THE EFFECT ON GNSS AMBIGUITY RESOLUTION OF USING AN APPROXIMATE WEIGHT MATRIX

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ABSTRACT

A theorem of Teunissen (1999) states that the integer least-squares estimator maximizes the probability of correct integer estimation within the class of admissible integer estimators. Applied to the problem of GNSS ambiguity resolution, this implies that the largest success rate is obtained when the integer least-squares principle is used for estimating the integer carrier phase ambiguities. No other integer ambiguity estimator will produce higher success rates. In this contribution it will be shown how the success rate is affected when a too optimistic or a too pessimistic precision description of the ambiguities is used in the ambiguity estimation process.

Keywords: GNSS, integer least-squares, ambiguity precision, ambiguity success rate

1 Introduction

GNSS models on which ambiguity resolution is based, can all be cast in the following conceptual frame of linear(ized) observation equations

$$y = Aa + Bb + e \tag{1}$$

where y is the given GNSS data vector of order m, a and b are the unknown parameter vectors respectively of order n and o, and where e is the noise vector. The matrices A and B are the corresponding design matrices. The data vector y will usually consist of the 'observed minus computed' single- or dual- frequency double-difference (DD) phase and/or pseudorange (code) observations accumulated over all observation epochs. The entries of vector a are then the DD carrier phase ambiguities, expressed in units of cycles rather than range. They are known to be *integers*, $a \in Z^n$. The entries of the vector b will consist of the remaining unknown parameters, such as for instance baseline components (coordinates) and possibly atmospheric delay parameters (troposphere, ionosphere). They are known to be real- valued, $b \in R^o$.

The structure of the above conceptual model applies to a great variety of GNSS models currently in use. They range from single-baseline models used for kinematic positioning to multi-baseline models used as a tool for studying geodynamic phenomena. An overview of these and other GNSS models, together with their application in surveying, navigation and geodesy, can be found in texbooks such as (*Hofmann-Wellenhof et al.*, 1997), (*Leick*, 1995), (*Parkinson and Spilker*, 1996), (*Strang and Borre*, 1997) and (*Teunissen and Kleusberg*, 1998).

The procedure which is usually followed for solving the GNSS model (1), can be divided into three steps [for more details we refer to e.g. (*Teunissen*, 1993) or (*de Jonge and Tiberius*, 1996)]. In the *first* step one simply disregards the integer constraints $a \in Z^n$ on the ambiguities and performs a standard least-squares adjustment. As a result one obtains the (real-valued) estimates of a and b, together with their variance-covariance (vc-) matrix

$$\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix}, \begin{bmatrix} Q_{\hat{a}} & Q_{\hat{a}\hat{b}} \\ Q_{\hat{b}\hat{a}} & Q_{\hat{b}} \end{bmatrix} \tag{2}$$

This solution is referred to as the 'float' solution. In the *second* step the 'float' ambiguity estimate $\hat{a}$ is used to compute the corresponding integer ambiguity estimate $\check{a}$. This implies that a mapping $F : R^n \mapsto Z^n$, from the n-dimensional space of reals to the n-dimensional space of integers, is introduced such that

$$\check{a} = F(\hat{a}) \tag{3}$$

Once the integer ambiguities are computed, they are used in the *third* step to finally correct the 'float' estimate of b. As a result one obtains the 'fixed' solution $\check{b} = \hat{b} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1}(\hat{a} - \check{a})$.

In this contribution the mapping F will be defined as

$$\check{a}_\Sigma = \arg \min_{z \in Z^n} \| \hat{a} - z \|_\Sigma^2 \tag{4}$$

with $\| \,.\, \|_\Sigma^2 = (.)^T \Sigma^{-1} (.)$ and Σ any positive-definite matrix of order n. To each choice of Σ will generally correspond a different integer ambiguity estimator $\check{a}_\Sigma$. It is the purpose of this contribution to prove which choice of Σ will lead to the largest ambiguity success rate. In the next section we will present our main result in the form of a theorem. This theorem will be proven in section 3.

2 The result

It is well known from standard least-squares theory, that the linear unbiased (real-valued) least-squares estimator may loose its optimality property of minimum variance, when a positive-definite matrix other than the vc-matrix is used for the weighting of the data. One may expect that a similar situation will hold true for the integer least-squares estimator. However, instead of considering the variance of the integer estimator as criterion, we will use the probability of correct integer estimation - the success rate - as our criterion. The reason for this is twofold. First, in the practical context of GNSS ambiguity resolution, one is first and foremost interested in the success rate performance of the ambiguity estimation process. Second, also the information on the vc-matrix of the *integer* ambiguity estimator is not sufficient for making probabilistic statements, even if the real-valued ambiguity estimator is normally distributed.

Our main result is summarized in the following theorem.

Theorem

Let the real-valued 'float' ambiguity vector be normally distributed as $\hat{a} \sim N(a, Q_{\hat{a}})$ with $a \in Z^n$ and the corresponding integer ambiguity vector be defined as $\check{a}_\Sigma = \arg\min_{z \in Z^n} \| \hat{a} - z \|_\Sigma^2$. Then the probability of correct integer estimation will be at its maximum when $\Sigma = Q_{\hat{a}}$, or

$$P(\check{a}_{Q_{\hat{a}}} = a) \geq P(\check{a}_\Sigma = a) \tag{5}$$

for any any positive-definite matrix Σ of order n.

As the theorem shows, a proper choice of the data weight matrix is also of importance for ambiguity resolution. The choice of weights is optimal when the weight matrix equals the inverse of the ambiguity vc-matrix. A too optimistic precision description or a too pessimistic precision description, will both result in a less than optimal ambiguity success rate. In case of GPS, the observation equations (the functional model) are sufficiently known and well documented. However, the same can not be said of the vc-matrix of the GPS data. In the many GPS-textbooks available, one will usually find only a few comments, if any, on this vc-matrix. Only a few studies have been reported in the literature. Examples are (*Euler and Goad*, 1991), (*Jin and de Jong*, 1996), (*Gerdan*, 1995) and (*Gianniou*, 1996), who studied the elevation dependence of the observation variances, and (*Jonkman*, 1997) and (*Tiberius*, 1998), who considered time-correlation and cross-correlation of the pseudo ranges and carrier phases as well.

A systematic study of the stochastic model is of course far from trivial. Not only do the noise characteristics depend on the mechanization of the measurement process, and therefore on the make and type of the receiver used, but the residual terms which are not captured by the observation equations, such as environmental effects, will also have their influence. Despite these difficulties though, we believe that the time has come to put more research effort into the stochastic model. Examples that show that improved stochastic modelling indeed pays off in terms of an increased ambiguity success rate, can be found in (*Jonkman*, 1997) and (*Teunissen et al.*, 1998).

Quite another aspect made clear by the above theorem, is the relation between 'integer rounding' and 'integer least-squares'. One of the simplest choices for Σ would be a diagonal matrix. In that case $\| \hat{a} - z \|_\Sigma^2$ reduces to a sum of squares and $\check{a}_\Sigma$ becomes the integer estimator that follows from a rounding to the nearest integer of the entries of $\hat{a}$. Thus $\check{a}_\Sigma = [\hat{a}]$, where $'[.]'$ denotes the operation of componentwise rounding, and

$$P(\check{a}_{Q_{\hat{a}}} = a) \geq P([\hat{a}] = a) \tag{6}$$

We can generalize this result to a whole class of integer estimators based on rounding, when the choice $\Sigma = (Z^T D Z)^{-1}$ is made, where D is a diagonal matrix with positive entries and Z is an admissible ambiguity transformation. Ambiguity transformations are said to be admissible when all the entries of both Z and its inverse are integer (*Teunissen*, 1995). For this particular choice of Σ, we have $\| \hat{a} - z \|_\Sigma^2 = (Z\hat{a} - u)^T D (Z\hat{a} - u)$, with $u = Zz \in Z^n$. Hence, when parametrized in u, $\| \hat{a} - z \|_\Sigma^2$ again reduces to a sum of squares. Thus $\check{u} = [Z\hat{a}]$ and $\check{a}_\Sigma = Z^{-1}[Z\hat{a}]$. In this case the integer estimator is computed by first transforming the 'float' solution, then applying the componentwise rounding scheme, followed by the back-transformation. For the probability of correct integer estimation, we thus have

$$P(\check{a}_{Q_{\hat{a}}} = a) \geq P(Z^{-1}[Z\hat{a}] = a) \tag{7}$$

for any admissible ambiguity transformation Z.

Note that (6) is a special case of (7). The choice $Z = I_n$, however, is usually not the best one. That is, one can often improve upon the ambiguity success rate by choosing an appropriate transformation matrix Z. This is particularly true in case of GPS, when the DD ambiguities are used. Since the equality in (7) will hold true in case the vc-matrix of $Z\hat{a}$ is diagonal, an ambiguity transformation should be used that results in a close to diagonal form as possible. This is achieved when using the decorrelation process of the LAMBDA method. Hence, when one decides to use the integer estimator based on rounding, one should at least decorrelate the ambiguities first, before applying the integer rounding scheme. In this way one will obtain a success rate which is higher than the one obtained without using the decorrelation process.

3 The proof

In order to prove the above theorem we will make use of a result of Teunissen (1999). This result gives a probabilistic justification of the integer least-squares estimator for a wide class of integer estimators. When the pdf of $\hat{a}$ is given as the elliptically contoured distribution $p_a(x) = \sqrt{\det(Q_{\hat{a}}^{-1})}G(\| x - a \|_{Q_{\hat{a}}}^2)$ where $G : R \mapsto [0, \infty)$ is decreasing and $Q_{\hat{a}}$ is positive-definite, the result states that

$$P(\check{a}_{Q_{\hat{a}}} = a) \geq P(\check{a} = a) \tag{8}$$

for *any* admissible estimator $\check{a}$. The class of admissible ambiguity estimators is defined as follows.

Definition

Let $S_z \subset R^n$, $z \in Z^n$ be the pull-in regions of the integer estimator, i.e. $\hat{a} \in S_z \Leftrightarrow \check{a} = z$. The integer estimator $\check{a}$ is then said to be *admissible* if

$$\begin{array}{ll} (i) & \bigcup_{z \in Z^n} S_z = R^n \\ (ii) & S_{z_1} \bigcap S_{z_2} = \emptyset, \;\; \forall z_1, z_2 \in Z^n, z_1 \neq z_2 \\ (iii) & S_z = z + S_0, \;\; \forall z \in Z^n \end{array}$$

This definition is motivated as follows. Each one of the above three conditions describe a property of which it seems reasonable that it is possessed by an arbitrary integer ambiguity estimator. The first condition states that the pull-in-regions should not leave any gaps and the second that they should not overlap. The absence of gaps is needed in order to be able to map any 'float' solution $\hat{a} \in R^n$ to Z^n, while the absence of overlaps is needed to guarantee that the 'float' solution is mapped to just one integer vector. To be more precise, the second condition applies to the interior of the pull-in regions. Hence, we allow the pull-in-regions to have common boundaries. This is permitted if we assume to have zero probability that $\hat{a}$ lies on one of the boundaries. This will be the case when the probability density function (pdf) of $\hat{a}$ is continuous.

The third and last condition states that when the 'float' solution is perturbed by $z \in Z^n$, the corresponding integer solution is perturbed by the same amount. It therefore allows one to work with the fractional parts of the entries of $\hat{a}$, instead of with its complete entries.

We are now in a position to prove the theorem of the previous section. First note that the normal distribution is a member of the family of elliptically contoured distributions. Hence, inequality (8) applies to the case of the theorem as well. We therefore only need to show that $\check{a}_\Sigma$ is an admissible estimator for any positive-definite Σ. Once this has been established, the theorem follows from applying (8). The admissibility can be shown as follows. The first two conditions of the definition are satisfied, since $\check{a}_\Sigma$ is a unique integer vector - apart from boundary ties - for any 'float' solution $\hat{a} \in R^n$. And since $\check{a}_\Sigma = \arg\min_{z \in Z^n} \parallel \hat{a} - u - z \parallel_\Sigma^2 + u$ holds true for any integer $u \in Z^n$, also the third condition is satisfied.

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