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A BRIEF ACCOUNT ON THE EARLY HISTORY OF ADJUSTING GEODETIC AND ASTRONOMICAL OBSERVATIONS[1]

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Samenvatting

Als onderdeel van de mathematische geodesie, houdt de vereffeningstheorie zich bezig met het optimaal combineren van overtallige waarnemingen ter bepaling van onbekende parameters. Het is een fundamenteel vak, de betekenis ervan vergelijkbaar met die van de mechanica voor een civielingenieur of werktuigbouwkundige. De twee hoofdredenen voor het verrichten van overtallige waarnemingen zijn verhoging van de nauwkeurigheid en het mogelijk maken van controles. Door de onvermijdelijke onzekerheid in de waarnemingen, leidt overtalligheid in de regel tot een inconsistent stelsel van vergelijkingen. Zonder additionele criteria is een dergelijk stelsel niet eenduidig oplosbaar.
De zoektocht naar het vinden van methoden waarmee inconsistente stelsels vergelijkingen kunnen worden opgelost, had halverwege de 18e eeuw de aandacht van verschillende wetenschappers van naam. De eerste methoden voor het vereffenen van overtallige waarnemingen vinden hun oorsprong in geodetische en astronomische studies, namelijk bij studies ter bepaling van de vorm en grootte van de aarde en bij studies ter bepaling van het bewegingsgedrag van de maan. Vanaf de ontdekking van de kleinste-kwadraten methode, nu bijna 200 jaar geleden, is deze methode de meest populaire vereffeningsmethode gebleken. Hoewel het kleinste-kwadraten principe voor een moderne student van de vereffeningstheorie voor de hand zal liggen, volgt de ontdekking ervan een reeks van daar aan voorafgaande methoden. In deze bijdrage wordt de historische ontwikkelingslijn van deze vereffeningsmethoden in het kort beschreven.

I. Introduction

Adjustment theory can be regarded as the part of mathematical geodesy[2] that deals with the optimal combination of redundant observations for the purpose of determining values for parameters of interest. It is essential for a geodesist, its meaning comparable to what mechanics means to a civil engineer or a mechanical engineer. The two main reasons for performing redundant measurements are the wish to increase the accuracy of the results computed and the requirement to be able to check for errors. Due to the intrinsic uncertainty in observations, redundancy generally leads to an inconsistent system of equations. Without additional criteria, such a system is not uniquely solvable.
The problem of solving an inconsistent system of equations has attracted the attention of leading scientists in the middle of the 18th century. Historically, the first methods of combining redundant observations originate from studies in geodesy and astronomy, namely from the problem of determining the size and shape of the Earth, and the problem of finding a mathematical representation of the motions of the moon. Since its discovery almost 200 years ago, least-squares has been the most popular method of adjustment. Although the

method of least-squares may seem 'natural' for a modern student of adjustment theory, its discovery evolved only slowly from earlier methods of combining redundant observations[3]. In this contribution we sketch the historical line of development of these adjustment methods in the second half of the 18th century.

II. *The method of selected points*

It is convenient to cast the problem of combining redundant observations in terms of vectors and matrices. Suppose we are given a set of linear equations of the form

$$\underset{m\times 1}{y} = \underset{m\times n}{A}\ \underset{n\times 1}{x}$$

where y is a vector of observations, A is a given matrix of full rank and x is the vector of unknown parameters. This set of linear equations is said to be overdetermined when there are more observations than unknowns, $m>n$. The problem is to combine the m observations so that one can solve for the n unknown parameters. If we restrict ourselves to linear combinations of the observations, we can write the general solution in the following form

$$\underset{n\times 1}{x} = \underset{n\times m}{B}\ \underset{m\times 1}{y} \quad \text{with} \quad \underset{n\times m}{B} = \underset{n\times n}{(LA)^{-1}}\ \underset{n\times m}{L}$$

and where matrix L is a suitably chosen matrix defining the linear combinations. Different choices of L give different linear combinations and therefore different solutions. In modern terminology, matrix B is called a left-inverse of A, since B times A equals the identity matrix. Before 1750 a popular, albeit subjective, method of solving an overdetermined set of linear equations was *the method of selected points*. It consists of choosing n out of the m observations (referred to as the selected points) and using their equations to solve for x. If the choice falls on the first n observations, the corresponding L matrix takes the form

$$L = \begin{bmatrix} \underset{n\times n}{I} & \underset{n\times(m-n)}{0} \end{bmatrix}$$

where I is the identity matrix. For the method of selected points, n residuals (the difference between the observed and adjusted observations) are by definition equal to zero. Many scientists using this method calculated the remaining m-n residuals and studied their sign and size to get an impression of the goodness of fit between observations and the proposed law.
The method is subjective because no clear rule is given which observations to select and which to throw out. Selecting another set of n observations leads to a different solution for x. Although the disadvantage of not using all observations was recognized, no simple method existed to tackle this shortcoming. Sometimes all possible combinations of n observations were considered and then averaged to obtain the final result. But since this approach requires handling m over n combinations, it was only practical for problems of low dimensions.

III. *The method of averages*

Tobias Mayer (1723-1762), professor of mathematics and head of the Göttingen observatory, made numerous observations of the moon with the purpose of determining the characteristics of the moon's orbit. In 1750[4] Mayer proposed a new method for adjusting his moon data, a method which solved the above mentioned pitfall of the method of selected points. Apart from his adjustment method, Mayer is also known for his other contributions to surveying and navigation. In 1752 he invented the Repeating or Reflecting Circle, an instrument for observing the angle between two celestial bodies. The accuracy of Mayer's instrument was comparable to John Hadley's reflecting octant (1731), but had the advantage that it could be used to measure angles of over 90 degrees[5]. Mayer also contributed to solving the mariner's 'longitude problem'. It was the British Parliament, which in 1714, offered the

'Longitude Prize' to those who could find a 'useful and practicable' method for determining longitude at sea. To determine longitude of a ship at sea, the mariner needs to know both his local time and the time at some standard location. Local time was readily determined, but the determination of standard time at sea was more complicated. Mayer's detailed lunar tables (1755) made it possible to translate the instrument readings into longitude positions. The use of Mayer's lunar tables was later superseded by John Harrison's marine chronometer H-4 (1759). In recognition of their contributions, both men were awarded part of the 'Longitude Prize', with the larger sum going to Harrison[6].

Mayer studied the libration of the moon by observing the changing position of the crater Manilius as seen from the Earth. Using spherical geometry, he found a linearized relationship between his observables and some location parameters of Manilius and the moon's pole. This gave him an inconsistent system of 27 linear equations in 3 unknown parameters

$$\underbrace{\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}}_{y} = \underbrace{\begin{bmatrix} 1 & a_{12} & a_{13} \\ \vdots & \vdots & \vdots \\ 1 & a_{m2} & a_{m3} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x}$$

Mayer proposed to divide the 27 equations into 3 groups of 9 each, to sum the equations within each group, and to solve the resulting 3 equations in the 3 unknowns. For a general set of m equations in n unknowns, this approach amounts to a separation of the m equations into n groups, followed by a groupwise summation. For example in case n=2, the correspon-ding L matrix takes the form

$$L = \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix}$$

where the e_1 and e_2 are row vectors having only 1's as their entries. Since one may use averages instead of sums, the method became later known as *the method of averages*. Mayer's method of averages soon became popular. It used all observations and it was very simple to apply. However, due to the lack of an objective criterion of how to group the observations, the method was still a subjective one.

IV. The method of least absolute deviations

To determine the Earth's flattening as predicted by Newton's theory of gravitation (Principia[7], 1687), the French Academy of Sciences organized arc-measurement expeditions to Peru, Lapland and the Cape of Good Hope in the period 1735-1754. These expeditions aroused the interest in other countries and in 1750 Pope Benedict XIV commissioned, the Jesuit and professor of mathematics, Roger Joseph Boscovich (1711-1787) to perform a similar geodetic survey near Rome, the results of which were published in 1755. In a summary of this report, published in 1757[8], Boscovich formulated his new method, now known as the method of least absolute deviations, and applied it to the data of the French and Italian arc measurements.

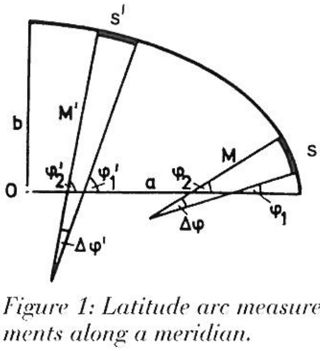

Figure 1: Latitude arc measurements along a meridian.

In order to understand the equations used by Boscovich, we first need to introduce some elements from ellipsoidal geodesy. For short meridian arcs, the arc length s (see figure 1) can be written as $s=M(\varphi)\Delta\varphi$, with $M(\varphi)$ the meridian radius of curvature, φ the geodetic latitude of the midpoint of the arc, and $\Delta\varphi$ the latitude difference of the two arc endpoints. The meridian curvature and its expansion are given as

$$M = \frac{a(1-e^2)}{(1-e^2\sin^2\varphi)^{3/2}} = a(1-e^2)\{1+\tfrac{3}{2}e^2\sin^2\varphi+\cdots\}$$

with $e^2=(a^2-b^2)/a^2$ the eccentricity, a and b the half lengths of the major and minor axis, and $a(1-e^2)$ the length of a degree at the equator. Using only the first two terms in the expansion, the length of a one-degree arc can be written as

$$s = x_1 + \sin^2\varphi\, x_2 \quad \text{with} \quad x_1 = a(1-e^2) \quad \text{and} \quad x_2 = \tfrac{3}{2}ae^2(1-e^2)$$

This is one equation in two unknowns, x_1 and x_2. The arc length s and geodetic latitude φ are determined from astronomical and geodetic measurements, while x_1 and x_2 contain the unknown dimensions of the ellipsoid of revolution. Although a minimum of two arcs is needed to solve for the two unknowns, it is preferable to use more than two arcs. As a result one obtains the following system of linear equations

$$\underbrace{\begin{bmatrix} s_1 \\ \vdots \\ s_m \end{bmatrix}}_{y} = \underbrace{\begin{bmatrix} 1 & \sin^2\varphi_1 \\ \vdots & \vdots \\ 1 & \sin^2\varphi_m \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x}$$

This is the system of equations which formed the start of Boscovich' analysis. Note that the A-matrix becomes near rank defect, when all arcs are close to the same latitude. For an accurate determination of the two unknowns, it is therefore preferable to choose arcs at widely differing latitudes. From the data available, Boscovich choose five such arcs (m=5). In his first analysis, Boscovich used the method of selected points. He choose the two arcs with the largest difference in latitude. Not satisfied with the result obtained (the 3 residuals were considered too large), he considers all possible pairs of measured arcs. This gave him 10 selected points to solve, but again he is not satisfied with the results obtained. After having struggled for some time on how to proceed, Boscovich finally formulates his new method of solution in 1757. He states that the parameters x_1 and x_2 should be chosen in such a way that the residuals sum up to zero and have minimum absolute sum. In formula form these two conditions read

$$\sum_{i=1}^{m}(s_i - x_1 - x_2\sin^2\varphi_i) = 0 \quad \text{and} \quad \sum_{i=1}^{m}\left|s_i - x_1 - x_2\sin^2\varphi_i\right| = \min$$

The first condition (although not essential) was motivated by the assumed symmetry in the error distribution, while the second was chosen to get the adjusted values 'as close as possible' to the observed ones. Boscovich gave a graphical algorithm for solving his problem, but no analytical one. The analytical proof of the solution was first given by Laplace in 1793. Using his principle, Boscovich first determined the two parameters $x1$ and $x2$, and from them the flattening as $f=x_2/3x_1$. Here he only used the first term of the expansion

$$f = (a-b)/a = \tfrac{1}{2}e^2 + \tfrac{1}{8}e^4 + \tfrac{1}{16}e^6 + \cdots$$

The value obtained by Boscovich equals f=1/246, which was smaller than the flattening predicted by Newton. Based on a rotational ellipsoid as an equilibrium figure for a homogeneous, fluid, rotating Earth, Newton obtained the value f=1/230. Boscovich' value is however larger than the value known today (International Association of Geodesy (1980): f=1/298.257). Boscovich' method was the first adjustment method that started from the principle of minimizing a function of the residuals. However, although the method is objective and uses all the observations, it did not reach the same level of popularity as Mayer's method. The method, being nonlinear, was difficult to apply, while the at that time available algorithm could only handle a system of equations with a maximum of two unknowns. In the second half of the 20th century the method gained in popularity due to its property of being resistent (robust) against outliers. Nowadays Boscovich adjustment method is usually referred to as an L_1-adjustment, since the L_1-norm of a vector is the sum of absolute values of its entries.

V. *The method of least-squares*

Adrien-Marie Legendre (1752-1833), a professor of mathematics at the École Militaire in Paris, was appointed by the French Academy of Sciences as member of various committees on astronomical and geodetic projects, among them the committee on the standardization of weights and measures. The committee proposed to define the meter as 10^{-7} times the length of the terrestrial meridian quadrant through Paris at mean sealevel. The arc-measurements took place in the period 1792-1795 and were analyzed, among others[9], by both Laplace and Legendre. Legendre's 1805[10] publication on the determination of the orbits of comets, contains a nine-page appendix in which for the first time *the method of least-squares*[11] is described, together with an application of the method to the arc measurements. Legendre used a different equation then Boscovich. Boscovich used the equation

$$s = [a(1-e^2) + \tfrac{3}{2}ae^2(1-e^2)\sin^2\varphi]\Delta\varphi$$

with $\Delta\varphi=1$. The arcs used by Legendre were not of one degree. Moreover, Legendre used a parametrization which differed from the one used by Boscovich. Since Legendre had the determination of the meter in mind, he parametrized his equation in the length of a one degree arc at 45 degree latitude. To obtain his equation, substitute

$\varphi = \tfrac{1}{2}(\varphi_1 + \varphi_2) =$ latitude of midpoint arc

$\mathrm{d} = a(1-e^2) + \tfrac{1}{2}\tfrac{3}{2}ae^2(1-e^2) =$ length of one degree arc at 45 degree latitude

$$\sin^2(\tfrac{1}{2}(\varphi_1+\varphi_2)) = \tfrac{1}{2}(1-\cos(\varphi_1+\varphi_2)), \quad \Delta\varphi = \varphi_{12}, \quad s = s_{12}$$

and use the approximations $\Delta\varphi \cong \sin(\Delta\varphi), f \cong \tfrac{1}{2}e^2, fd \cong \tfrac{1}{2}a(1-e^2)e^2$. As a result, we obtain

$$\varphi_{12} = s_{12}d^{-1} + \tfrac{3}{2}f\sin\varphi_{12}\cos(\varphi_1+\varphi_2)$$

This is one equation in two unknowns, d and f. Legendre understood that the observed latitude differences would correlate in case the arcs were connected. He therefore transformed the above equation of differences into an equivalent undifferenced form. This can be achieved by introducing an appropriate additional equation with an additional unknown. As a result, we obtain Legendre's linear system of equations as

$$\underbrace{\begin{bmatrix}\varphi_1\\ \vdots\\ \varphi_l\\ \vdots\\ \varphi_m\end{bmatrix}}_{y} = \underbrace{\begin{bmatrix}1 & s_{l1} & \tfrac{3}{2}\sin\varphi_{l1}\cos(\varphi_1+\varphi_l)\\ \vdots & \vdots & \vdots\\ 1 & 0 & 0\\ \vdots & \vdots & \vdots\\ 1 & s_{lm} & \tfrac{3}{2}\sin\varphi_{lm}\cos(\varphi_m+\varphi_l)\end{bmatrix}}_{A} \underbrace{\begin{bmatrix}\varphi_l\\ d^{-1}\\ f\end{bmatrix}}_{x}$$

Since Legendre had four connected arcs (m=5) at his disposal, he had to solve 5 equations in 3 unknowns. In order to solve his overdetermined linear system of equations, Legendre proposed to determine x such that the sum of the squares of the residuals is minimized. In vector-matrix form $(y-Ax)^T(y-Ax) = \min$ •
By setting the derivatives of this quadratic form equal to zero, he shows that the solution satisfies the consistent system of linear equations (nowadays referred to as 'the normal equations')

$A^T y = A^T A x$
Note that this result corresponds to the following choice of the L matrix
$L = A^T$
After solving his set of equations for the three unknowns φ_3, d and f, Legendre obtained for the flattening the value f=1/148, which he recognizes as being too large. He therefore recomputed his least-squares adjustment, but now with f constrained to the at that time adopted value for the Earth's flattening. As a result he obtained a value for d, which was now close to the value obtained earlier by Laplace and on which the actual definition of the meter was based. The reason for Legendre having to constrain f lies in the poor resolution of his data. The total arc length of his data covered only about 10 degrees.
Although Legendre did not give a clear motivation for his 'least-squares' criterion, he did realize its potential. His method used all the observations, had an objective criterion and most importantly, resulted - as opposed to Boscovich' method - in a solvable *linear* system of equations. The method met with almost immediate success. Within ten year after Legendre's publication, the method of least-squares became a standard tool in astronomy and geodesy in various European countries, and within twenty years, also the probabilistic foundations of the method were largely completed, the main contributors being Laplace and Gauss.

1 Based on a presentation given at the occasion of the official opening of the Geodetic-Astronomical Observatory Westerbork, 24 Sept. 1999.

2 Mathematical geodesy covers the development of theory and its implementation as is needed in order to process, analyse, integrate and validate the various geodetic data. It concerns itself with the calculus of observations (adjustment and estimation theory), with the validation of mathematical models (testing and reliability theory) and with the analysis of spatial and temporal phenomena (interpolation and prediction theory). Founders of the Dutch School of Mathematical Geodesy, internationally also known as the 'Delft School', are the professors J.M. Tienstra (1895-1951) and W. Baarda.

3 Stigler, S.M. (1986): *The History of Statistics*, Belknap, Harvard. Hald, A. (1998): *A History of Mathematical Statistics*, Wiley-Interscience.

4 Mayer, T. (1750): Abhandlung ueber die Umwalzung des Monds um seine Axe und die scheinbare Bewegung der Mondsflecten. *Kosmogr. Nachr. Samml. auf das Jahr 1748*, 1, 52-183.

5 Forbes, E.G. (1974): *The Birth of Scientific Navigation*. Maritime Monographs and Reports, No. 10, National Maritime Museum, Greenwich, London.

6 Sobel, D. (1995): *Longitude*. Walker, New York. Dutch translation by E. van Altena (1997): Dava Sobel: *Lengtegraad*. Ambo.

7 Newton, I. (1687): *Philosophiae Naturalis Principia Mathematica*.

8 Boscovich, R.J., Maire, C. (1770): *Voyage astronomique et géographique dans l'etat de l'église*. French translation of original 1755 publication. For Boscovich' contributions, see also Sheynin, O.B. (1973): *Arch. History Exact Sci.*, 9: 306-324.

9 In 1795 the French government invited other European governments to delegate scientists to Paris for completing and checking the computations for the standardization of weights and measures. The Dutch delegates were professor Jan Hendrik van Swinden and the navy officer Henricus Aeneae. The European scientists met in Paris in 1798 and reported on their findings in 1799. The report on the meter was given by van Swinden. The standard meter, a bar of platinum with rectangular section of 254 millimeters, was placed in the French State Archives, Mètre et Kilogramme des Archives. In 1983 the standard meter was defined as the length traveled by light in vacuum in 1/299.792.458 seconds.

10 Legendre, A.M. (1805): *Nouvelles méthodes pour la détermination des orbites des comètes*. (Appendix: *Sur la méthode des moindre carrés*).

11 When Carl Friedrich Gauss published his first probabilistic version of the method of least-squares in 1809, he claimed that he had been using the method ('our principle') already since 1795. This claim resulted in a priority dispute between Legendre and Gauss, for a discussion see e.g. Placket, R.L. (1972): The discovery of the method of least-squares. *Biometrica* 59:239-251.