

ADOP BASED UPPERBOUNDS FOR THE
BOOTSTRAPPED AND THE LEAST-SQUARES
AMBIGUITY SUCCESS RATES

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ABSTRACT: The success or failure of carrier phase ambiguity resolution can be predicted by means of the probability of correct integer estimation, also referred to as the ambiguity success-rate. In this contribution two easy-to-compute upperbounds of the ambiguity success-rate are described, one for the integer bootstrapped success-rate and one for the integer least-squares success-rate. The relationship between the two upperbounds is also shown.

Keywords GPS, ADOP, integer bootstrapping, integer least-squares, ambiguity success-rate upperbound.

1. INTRODUCTION

Carrier phase ambiguity resolution is the key to fast and high precision GPS positioning. Critical in the application of ambiguity resolution is the quality of the computed integer ambiguities. Unsuccessful ambiguity resolution, when passed unnoticed, will too often lead to unacceptable errors in the positioning results. The success or failure of carrier phase ambiguity resolution can be predicted by means of the probability of correct integer estimation, also referred to as the ambiguity success-rate. Upperbounds of the success-rate can be used to decide that ambiguity resolution has become unreliable. In this contribution we describe and discuss two easy-to-compute upperbounds, one for the bootstrapped success-rate and one for the integer least-squares success-rate. Both are driven by the easy-to-compute Ambiguity Dilution of Precision (ADOP).

This contribution is organized as follows. In section 2 we start with the class of admissible integer estimators and show that the integer bootstrapped estimator, \tilde{a}_B , and the integer least-squares estimator, \tilde{a}_{LS} , are both members of this class. Since the integer least-squares estimator is shown to have the highest success-rate of all admissible integer estimators, we have

$$P(\tilde{a}_B = a) \leq P(\tilde{a}_{LS} = a)$$

In section 3 we define the ADOP and describe some of its properties. The ADOP is then used to determine two different upperbounds of respectively $P(\hat{a}_B = a)$ and $P(\hat{a}_{LS} = a)$. The relation between the two upperbounds is also given. The upperbounds are attractive since they are so easy to compute and since they are invariant for the class of admissible ambiguity transformations.

2. INTEGER BOOTSTRAPPING AND INTEGER LEAST-SQUARES

2.1 Admissible integer estimators

There are many ways of computing an integer ambiguity vector \tilde{a} from its real-valued counterpart \hat{a} . To each such method belongs a mapping $S : R^n \mapsto Z^n$ from the n -dimensional space of real numbers to the n -dimensional space of integers. Due to the discrete nature of Z^n , the map S will not be one-to-one, but instead a many-to-one map. This implies that different real-valued ambiguity vectors will be mapped to the same integer vector. One can therefore assign a subset $S_z \subset R^n$ to each integer vector $z \in Z^n$:

$$S_z = \{x \in R^n \mid z = S(x)\}, \quad z \in Z^n \quad (1)$$

The subset S_z contains all real-valued ambiguity vectors that will be mapped by S to the same integer vector $z \in Z^n$. This subset is referred to as the *pull-in region* of z . It is the region in which all ambiguity 'float' solutions are pulled to the same 'fixed' ambiguity vector z . Using the pull-in regions, one can give an explicit expression for the corresponding integer ambiguity estimator. It reads

$$\tilde{a} = \sum_{z \in Z^n} z s_z(\hat{a}) \quad \text{with the indicator function } s_z(\hat{a}) = \begin{cases} 1 & \text{if } \hat{a} \in S_z \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

Since the pull-in regions define the integer estimator completely, one can define classes of integer estimators by imposing various conditions on the pull-in regions. In [1] we defined one such class, which we called the class of admissible integer estimators. These integer estimators are defined as follows.

Definition

The integer estimator $\tilde{a} = \sum_{z \in Z^n} z s_z(\hat{a})$ is said to be *admissible* if

- (i) $\bigcup_{z \in Z^n} S_z = R^n$
- (ii) $\text{int}S_{z_1} \cap \text{int}S_{z_2} = \emptyset, \quad \forall z_1, z_2 \in Z^n, z_1 \neq z_2$
- (iii) $S_z = z + S_0, \quad \forall z \in Z^n$

The first two conditions of the definition state that the pull-in regions of admissible integer estimators need to cover the whole space R^n without gaps and overlaps. The last condition states that the pull-in regions need to be translated copies of one another. In the following we will single out two particular integer estimators, namely the integer bootstrapped estimator and the integer least-squares estimator.

2.2 The bootstrapped pull-in region

The bootstrapped estimator follows from a sequential conditional least-squares adjustment and it is computed as follows. If n ambiguities are available, one starts with the first ambiguity \hat{a}_1 , and rounds its value to the nearest integer. Having obtained the integer value of this first ambiguity, the real-valued estimates of all remaining ambiguities are then corrected by virtue of their correlation with the first ambiguity. Then

the second, but now corrected, real-valued ambiguity estimate is rounded to its nearest integer. Having obtained the integer value of the second ambiguity, the real-valued estimates of all remaining $n - 2$ ambiguities are then again corrected, but now by virtue of their correlation with the second ambiguity. This process is continued until all ambiguities are considered. The components of the bootstrapped estimator \tilde{a}_B are given as

$$\begin{aligned}\tilde{a}_{B,1} &= [\hat{a}_1] \\ \tilde{a}_{B,2} &= [\hat{a}_{2|1}] = [\hat{a}_2 - \sigma_{\hat{a}_2\hat{a}_1}\sigma_{\hat{a}_1}^{-2}(\hat{a}_1 - \tilde{a}_{B,1})] \\ &\vdots \\ \tilde{a}_{B,n} &= [\hat{a}_{n|N}] = [\hat{a}_n - \sum_{i=1}^{n-1} \sigma_{\hat{a}_n\hat{a}_i|I}\sigma_{\hat{a}_i|I}^{-2}(\hat{a}_i - \tilde{a}_{B,i})]\end{aligned}\quad (3)$$

where the shorthand notation $\hat{a}_{i|I}$ stands for the i th least-squares ambiguity obtained through a conditioning on the previous $I = \{1, \dots, (i-1)\}$ sequentially rounded ambiguities, $[\cdot]$ denotes the operation of integer rounding, and $\sigma_{\hat{a}_i\hat{a}_j}$ and $\sigma_{\hat{a}_i}^2$ denote respectively the ambiguity covariance and ambiguity variance.

The bootstrapped estimator is admissible. The first two conditions of the definition are satisfied, since - apart from ties in rounding - any 'float' solution gets mapped to a unique integer ambiguity vector. Also the third condition of the definition applies. To see this, let \tilde{a}'_B be the bootstrapped estimator which corresponds with $\hat{a}' = \hat{a} - z$. It follows then from (3) that $\tilde{a}_B = \tilde{a}'_B + z$.

The real-valued sequential conditional least-squares solution can be obtained by means of the triangular decomposition of the ambiguity variance-covariance matrix. Let the triangular decomposition of the variance-covariance matrix be given as $Q_{\hat{a}} = LDL^T$, with L a unit lower triangular matrix and D a diagonal matrix. Then $(\hat{a} - z) = L(\hat{a}^c - z)$, where \hat{a}^c denotes the conditional least-squares solution obtained from a sequential conditioning on the entries of z . The variance-covariance matrix of \hat{a}^c is given by the diagonal matrix D . This shows, when a componentwise rounding is applied to \hat{a}^c , that z is the integer solution of the bootstrapped method. Thus \tilde{a}_B satisfies $[L^{-1}(\hat{a} - \tilde{a}_B)] = 0$. Hence, if c_i denotes the i th canonical unit vector having a 1 as its i th entry, the bootstrapped pull-in regions $S_{B,z}$ follow as

$$S_{B,z} = \cap_{i=1}^n \{x \in R^n \mid |c_i^T L^{-1}(x - z)| \leq \frac{1}{2}\}, \quad \forall z \in Z^n \quad (4)$$

2.3 The least-squares pull-in region

The integer least-squares estimator is defined as

$$\tilde{a}_{LS} = \arg \min_{z \in Z^n} \|\hat{a} - z\|_{Q_{\hat{a}}}^2 \quad (5)$$

where $\|\cdot\|_{Q_{\hat{a}}}^2 = (\cdot)^T Q_{\hat{a}}^{-1}(\cdot)$ and $Q_{\hat{a}}$ is the ambiguity variance-covariance matrix of the 'float' solution. This ambiguity estimator was first introduced [2]. Also this estimator is admissible. Apart from boundary ties, it produces a unique integer vector for any 'float' solution $\hat{a} \in R^n$. And since $\tilde{a}_{LS} = \arg \min_{z \in Z^n} \|\hat{a} - u - z\|_{Q_{\hat{a}}}^2 + u$ holds true for any integer vector u , also the third condition of the definition applies.

It follows from (5) that the 'float' solutions $\hat{a} \in R^n$ which are mapped to the same integer vector \tilde{a}_{LS} are those that lie closer to this integer vector than to any other integer vector $z \in Z^n$. Hence, the integer least-squares pull-in regions are given as

$$S_{LS,z} = \{x \in R^n \mid \|x - z\|_{Q_{\hat{a}}}^2 \leq \|x - u\|_{Q_{\hat{a}}}^2, \forall u \in Z^n\}, \quad \forall z \in Z^n \quad (6)$$

These pull-in regions can also be represented in a way that looks more like the representation of the bootstrapped pull-in regions (4). The integer least-squares pull-in regions $S_{LS,z}$ consist of intersecting half-spaces, each one of which is bounded by the plane orthogonal to $(c - z)$, $c \in Z^n$ and passing through the mid-point $\frac{1}{2}(z + c)$. Here, orthogonality is taken with respect to the metric as defined by the ambiguity variance-covariance matrix. Since \hat{a} lies in one of these half-spaces when the length of the orthogonal projection of $(\hat{a} - z)$ onto $(c - z)$ is less than or equal to half the distance between c and z , it follows that

$$S_{LS,z} = \bigcap_{c \in Z^n} \{x \in R^n \mid |c^T Q_{\hat{a}}^{-1}(x - z)| \leq \frac{1}{2} \|c\|_{Q_{\hat{a}}}^2\}, \forall z \in Z^n \quad (7)$$

Upon comparing (4) and (7), we note that the two pull-in regions become identical when the ambiguity variance-covariance matrix is diagonal. Hence, the two integer estimators \tilde{a}_B and \tilde{a}_{LS} are the same when no correlation exists between the ambiguities of the 'float' solution. The two estimators are also identical when all matrix entries of the triangular factor L are integer. This is the case when L is an admissible ambiguity transformation [3]. In that case we have $\tilde{a}_{LS} = \tilde{a}_B = L[L^{-1}\hat{a}]$.

2.4 Integer least-squares is optimal

So far, we introduced a class of admissible integer estimators and discussed two of its members. We thus now have a variety of reasonable integer estimators available. The question which comes up next is which of these estimators to choose? Does there exist an estimator which one can single out as being the 'best'? And how do we want to define the qualification 'best'? The approach that will be followed here is a probabilistic one. That is, we will use the probability distribution of the integer estimator for determining which estimator is optimal. Since the integer estimator \tilde{a} is by definition of the discrete type, its distribution will be a probability mass function (pmf). It will be denoted as $P(\tilde{a} = z)$, with $z \in Z^n$. In order to determine this distribution, one first needs the probability density function (pdf) of \hat{a} . The pdf of \hat{a} will be denoted as $p_a(x)$, with $x \in R^n$. The subscript is used to show that the pdf still depends on the unknown ambiguity vector $a \in Z^n$.

The pmf of \tilde{a} can now be obtained as follows. Since the integer estimator is defined as $\tilde{a} = z \Leftrightarrow \hat{a} \in S_z$ it follows that $P(\tilde{a} = z) = P(\hat{a} \in S_z)$. The pmf of \tilde{a} follows therefore as

$$P(\tilde{a} = z) = \int_{S_z} p_a(x) dx, \forall z \in Z^n \quad (8)$$

The probability that \tilde{a} coincides with z is therefore given by the integral of the pdf $p_a(x)$ over the pull-in region $S_z \subset R^n$. The pmf of \tilde{a} can be used to study various properties of the integer estimator. Of the pmf (8), the probability of correct integer estimation, $P(\tilde{a} = a)$, is particularly of interest for GPS. It describes the frequency with which one can expect to have a successful ambiguity resolution. This probability is also referred to as the *ambiguity success-rate*. In the case of GPS one usually requires a very high success-rate, the rationale being that if $P(\tilde{a} = a)$ is sufficiently close to 1, then \tilde{a} may be treated as being deterministic and consequently all carrier phase data will start to act as if they were very precise pseudo range data. The following theorem states that the integer least-squares estimator obtains the largest success-rate of all admissible integer estimators. This theorem is due to the author and was proven in [1].

Theorem (Optimality of integer least-squares)

Let the integer least-squares estimator be given as

$$\hat{a}_{LS} = \arg \min_{z \in \mathbb{Z}^n} \| \hat{a} - z \|_{Q_a}^2$$

and let the pdf of \hat{a} belong to the family of elliptically contoured distributions,

$$p_a(x) = \sqrt{\det(Q_a^{-1})} G(\|x - a\|_{Q_a}^2)$$

where $G: R \mapsto [0, \infty)$ is nonincreasing and Q_a is positive-definite. Then

$$P(\hat{a}_{LS} = a) \geq P(\tilde{a} = a) \quad (9)$$

for any admissible estimator \tilde{a} .

From this theorem it follows that \hat{a}_{LS} is better than the bootstrapped estimator \tilde{a}_B . Thus

$$P(\tilde{a}_B = a) \leq P(\hat{a}_{LS} = a) \quad (10)$$

A very useful application of this result is that it shows how one can *lower bound* the probability of correct integer least-squares estimation. This is particularly useful since the probability $P(\tilde{a}_B = a)$ can be computed exactly and rather easily in case the pdf $p_a(x)$ is normal. As shown in [4], it can be computed as

$$P(\tilde{a}_B = a) = \prod_{i=1}^n \left(2\Phi\left(\frac{1}{2\sigma_{\hat{a}_{i|I}}}\right) - 1 \right) \quad \text{with} \quad \Phi(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}v^2\right\} dv \quad (11)$$

Note that this probability is driven by the ambiguity conditional variances $\sigma_{\hat{a}_{i|I}}^2$. In the next section we will consider upperbounds of respectively $P(\tilde{a}_B = a)$ and $P(\hat{a}_{LS} = a)$, both of which are also driven by these conditional variances.

3. THE ADOP BASED UPPERBOUNDS

Various ways of evaluating or approximating the ambiguity success-rate are given in [5]. In this section we will give two upperbounds of the success-rate, both based on the ADOP (Ambiguity Dilution of Precision). The bootstrapped upperbound has been used in [4] and [6], and the least-squares upperbound in [7] and [8]. We also give a proof of the ADOP-based integer least-squares upperbound, since no such proof has yet been given.

3.1 The ADOP

The ADOP is defined as

$$\text{ADOP} = \sqrt{\det Q_a}^{\frac{1}{n}} \quad (\text{cycle}) \quad (12)$$

The ADOP was introduced in [9] and it has the following properties [10]. It is invariant for the class of admissible ambiguity transformations. Thus the same ADOP-value is obtained, irrespective of which satellite is chosen as reference in the DD definition of the ambiguities. Similarly, the same ADOP-value is also obtained when one uses, instead of the original variance matrix, the variance matrix of the transformed ambiguities, as produced by the LAMBDA method. For more information on the LAMBDA method and its applications, we refer to e.g. [11-18].

When the ambiguities are completely decorrelated, the ADOP equals the geometric mean of the standard deviations of the ambiguities. This follows from $\det(Q_{\hat{a}}) = \prod_{i=1}^n \sigma_{\hat{a}_i}^2 \det(R_{\hat{a}})$, where $R_{\hat{a}}$ is the ambiguity correlation matrix. Since the LAMBDA method produces ambiguities that are largely decorrelated, the ADOP approximates the average precision of the transformed ambiguities.

Different approaches can be used for computing the ADOP. First, one may use the variance matrix of the original DD ambiguities or the variance matrix of the LAMBDA-transformed ambiguities. Second, for computing the determinant, one may use eigenvalues, conditional variances or, if applicable, the analytical closed form expressions as given in [19].

When the eigenvalues $\lambda_{\hat{a}_i}$ of the ambiguity variance matrix are used, we have

$$\text{ADOP} = \prod_{i=1}^n \lambda_{\hat{a}_i}^{\frac{1}{2n}} \quad (13)$$

Instead of working with eigenvalues, a cheaper way would be to make use of the conditional variances. This approach is based on using a triangular decomposition or a Cholesky decomposition of the ambiguity variance matrix or its inverse. The entries of the diagonal matrix D in the LDL^T decomposition of the variance matrix are the sequential conditional variances of the ambiguities. Since the determinant of the diagonal matrix D equals the determinant of the variance matrix, the ADOP becomes

$$\text{ADOP} = \prod_{i=1}^n \sigma_{\hat{a}_{i|j}}^{\frac{1}{2n}} \quad (14)$$

The conditional variances $\sigma_{\hat{a}_{i|j}}^2$ are usually already available or otherwise very cheap to come by. They are available when the search for the integer least-squares ambiguities is based on a sequential conditional least-squares adjustment, as it is the case with the LAMBDA method. Alternatively, they can be obtained from the Cholesky decomposition (note: when solving least-squares problems, the Cholesky decomposition of the inverse of the variance matrix is usually already available).

3.2 The ADOP-based integer least-squares upperbound

The following result shows how the ADOP can be used to upper bound the success-rate of the integer least-squares estimator.

Corollary 1

The integer least-squares ambiguity success-rate is bounded from above as

$$P(\hat{a}_{LS} = a) \leq P\left(\chi_n^2 \leq \frac{c_n}{\text{ADOP}^2}\right) \quad (15)$$

in which χ_n^2 has a central Chi-square distribution with n degrees of freedom, $c_n = (\frac{n}{2}\Gamma(\frac{n}{2}))^{2/n}/\pi$ and Γ denotes the gamma-function.

Proof

In order to prove (15) we will make use of the following result [20, p. 265]. The region

$$\Omega = \{x \in R^n \mid F(x) \geq \lambda\} \subset R^n \text{ with } F: R^n \rightarrow R \quad (16)$$

where λ is chosen to satisfy the constraint $\int_{\Omega} dx = 1$, maximizes $\int_{\Omega} F(x) dx$. In order to apply this result, recall that the integer least-squares success-rate is given as $P(\hat{a}_{LS} =$

