On the GNSS integer ambiguity success rate

P.J.G. Teunissen
Mathematical Geodesy and Positioning
Faculty of Civil Engineering and Geosciences

Introduction

Global Navigation Satellite System (GNSS) ambiguity resolution is the process of effectively accounting for the integerness of the unknown cycle ambiguities of the double-difference (DD) carrier phase data. It applies to a great variety of GNSS models. This holds true not only for the current Global Positioning System (GPS), but also for the future modernized GPS and the European Galileo system. The GNSS models range from single-baseline models used for kinematic positioning to multi-baseline models used as a tool for monitoring and studying geodynamic phenomena. The models may have the relative receiver-satellite geometry included (referred to as geometry-based) or excluded (referred to as geometry-free). The geometry is included through the unit direction vectors in the design matrix. When the geometry is excluded, the baseline components are not involved as unknowns in the model, but instead the receiver-satellite ranges themselves. The models may also be discriminated as to whether the slave receiver(s) are in motion or not. When the receivers are in motion, one solves for one or more trajectories, because with the receiver-satellite geometry included, one will have new coordinate unknowns for each new epoch. One may also discriminate as to whether the differential atmospheric delays are included as unknowns or not. In case of sufficiently short baselines these delays are often neglected.

Despite the differences in application of the various GNSS models, their ambiguity-resolution problems are intrinsically the same. In all cases the aim is to incorporate the integerness of the ambiguities into the adjustment so as to improve the precision of the results. Once the integer ambiguities are known, the corresponding carrier phase measurements will act as if they are high precision pseudo-range measurements, thereby allowing the remaining parameters, such as baseline coordinates, to be estimated with a comparable high precision.

For the purpose of ambiguity resolution, GNSS data processing is usually carried out in three different steps (see Figure 1). In the first step no distinction is made between ambiguities and other parameters, like baseline coordinates and atmospheric delays. The parameter estimation problem is solved without taking into account the special integer nature of the ambiguities. The result so obtained is often referred to as the ‘float solution’.

In the second step, the float solution of the ambiguities is used to estimate the integer ambiguity values. Here one could choose from a wide variety of integer estimation methods. These methods range from simple rounding schemes to more advanced methods based on integer searches. One popular method is the LAMBDA method, developed at the Department of Mathematical Geodesy of the Delft University of Technology [1]. With this method, the ambiguities are estimated by means of integer least-squares using a very efficient search procedure (For details on the method consult `http://www.geo.tudelft.nl/mgp/`. Available on request are a FORTRAN and a Matlab implementation of the LAMBDA-method. Directly available for download is an extensive description of the method and its implementation in the report [2]. The Matlab-implementation comes with a separate guide and also includes a user-friendly demo-application which can be used for solving small problems interactively.)

Finally in the third step, the computed integer ambiguities are used to improve the first-step solution for the remaining parameters, like baseline-coordinates and/or atmospheric delays. These parameters are recomputed, but this time with the ambiguities constrained to the integer values as obtained from the second step. This final result is referred to as the ‘fixed’ solution and it generally inherits a much higher precision than the previously obtained ‘float solution’.

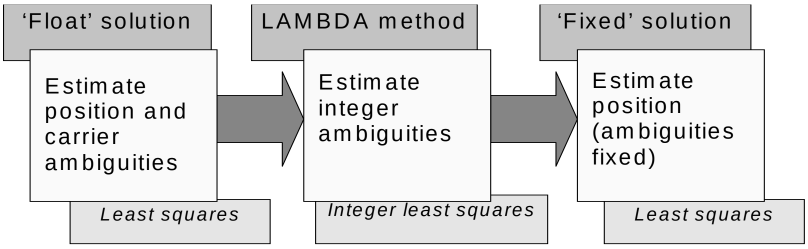

Figure 1: The three steps involved in GNSS data processing for precise relative positioning and the corresponding optimal estimation methods.

Estimated integer ambiguities are stochastic

When computing the 'fixed' baseline, the integer ambiguities are usually assumed to be known with certainty. But how sure can one be? After all, the integer ambiguities are determined from noisy data. Only in the hypothetical case of perfect observations without any noise or errors, would the float solution always yield the correct integer ambiguity values. In reality, however, this is not the case. Any uncertainty in the observations will propagate and manifest itself as uncertainty in the integer ambiguities.

Figure 2 shows a single-frequency example based on the geometry-free GNSS model. The figure illustrates empirically how uncertainty in the data (left) propagates into the ambiguity float estimate (middle) and finally into the integer ambiguity estimate (right). The correct integer for the ambiguity is 4 in this case, but as one can see from the graph at right, also other integer values are frequently obtained.

In order to capture the integer ambiguity uncertainty, one will have to treat the estimated integer ambiguities as stochastic (random) variates. This is not too different from standard adjustment practice. In standard adjustments, where all parameters are real-valued, one also propagates the observational uncertainty so as to obtain the uncertainty of the estimated parameters. This uncertainty is then captured by the probability distribution of these parameters. The real difference between a standard and an integer adjustment lies in the type of probability distribution. In the standard case the distribution will be continuous, whereas in the integer case it will be of discrete type, cf. Figure 2 at right. That is, the distribution of the estimated integer ambiguities will be a probability mass function.

Without any knowledge of the probability mass function of the integer ambiguities, one has no way of knowing how often to expect the computed ambiguity solution to coincide with the correct but unknown integers. Is this 9 out of 10 times, 99 out of 100, or a higher percentage? In the example shown in figure 2 it is less than 45%. This implies that when carrying out an experiment according to the assumption made in the example, one has about 55% chance of computing a wrong integer ambiguity.

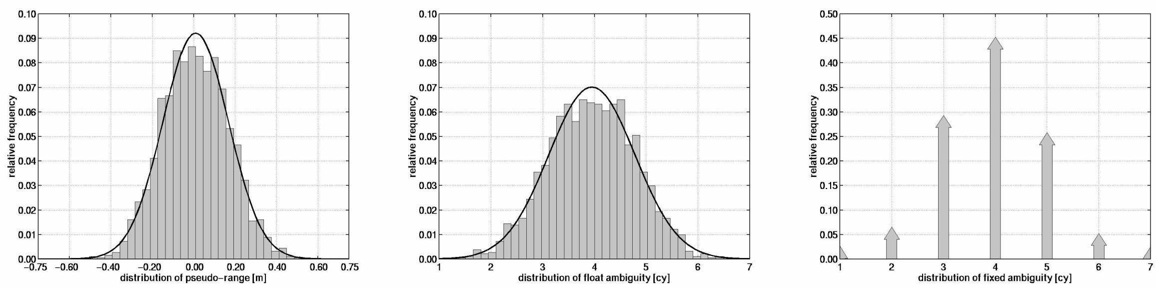

Figure 2: Using single frequency pseudo-range and carrier phase data, the phase ambiguity of the geometry-free GPS model is estimated in 1800 single epoch experiments at a 1 second interval. The histogram at left shows the residuals of the (double difference) pseudo-range measurements; the noise is at the decimeter level. The histogram in the middle concerns the float ambiguity. It is primarily the noise in the pseudo-range which is reflected in the noise of the float ambiguity, and as the L1-wavelength is about 2 decimeter, the corresponding uncertainty in the float ambiguity is at the cycle level. In both the graph at left and in the middle, the formal Gaussian probability distribution is also shown. Finally the integer ambiguity was computed for each experiment, and yields the histogram at right. In this case the integer ambiguity is estimated correctly (value 4) in only 43% of the experiments.

Ambiguity success-rate

If one wants to treat the computed integer ambiguities as deterministic variates, as is done in practice, one will have to ensure that their uncertainty is sufficiently small to be indeed neglected. This is the case when the frequency with which estimated integer ambiguities coincide with the correct but unknown values, is sufficiently large. This concept is formalized in a probabilistic measure, referred to as the *ambiguity success-rate*. The success-rate is a number between 0 and 1, or 0% and 100%, and it expresses the chance, or probability, that the integer ambiguities are correctly estimated.

The ambiguity success-rate depends on three contributing factors: the observation equations (functional model), the precision of the observables (the stochastic model), and the chosen method of integer estimation. Changes in any one of these will affect the success-rate. The first two contributing factors reflect the strength of the data model and they are given once the measurement set-up is known. As to the method of integer estimation, one has a variety of options available. However, since different methods of integer estimation will generally result in different success-rates, one might wish to use the method that maximizes the success-rate. It has recently been proven, see [3], that the integer least-squares estimator has the largest success-rate of all admissible integer estimators. The success-rate of the LAMBDA method is therefore larger than, or at least as large as any other integer ambiguity estimator.

Figure 3 shows a two-dimensional example of how the success-rate of the integer least-squares ambiguities is to be obtained. The figure shows the probability density function of the float ambiguities at left, and the corresponding discrete distribution of the integer least-squares ambiguities at right. The probability density function can be computed once the GNSS functional- and stochastic model are known. In case the GNSS data are assumed to be Gaussian distributed, the shape of the distribution is completely specified by the variance-covariance matrix of the float ambiguities. In the example the standard deviations of the two ambiguities are about 0.3 cycle. The corresponding success-rate follows then as the integral of the probability density function over the area of the convex polygon. This area is referred to as the *ambiguity pull-in region*. It contains all locations of the float ambiguities which get pulled to the correct integer solution. Different integer estimators will have different pull-in regions. The pull-in region of integer rounding, for instance, equals a square. The optimal pull-in region, the one of integer least-squares, is shown in the figure. If we denote the probability density function of the float ambiguities as $p_a(x)$ and the pull-in region of the correct integer ambiguity vector as R_a, the ambiguity success-rate can be written in formula form as

$$\text{Success} - \text{rate} = \int_{R_a} p_a(x)dx$$

There exist various ways of computing or approximating the optimal success-rate, two of which will be given. One way of obtaining the success-rate is by simulation. Using a random generator, a large number of real-valued ambiguity vectors are generated from the origin-centered probability distribution $p_a(x)$ of the 'float' solution. For each of these generated vectors, the corresponding integer least-squares solution is computed using the LAMBDA method. The percentage of integer solutions that coincide with the origin yields the success-rate. The number of generated samples must be large enough in order to obtain a close enough approximation to the success-rate.

A second option of inferring the success-rate is to compute a sharp lower bound of the probability of correct integer least-squares estimation. A sharp and *easy-to-compute* lower bound (*LB*) is given in [4] and reads:

$$LB = \prod_{i=1}^{n}\left[2\Phi\left(\frac{1}{2\sigma_{i|I}}\right)-1\right] \leq success - rate \quad \text{with} \quad \Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

It equals a product of *n* terms (the number of ambiguities). Φ is the standard normal cumulative probability distribution and $\sigma_{i|I}$ is the standard deviation of ambiguity *i*, conditioned on all previous ambiguities, indicated by *I*. The conditional standard deviations follow directly from the triangular decomposition of the float ambiguity variance-covariance matrix $Q_{\hat{a}}^{-1} = LDL^T$ as one over the square root of the elements of diagonal matrix *D*. This decomposition is already made in the computations for the LAMBDA method, and hence available at no extra cost.

For this lower bound to be sharp, it is essential that the variance-covariance matrix of the LAMBDA-transformed ambiguities is used for the computation of the conditional standard deviations, as they have an improved precision and decreased correlation over the original double difference ambiguities.

This approximation to the success-rate can be computed straightforwardly and if it is sufficiently large, say 0.99 or 0.999, it is guaranteed that the actual success-rate of the integer least-squares method is at least equally high and thus very close to 1.0. As it provides a lower bound, one can safely rely on this approximation.

It is clear that the ambiguity success-rate can be evaluated once the GNSS functional- and stochastic model are known. Hence, similar to the usage of Dilution Of Precision (DOP) measures, it can be computed without having the actual measurements available, thus prior to actual field operation. By means of the success-rate the user is given a rigorous way of assessing how often he or she can expect ambiguity resolution to be successful. Only when the success-rate is close enough to one, is one allowed to proceed as if the estimated integer ambiguities are non-stochastic.

The success-rate depends of course, as any other formal reliability measure, on the correctness of the assumptions which underly the model used. Misspecifications in the model may lead to unrealistic values for the success-rate. For instance, even with a high enough success-rate, fixing to the wrong integer ambiguities is still possible when one or more observations are erroneous. A success-rate close enough to one does therefore not release one from the obligation of performing statistical tests for model validation. It does however make it much more easier to perform such tests. The higher the success-rate, the sooner one is allowed to apply the classical theory of statistical hypothesis testing.

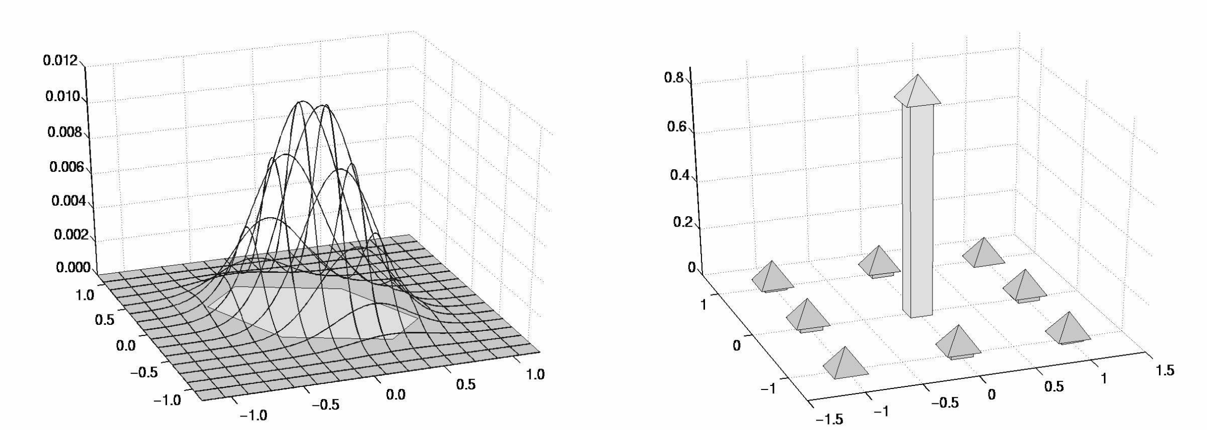

Figure 3: By taking the integral of the probability density function (at left) over the pull-in region for each integer vector, the probability is obtained that this vector will result as the integer least-squares solution. The probabilities are given at right for the integer vectors between -1 and +1. The integral over the area for the *correct* integer vector, in this case (0,0), gives the success-rate. It is about 0.85 in this example.

An example: dual versus triple frequency GPS

The success-rate can also serve as a tool in analysing the benefits of three frequency GPS or as a design-tool in choosing the three frequencies for the planned European ‘Galileo’, see [5]. As such an example, we will compare the performance of the current dual-frequency GPS with the future triple-frequency GPS, also referred to as ‘modernized GPS’. This comparison will be based on the so-called geometry-free model. This model is the simplest possible GNSS model that still allows the estimation of integer carrier phase ambiguities. In its most basic form the model consists of the double-differenced pseudo range and carrier phase observations of two receivers to two satellites, parametrized in terms of an unknown double-differenced satellite-receiver range, unknown ambiguities and an unknown ionospheric delay. The ionospheric delay is included so as to make the model applicable for long baselines.

We will first study the dual-frequency success rate in its dependence on a varying second frequency, whilst the first frequency is kept fixed to the GPS L1 frequency. This is shown in figure 4, at left. First note that the success rate fails to exceed the very small value of 0.025 within the frequency range shown. This stipulates the poor performance of instantaneous dual-frequency ambiguity resolution for long baselines. Hence, for these cases one can not expect dual-frequency ambiguity resolution to be successful. The figure also shows that the success rate reaches its minimum when the two frequencies coincide and that the success rate gets larger when the frequency separation gets larger. This contradicts the popular belief that ambiguity resolution would benefit from choosing the frequencies close together. It is of course still true that frequencies with litlle separation would allow one to construct a wide-lane with a corresponding very large wave length. However, as the figure shows this turns out to be counterproductive as far as the overall success rate is concerned. In fact, as the figure shows, the success rate will be identical to zero when the two frequencies coincide. This is understandable when one recognizes that a nonzero frequency separation is needed per se in order to be able to estimate the ionospheric delays. When the two frequencies coincide, the ionospheric delay becomes non-estimable and the variance-covariance matrix of the ambiguities becomes singular. As a consequence, the success rate reduces to zero.

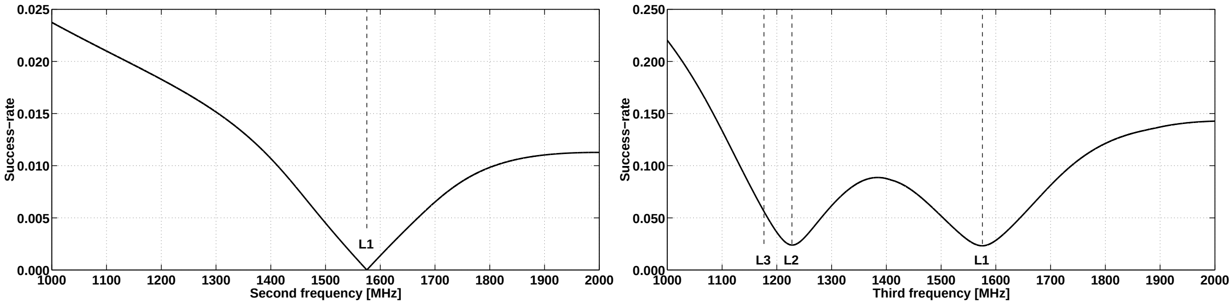

Figure 4. The long baseline, single epoch, dual-frequency ambiguity success rate as function of the second frequency at left and the corresponding triple-frequency ambiguity success rate as function of the third frequency at right. For the dual-frequency case, the first frequency was fixed at the GPS L1 value and for the triple-frequency case, the first two frequencies were fixed at the GPS L1 and L2 values. The dashed vertical lines indicate the current L1- and L2-frequency, as well as the chosen third GPS frequency.

We will now consider the triple-frequency case. Figure 4, at right, shows the long baseline, single epoch, triple-frequency, ambiguity success rate as function of a varying third frequency. The first two frequencies were fixed at the GPS L1 and L2 values. When compare to figure 4, at left, the figure shows that the addition of a third frequency indeed improves the success rate. The maximum value is about 10 times larger. The success rates however, are still too small for single epoch ambiguity resolution to be successful. This not only holds true for modernized GPS, for which the third frequency equals the L3 value. It would hold true for any triple-frequency system for which the third frequency lies in the frequency range shown. The conclusion reads therefore that, although one can significantly improve upon the third frequency choice of modernized GPS, the improvement will still not make successful instantaneous long baseline ambiguity resolution feasible.

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