace}(Q_{x|y} Q^{-1})$.

5.2. The marginal posterior pdf's

In order to apply (38) to our ambiguity resolution problem, we first need an expression for the posterior density $p(x \mid y) = p(a, b \mid y)$. In the Bayesian approach to GNSS ambiguity resolution, a and b are assumed to be independent, with the following improper priors

$$\begin{cases} p(a) & \propto & \sum_{z \in Z^n} \delta(a - z) \;\; \text{(pulsetrain)} \\ p(b) & \propto & constant \end{cases} \tag{40}$$

where δ denotes the Dirac function. From the orthogonal decomposition (22), the likelihood function can be seen to be proportional to $p(y \mid a, b) \propto \exp -\frac{1}{2}\{\| \hat{a} - a \|_{Q_{\hat{a}}}^2 + \| \hat{b}(a) - b \|_{Q_{\hat{b}|\hat{a}}}^2\}$. The posterior density follows therefore as

$$\begin{array}{l} p(a, b \mid y) \propto \exp -\frac{1}{2}\{\| \hat{a} - a \|_{Q_{\hat{a}}}^2 \\ + \| \hat{b}(a) - b \|_{Q_{\hat{b}|\hat{a}}}^2\} \sum_{z \in Z^n} \delta(a - z) \end{array} \tag{41}$$

The required marginal posterior densities, $p(a \mid y)$ and $p(b \mid y)$, follow from integrating the joint posterior density over the domains of respectively a and b. Note that in the present case, the domain of a is taken as R^n and not as Z^n. In the Bayesian approach, the discrete-like nature of a is thought to be captured by assuming the prior to be a pulsetrain. Once the integrations are carried out and the normalizing constants are restored, the marginals are obtained as follows.

Theorem 8 (*Marginal posterior pdf's*)
The posterior pdf's of the ambiguities and baseline are given as

$$\begin{cases} p(a \mid y) & = & w_a(\hat{a}) \sum_{z \in Z^n} \delta(a - z) \\ \\ p(b \mid y) & = & \sum_{z \in Z^n} p_{b|a}(b \mid a = z, y) w_z(\hat{a}) \end{cases} \tag{42}$$

with the weight function

$$w_z(\hat{a}) = \frac{\exp -\frac{1}{2}\{\| \hat{a} - z \|_{Q_{\hat{a}}}^2\}}{\sum_{u \in Z^n} \exp -\frac{1}{2}\{\| \hat{a} - u \|_{Q_{\hat{a}}}^2\}} \;,\; z \in Z^n \tag{43}$$

and the conditional posterior

$$p_{b|a}(b \mid a, y) = \frac{1}{\sqrt{\det Q_{\hat{b}|\hat{a}}} (2\pi)^{\frac{1}{2}p}} \exp -\frac{1}{2} \| b - \hat{b}(a) \|_{Q_{\hat{b}|\hat{a}}}^2 \tag{44}$$

It is now interesting to observe how the above posterior marginal pdf for the baseline, $p(b \mid y)$, compares with the pdf of the 'fixed' baseline, $p_{\check{b}}(x)$, as given in (10). Both pdf's are very similar in structure. Both equal an infinite sum of weighted conditional baseline distributions. The two type of conditional baseline distributions, $p_{\hat{b}|\hat{a}}(x \mid z)$ and $p_{b|a}(b \mid z, y)$, have an identical shape but differ in their point of symmetry. The first is symmetric about $b(z) = b - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1}(a - z)$, while the second is symmetric about $\hat{b}(z) = \hat{b} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1}$
$(\hat{a} - z)$. Also the weights share some resemblance. This can be seen if we consider the probability

$$P(\check{a} = a - z) = \int_{S_{a-z}} (\sqrt{\det Q_{\hat{a}}} (2\pi)^{\frac{1}{2}n})^{-1} \exp -\frac{1}{2} \| x - a \|_{Q_{\hat{a}}}^2 \, dx$$

This probability can be worked out to give

$$P(\check{a} = a - z) = \frac{\int_{S_0} \exp -\frac{1}{2} \| x - z \|_{Q_{\hat{a}}}^2 \, dx}{\sum_{u \in Z^n} \int_{S_0} \exp -\frac{1}{2} \| x - u \|_{Q_{\hat{a}}}^2 \, dx} \tag{45}$$

which shows the resemblance with (43).

5.3. The Bayes baseline

With the posterior baseline distribution available, one can now study the corresponding confidence regions as well as determine the Bayes estimate of the baseline,

$$\hat{b}_{Bayes} = \int b p(b \mid y) db$$

For a discussion on how to approximate the confidence regions of the posterior baseline, we refer to [*Gundlich and Koch*, 2001].

Using the results of theorem 8, the Bayes baseline follows as

$$\hat{b}_{Bayes} = \hat{b} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} \left(\hat{a} - \sum_{z \in Z^n} z w_z(\hat{a}) \right) \tag{46}$$

Again there is a striking resemblance with the results of section 2. From (4) and (2.2) it follows that the 'fixed' baseline can be written as

$$\check{b} = \hat{b} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} \left(\hat{a} - \sum_{z \in Z^n} z s_z(\hat{a}) \right) \tag{47}$$

We thus see that the two solutions differ in the way the 'float' solution $\hat{a}$ is used to weigh all integer grid points $z \in Z^n$. In case of the Bayes baseline the smooth weights $w_z(\hat{a})$ are used, while in case of the 'fixed' baseline, the $0-1$ values of the indicator function $s_z(\hat{a})$ are used. Although both baseline solutions contain an infinite sum, the one of the 'fixed' baseline can be computed exactly, while the one of the Bayes baseline can only be approximated.

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