A THEOREM ON MAXIMIZING THE PROBABILITY OF CORRECT INTEGER ESTIMATION

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ABSTRACT

High ambiguity success rates are required for GPS ambiguity resolution to be successful. It is therefore of importance to be able to identify the integer estimators which maximize these success rates. In this contribution we present a theorem which shows when the success rate is maximized. This theorem generalizes a result of (*Teunissen*, 1998), which states that, in case of elliptically contoured distributions, it is the integer least-squares estimator that provides the largest probability of correct integer estimation.

Keywords: admissible integer estimation, maximum success rate, GPS ambiguity resolution

1 Introduction

GPS models on which ambiguity resolution is based, can all be cast in the following conceptual frame of linear(ized) observation equations, see e.g. (*Hofmann-Wellenhof et al.*, 1997), (*Leick*, 1995), (*Parkinson and Spilker*, 1996), (*Strang and Borre*, 1997) and (*Teunissen and Kleusberg*, 1998):

$$y = Aa + Bb + e \tag{1}$$

where y is the given GPS data vector of order m, a and b are the unknown parameter vectors respectively of order n and o, and where e is the noise vector. The matrices A and B are the corresponding design matrices. They are assumed to be of full rank. The data vector y will usually consist of the 'observed minus computed' single- or dual- frequency double-difference (DD) phase and/or pseudorange (code) observations accumulated over all observation epochs. The entries of vector a are then the DD carrier phase ambiguities, expressed in units of cycles rather than range. They are known to be *integers*, $a \in Z^n$. The entries of the vector b will consist of the remaining unknown parameters, such as for instance baseline components (coordinates) and possibly atmospheric delay parameters (troposphere, ionosphere). They are known to be real- valued, $b \in R^o$.

The procedure which is usually followed for solving the GPS model (1), can be divided into three steps [for more details we refer to e.g. (*Teunissen*, 1993) or (*de Jonge and Tiberius*, 1996)]. In the *first* step one simply disregards the integer constraints $a \in Z^n$ on the ambiguities and performs a standard adjustment. As a result one obtains the (real-valued) estimates of a and b, together with their variance-covariance (vc-) matrix

$$\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix}, \begin{bmatrix} Q_{\hat{a}} & Q_{\hat{a}\hat{b}} \\ Q_{\hat{b}\hat{a}} & Q_{\hat{b}} \end{bmatrix} \tag{2}$$

This solution is referred to as the 'float' solution. In the *second* step the 'float' ambiguity estimate $\hat{a}$ is used to compute the corresponding integer ambiguity estimate $\check{a}$. This implies that a mapping $F : R^n \mapsto Z^n$ is introduced, from the n-dimensional space of reals to the n-dimensional space of integers, such that

$$\check{a} = F(\hat{a}) \tag{3}$$

Once the integer ambiguities are computed, they are used in the *third* step to finally correct the 'float' estimate of b. As a result one obtains the 'fixed' solution $\check{b} = \hat{b} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1}(\hat{a} - \check{a})$.

In this contribution we will study the choice of $F : R^n \mapsto Z^n$ from the viewpoint of maximizing the probability of correct integer estimation.

2 The admissible integer estimators

There are many ways of computing an integer ambiguity vector $\check{a}$ from its real-valued counterpart $\hat{a}$. To each such method belongs a mapping $F : R^n \mapsto Z^n$ from the n-dimensional space of real numbers to the n-dimensional space of integers. Due to the discrete nature of Z^n, the map F will not be one-to-one, but instead a many-to-one map. This implies that different real-valued ambiguity vectors will be mapped to the same integer vector. One can therefore assign a subset $S_z \subset R^n$ to each integer vector $z \in Z^n$:

$$S_z = \{x \in R^n \mid z = F(x)\}, \quad z \in Z^n \tag{4}$$

The subset S_z contains all real-valued ambiguity vectors that will be mapped by F to the same integer vector $z \in Z^n$. This subset is referred to as the *pull-in-region* of z (*Jonkman*, 1998), (*Teunissen*, 1998). It is the region in which all ambiguity 'float' solutions are pulled to the same 'fixed' ambiguity vector z. Using the pull-in-regions, one can give an explicit expression for the corresponding integer ambiguity estimator. It reads

$$\check{a} = \sum_{z \in Z^n} z s_z(\hat{a}) \text{ with the indicator function } s_z(\hat{a}) = \begin{cases} 1 & \text{if } \hat{a} \in S_z \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

Since the pull-in-regions define the integer estimator completely, one can define classes of integer estimators by imposing various conditions on the pull-in-regions. In (*ibid*) we defined one such class, which we called the class of admissible integer estimators. These integer estimators are defined as follows.

Definition
The integer estimator $\check{a} = \sum_{z \in Z^n} z s_z(\hat{a})$ is said to be *admissible* if

$$\begin{array}{ll} (i) & \bigcup_{z \in Z^n} S_z = R^n \\ (ii) & S_{z_1} \bigcap S_{z_2} = \{0\}, \;\; \forall z_1, z_2 \in Z^n, z_1 \neq z_2 \\ (iii) & S_z = z + S_0, \;\; \forall z \in Z^n \end{array} \tag{6}$$

This definition was motivated as follows. Each one of the above three conditions describe a property of which it seems reasonable that it is possessed by an arbitrary integer ambiguity estimator. The first condition states that the pull-in-regions should not leave any gaps and the second that they should not overlap. The absence of gaps is needed in order to be able to map any 'float' solution $\hat{a} \in R^n$ to Z^n, while the absence of overlaps is needed to guarantee that the 'float' solution is mapped to just one integer vector. Note that we allow the pull-in-regions to have common boundaries. This is permitted if we assume to have zero probability that $\hat{a}$ lies on one of the boundaries. This will be the case when the probability density function (pdf) of $\hat{a}$ is continuous.

The third and last condition follows from the requirement that $F(x+z) = F(x) + z, \forall x \in R^n, z \in Z^n$. Also this condition is a reasonable one to ask for. It states that when the 'float' solution is perturbed by $z \in Z^n$, the corresponding integer solution is perturbed by the same amount. This property allows one to apply the *integer remove-restore* technique: $F(\hat{a} - z) + z = F(\hat{a})$. It therefore allows one to work with the fractional parts of the entries of $\hat{a}$, instead of with its complete entries.

3 Maximizing the probability of correct integer estimation

In this section we come to the main result of this contribution. So far, we introduced a class of admissible integer estimators. We thus have now a variety of reasonable integer estimators available. The question which comes up next is which of these estimators to choose? The approach that we will follow is a probabilistic one. That is, we will use the probability distribution of the integer estimator for deciding which estimator to choose. Since the integer estimator $\check{a}$ is by definition of the discrete type, its distribution will be a probability mass function (pmf). It will be denoted as $P(\check{a} = z)$, with $z \in Z^n$. In order to determine this distribution, we first need the probability density function (pdf) of $\hat{a}$. The pdf of $\hat{a}$ will be denoted as $p_a(x)$, with $x \in R^n$. The subindex is used to show that the pdf still depends on the unknown parameter vector $a \in Z^n$.

The pmf of $\check{a}$ can now be obtained as follows. Since the integer estimator is defined as

$$\check{a} = z \;\; \Leftrightarrow \;\; \hat{a} \in S_z \tag{7}$$

it follows that $P(\check{a} = z) = P(\hat{a} \in S_z)$. The pmf of $\check{a}$ follows therefore as

$$P(\check{a} = z) = \int_{S_z} p_a(x) dx \;, \; \forall z \in Z^n \tag{8}$$

The probability that $\check{a}$ coincides with z is therefore given by the integral of the pdf $p_a(x)$ over the pull-in-region $S_z \subset R^n$.

Having the problem of GPS ambiguity resolution in mind, we focus our attention in this contribution to the chance of successful ambiguity resolution. That is, we consider the probability of correct integer estimation $P(\check{a} = a)$. Since unsuccessful ambiguity resolution, when passed unnoticed, will all too often lead to unacceptable errors in the positioning results, one requires high success rates and therefore a large value for $P(\check{a} = a)$. It is therefore not only of theoretical interest, but also of practical interest, to know which integer estimator maximizes the ambiguity success rate. The answer is given by the following theorem.

Theorem
Let the integer maximum likelihood estimator

$$\check{a}_{ML} = \arg\max_{z \in Z^n} p_z(\hat{a})$$

be admissible. Then

$$P(\check{a}_{ML} = a) \geq P(\check{a} = a) \tag{9}$$

for any admissible estimator $\check{a}$.

Proof
The pull-in-regions of $\check{a}_{ML}$ are given as $S_{ML,z} = \{x \in R^n \mid z = \arg\max_{u \in Z^n} p_u(x)\}$ and they are assumed to satisfy all three conditions of the definition (6). From $S_{ML,z} = \{x \in R^n \mid p_z(x) \geq p_u(x)\ ,\ \forall u \in Z^n\}$ and the condition that pull-in-regions do not overlap, it follows that

$$p_a(x) \geq \sum_{z \in Z^n} s_z(x) p_z(x)\ ,\ \forall x \in S_{ML,a} \tag{10}$$

with the indicator function $s_z(x) = 1$, if $x \in S_z$, and zero otherwise, and the pull-in-regions S_z of an arbitrary admissible integer estimator. When taking the integral of (10) over $S_{ML,a}$, we get

$$\int_{S_{ML,a}} p_a(x)dx \geq \sum_{z \in Z^n} \int_{S_{ML,a} \cap S_z} p_z(x)dx \tag{11}$$

We now apply the change of variable $y = x + a - z$ and obtain the replacements: $p_z(x) \to p_z(y - a + z) = p_a(y)$, $S_{ML,a} \to S_{ML,2a-z}$ and $S_z \to S_a$. Hence

$$\int_{S_{ML,a}} p_a(x)dx \geq \sum_{z \in Z^n} \int_{S_{ML,2a-z} \cap S_a} p_a(y)dy = \int_{S_a} p_a(y)dy \tag{12}$$

where the last equality is a consequence of $\cup_{z \in Z^n} S_{ML,2a-z} = R^n$. On the left hand side of (12) we recognize the probability of correct integer estimation of the maximum likelihood estimator and on the right hand side the probability of correct integer estimation of any arbitrary admissible integer estimator. This concludes the proof of the theorem.

Note that, for the theorem to be applicable, one first needs to check whether the integer maximum likelihood estimator is admissible or not. This implies checking the three admissibility conditions. First, the union of the pull-in-regions $S_{ML,z} = \{x \in R^n \mid z = \arg\max_{u \in Z^n} p_u(x)\}$

needs to equal R^n. This will be the case when the domain of the pdf's $p_z(x)$ is R^n itself. Any $x \in R^n$ will then be allocated to one (or more) of the pull-in-regions. Second, the interiors of any two pull-in-regions should be disjoint. Thus, although the integer maximum likelihood solution is allowed to be non-unique, the probability of a non-unique solution should be zero. Third, the pull-in-regions should all be translated copies of one another, $S_{ML,z} = S_{ML,0} + z$. Note that this condition is automatically fulfilled when the pdf's have the translational property: $p_z(x+u) = p_{z-u}(x), \forall u \in Z^n$.

In the next section we will give an important example of a class of distributions for which the above theorem applies.

4 Elliptically contoured distributions

The above theorem generalizes a result of (*Teunissen*, 1998). In that contribution it was shown that the *integer least-squares* estimator is admissible and that it maximizes the success rate when the distribution is elliptically contoured. The family of elliptically contoured distributions is defined as (*Chmielewsky*, 1981):

Definition
The random vector $\hat{a} \in R^n$ is said to have an *elliptically contoured* distribution if its pdf is of the form

$$p_a(x) = \sqrt{\det(Q_{\hat{a}}^{-1})}G(\| x - a \|_{Q_{\hat{a}}}^2) \tag{13}$$

where $G : R \mapsto [0, \infty)$ is decreasing and $Q_{\hat{a}}$ is positive-definite.

Several important distributions belong to this family. The multivariate normal distribution can be shown to be a member of this family by choosing $G(x) = (2\pi)^{-\frac{n}{2}} \exp -\frac{1}{2}x$, $x \in R$. Another member is the multivariate t-distribution.

From the definition it immediately follows that

$$\arg \max_{z \in Z^n} \sqrt{\det(Q_{\hat{a}}^{-1})}G(\| \hat{a} - z \|_{Q_{\hat{a}}}^2) = \arg \min_{z \in Z^n} \| \hat{a} - z \|_{Q_{\hat{a}}}^2 \tag{14}$$

Hence, in case of an elliptically contoured distribution, the integer maximum likelihood estimator becomes identical to the integer least-squares estimator. The corresponding pull-in-regions read therefore: $S_{LSQ,z} = \{x \in R^n \mid \| x - z \|_{Q_{\hat{a}}}^2 \leq \| x - u \|_{Q_{\hat{a}}}^2, \forall u \in Z^n\}$. It is not too difficult to show that these regions indeed fulfill all three admissibility conditions. One may therefore apply the previous theorem to obtain the author's original result on the optimality of the integer least-squares estimator:

Theorem (*Teunissen*)
Let the pdf of $\hat{a}$ be elliptically contoured and the integer least-squares estimator be given as

$$\check{a}_{LSQ} = \arg \min_{z \in Z^n} \| \hat{a} - z \|_{Q_{\hat{a}}}^2$$

Then

$$P(\check{a}_{LSQ} = a) \geq P(\check{a} = a) \tag{15}$$

for any admissible estimator $\check{a}$.

This theorem gives a probabilistic justification for using the integer least-squares estimator. It particularly applies to GPS ambiguity resolution, for which often the multivariate normal distribution is assumed to hold true. For GPS ambiguity resolution one is thus better off using the integer least-squares estimator than any other admissible integer estimator, such as, for instance, the 'rounding' estimator or 'bootstrapped' estimator.

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