AMBIGUITY DILUTION OF PRECISION:
AN ADDITIONAL TOOL FOR GPS QUALITY CONTROL

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Abstract
Integer carrier phase ambiguity resolution is often a prerequisite for high precision GPS positioning. It applies to a great variety of GPS models, including those which are used in hydrographic applications and marine positioning. Since the quality of kinematic GPS positioning depends critically on whether the correct integer ambiguities are used or not, it is of importance to have easy-to-compute diagnostics available that measure the expected success rate of ambiguity resolution. In this contribution we will introduce and analyse such an ambiguity dilution of precision (*ADOP*) measure. In contrast to the traditional way in which *DOP*-measures are introduced, our *ADOP* is defined such that it is invariant for the class of admissible ambiguity transformations. It does not depend on the arbitrary choice of reference satellite when constructing the double differenced ambiguities. Since the GPS ambiguities are known to be highly correlated, the *ADOP* is constructed such that it not only captures the precision but also the correlation characteristics of the ambiguities. We will present the *ADOP*s for a variety of GPS models and show their behaviour by graphical means. These models include single-baselines as well as kinematic networks such as those used for attitude determination and seismic streamer positioning. It is also shown how the *ADOP* can be used to bound the success rate of ambiguity resolution.

1. Introduction

GPS ambiguity resolution is the process of resolving the unknown cycle ambiguities of the double-difference carrier phase data as integers. It is the key to high precision GPS relative positioning. Once the integer ambiguities are resolved, the carrier phase measurements will start to act if they were very precise pseudorange (code) measurements. As a consequence the remaining parameters in the model, such as the baseline components, can be estimated with a comparable high precision.

Ambiguity resolution applies to a great variety of GPS models currently in use. These models may range from single-baseline models used for kinematic positioning to multi-baseline models used for studying geodynamic phenomena. An overview of these and other GPS models, together with their applications in surveying, navigation and geodesy, can be found in textbooks such as (Hofmann-Wellenhof et al., 1997), (Kleusberg and Teunissen, 1996), (Leick, 1995), (Parkinson and Spilker, 1996) and (Strang and Borre, 1997). Also in hydrography and marine geodesy, the use of high precision GPS positioning, based on ambiguity resolution, has gained momentum. This not only holds true for the more traditional surveying tasks, but also for applications such as attitude determination and the positioning of seismic streamer networks, see e.g. (Lachapelle et al., 1994), (Zinn and Rapatz, 1995), (Cross, 1994).

Surveyors and hydrographers alike have always been aware of the importance of quality control (see e.g. the UKOOA recommendations). They know that a mere adjustment of redundant data is not enough. Proper testing procedures, enabling one to check for errors in the data and/or errors in the models, need to be included as well, (Baarda, 1968), (Teunissen, 1985). As a consequence the quality of the survey results can be described in terms of precision *and* reliability. These standard procedures of adjustment, testing and quality control are however not directly applicable to the problem of GPS ambiguity resolution. This is due to the *integer* nature of the carrier phase ambiguities. Only in recent years a rigorous theory has emerged for the estimation and validation of these integer GPS ambiguities (Teunissen, 1993). In this contribution we will focus on one aspect of this theory, namely the success rate of integer ambiguity estimation. For more details on the theory and its application, we refer to the list of the references.

2. The quality of ambiguity resolution

Ambiguity resolution is that part of the GPS data adjustment in which the ambiguities are constrained to integers. The idea is that the precision of the GPS baseline improves when use is made of these integer constraints. The procedure for incorporating these constraints can be divided into three steps. In the first step one simply performs a standard adjustment. Hence, in this first step the integer constraints are still disregarded. As a result one obtains the real-valued least-squares solution for both the ambiguities and the baseline(s). This solution is often referred to as the 'float' solution. In the second step the most likely integer values of the ambiguities are determined. They are determined from the real-valued least-squares ambiguities of the first step. Finally in the third step, these integer values of the ambiguities are used to adjust the 'float' baseline solution of the first step. As a result one obtains the so-called 'fixed' baseline solution. For more details on the computational intricacies of this procedure we refer to (Teunissen, 1993), (de Jonge and Tiberius, 1996).

It is of course not enough to perform the above computations and be done with it. One can always compute an integer ambiguity solution, whether it is of good quality or not. One therefore still needs to consider the quality of the solution so computed. In case of ambiguity resolution this is particularly critical. Unsuccessful ambiguity resolution, when passed unnoticed, will often lead to unacceptable errors in the positioning results. In the current practice of GPS positioning there are unfortunately no rigorous measures in place to diagnose the quality of ambiguity resolution. Although use is made of important precision- and reliability measures, such as *PDOP*s (Position Dilution of Precision) and *MDB*s (Minimal Detectable Biases), no measure is yet available that is particularly focussed on ambiguity resolution. Hence, the user has no way of knowing how often he can expect the computed ambiguity solution to coincide with the true, but unknown integer solution. Is this nine out of ten times, ninety-nine out of a hundred, or a higher percentage? It will surely never equal one hundred percent. After all, the integer ambiguities are computed from the data, and since the data are subject to uncertainty, so are the computed integer ambiguities. Although a success rate of a hundred percent is impossible, the user will clearly not accept a much smaller percentage. A succes rate of ninety percent may seem high, but it still means that only nine out of ten position fixes are based on correct integer ambiguities. This will not be acceptable in applications where one aims at a high productivity, such as in case of instantaneous (on-the-fly) GPS positioning.

In order to obtain a proper measure for the success rate of ambiguity resolution, one needs the probability distribution of the integer ambiguities (Teunissen, 1997a). Of this probability mass function, the probability of correct integer estimation is particularly of importance. This probability will be denoted as $P(\breve{a} = a)$. It describes how often the computed vector of integer ambiguities, $\breve{a}$, will equal the unknown, but true vector of ambiguities, a. This probability depends on three contributing factors: the observation equations (the functional model), the precision of the observables (the stochastic model) and the chosen method of integer ambiguity estimation. Changes in any one of these will affect the success rate.

3. Integer bootstrapping

As to the method of integer ambiguity estimation, one has a variety of options available (Teunissen, 1998a). Here we will restrict ourselves to one of the simpler methods, the method of integer bootstrapping. The integer bootstrapped ambiguity vector follows from applying a *sequential* rounding scheme to the real-valued least-squares ambiguities. It goes as follows. If *n* ambiguities are available, one starts with the first ambiguity and rounds its value to the nearest integer. Having obtained the integer value of this first ambiguity, the real-valued estimates of all remaining ambiguities are then corrected by virtue of their correlation with the first ambiguity. Then the second, but now corrected, real-valued ambiguity estimate is rounded to its nearest integer. Having obtained the integer value of the second ambiguity, the real-valued estimates of all remaining *n-2* ambiguities are then again corrected, but now by virtue of their correlation with the second ambiguity. This process is continued until all ambiguities are taken care of. In essence this ‘bootstrapping’ technique boils down to the use of a sequential conditional least-squares adjustment, with a conditioning on the integer ambiguity values obtained in the previous steps.

It can be shown that the success rate of this bootstrapped integer estimation method is given as

$$P(\breve{a}=a)=\prod_{i=1}^{n}(2\Phi(\frac{1}{2\sigma_{\hat{a}_{i|I}}})-1)$$

where $\Phi(x)$ denotes the integral of the standardized normal probability density function from minus infinity to *x*. This probability equals the product of *n* terms, where *n* equals the number of carrier phase ambiguities. In each term the Φ-function needs to be evaluated using one of the sequential conditional standard deviations

$$\sigma_{\hat{a}_{i|I}} \quad \text{for} \quad i=1,\cdots,n$$

This is the standard deviation of the *i*th ambiguity, conditioned on the assumption that all previous *I={1,..., (i-1)}* ambiguities are known. The above success rate was introduced in (Teunissen, 1997a) and has been used in (Teunissen et al., 1998a,b) to analyse the reliability of ambiguity resolution for different GPS models.

4. Bootstrapping and ambiguity decorrelation

Although the above given expression gives the exact value of the bootstrapped success rate, it should be noted that the method of integer bootstrapping is not invariant for a reordering or a permutation of the ambiguities. The result of integer bootstrapping may change when the ordering of the ambiguities is changed. Thus also the corresponding success rate may change when the ordering of the ambiguities is changed. This lack of invariance also applies when one changes the definition of the ambiguities. There are different ways of defining DD ambiguities. When defining DD ambiguities, one has to specify which satellite is taken as the reference satellite. Since each satellite can be chosen as reference, one already has as many definitions of the DD ambiguities. Since each such set of DD ambiguities will have a different variance-covariance matrix, also the sequential conditional variances and the bootstrapped success rate will differ for the different sets. This implies that one set of DD ambiguities will have a higher success rate than another set. Since the bootstrapped success rate gets larger when the sequential conditional variances get smaller, one should aim at using an ambiguity parametrization which gives the smallest possible sequential conditional variances. A method which has this aim in mind is the LAMBDA (Least-squares AMBiguity Decorrelation Adjustment) method. By means of the ambiguity decorrelation process of this method, the original DD ambiguities are transformed to new ambiguities which all have much smaller conditional variances. The method was introduced in (Teunissen, 1993) and has since then been used for various applications. A few examples are (Tiberius and de Jonge, 1995), (Han, 1995), (de Jonge and Tiberius, 1996), (Boon and Ambrosius, 1997), (Boon et al., 1997), (Jonkman, 1998). The conclusion reads therefore that one should first apply the LAMBDA method before commencing with the integer bootstrapping. Thus also the computation of the above given success rate should be based on the sequential conditional variances of the transformed ambiguities obtained by means of the LAMBDA method.

5. Ambiguity dilution of precision

Although the above given procedure for computing the bootstrapped success rate is the one that should be preferred, it would still be helpful if a simpler diagnostic measure could be found. Such a measure is the Ambiguity Dilution of Precision (*ADOP*). It was introduced in (Teunissen, 1997b) and it has been applied to different GPS models in (Teunissen and Odijk, 1997). The *ADOP* is based on the determinant of the variance-covariance matrix of the least-squares ambiguities and it is defined as

$$ADOP = \sqrt{\det Q_{\hat{a}}}^{\frac{1}{n}} \quad (cycle)$$

where $Q_{\hat{a}}$ denotes the variance-covariance matrix of the ambiguities and n its order. The *ADOP* is a scalar measure and it is expressed in the 'unit' of cycles. The *ADOP* has a number of interesting properties, three of which will be mentioned here. The *ADOP* equals the geometric average of the sequential conditional standard deviations of the least-squares ambiguities. This follows from the fact that the determinant of the variance-covariance matrix of the ambiguities equals the product of the n sequential conditional variances. The *ADOP* is thus a simple measure for the average precision of the ambiguities. It can be used as a design parameter. No actual measurements are needed to compute the *ADOP*. Only the variance-covariance matrix of the ambiguities is needed. This variance matrix can be computed from the design matrix and the variance matrix of the GPS observables.

An important property of the *ADOP* is its invariance against the choice of ambiguity parametrization. Since all admissible ambiguity transformations can be shown to have a determinant which equals one, the *ADOP* does not change when one changes the definition of the ambiguities. It therefore measures the intrinsic precision of the ambiguities.

The *ADOP* can also be used to provide an upperbound for the success rate of integer bootstrapping. This upperbound reads as

$$P(\breve{a} = a) \leq [2\Phi(\frac{1}{2ADOP}) - 1]^{n}$$

This relation shows how the *ADOP* can be used to infer whether the success rate of integer bootstrapping is not too small. As mentioned earlier the left-hand side of the inequality depends on the chosen ambiguity parametrization. The right-hand side, however, is invariant for this choice. Hence the above upperbound covers the success rates for all possible choices of ambiguity parametrization. This means that when using the *ADOP* in the above upperbound, one does not have to worry about the definition that has been used in the formation of the ambiguities.

6. Some examples

We will now show how the *ADOP* can be used to study the success rate of ambiguity resolution for various measurement set-ups. One can choose to use the *ADOP*s directly, or alternatively choose to use them for the computation of the upperbounds of the ambiguity success rates. Both approaches will be shown. To get a feeling of how the two approaches are related numerically, consider the following. Let us assume that one aims at an ambiguity success rate not smaller than 99%. To reach such a success rate for a single ambiguity requires a standard deviation of about 0.2 cycle. Hence, if *n*=1 and *ADOP*=0.2, then $P(\breve{a}=a)=99\%$. When we keep the *ADOP* fixed, this probability will get smaller when *n* gets larger. Thus for larger *n*, smaller *ADOP*-values are needed to get the same probability. Since $P(\breve{a}=a)=99.9\%$ in case *ADOP*=0.15 and *n*=1, the corresponding upperbound follows as $(99.9\%)^n$. This upperbound equals 99% in case *n*=10.

Example 1: short baseline geometry-free model

A GPS model is referred to as a 'short baseline' model when the ionospheric delays are assumed absent. The geometry-free GPS model is the simplest model one can think of for ambiguity resolution. In this model the DD observation equations are parametrized in terms of the DD receiver-satellite ranges instead of in the baseline components. Due to this parametrization, the information content of the relative receiver-satellite geometry is not used. Hence the term 'geometry-free'. The geometry-free model can of course not be used directly for positioning purposes. It can be used however for determining the integer values of the DD carrier phase ambiguities, see e.g. (Hatch, 1982), (Dedes and Goad, 1994), (Teunissen, 1996), (Teunissen and Odijk, 1997), (Jonkman, 1998). Once the integer ambiguities have been determined, the carrier phases will act as if they are very precise pseudoranges. Precise positioning is then possible by means of these 'resolved' carrier phases (de Jong, 1998).

Figure 1 shows the *ADOP*s for the short baseline geometry-free model as function of the number of epoch-samples used. The *ADOP*s are shown for the single frequency (L1) and for the dual-freqeuncy (L1+L2) case. They are also shown for a different number of satellites tracked. For the geometry-free model a minimum of 2 satellites is needed. The standard deviation of the undifferenced phase observation was set at 3mm and the undifferenced standard deviation of the pseudorange (code) observation was set at 30cm. Unless otherwise stated, these same values are also used in the remaining part of this contribution.

The figure shows that the improvement in the *ADOP*s, when increasing the number of satellites, is marginal. This is a consequence of not using the relative receiver-satellite geometry. The figure also shows that there is a significant difference between the single- and dual frequency case. The *ADOP*s of the dual-frequency case are significantly smaller than those of the single-frequency case (note the logarithmic scale). The figure shows that one can not expect to have a successful ambiguity resolution in the single-frequency case, when one aims at an *ADOP* of 0.15 cycle.

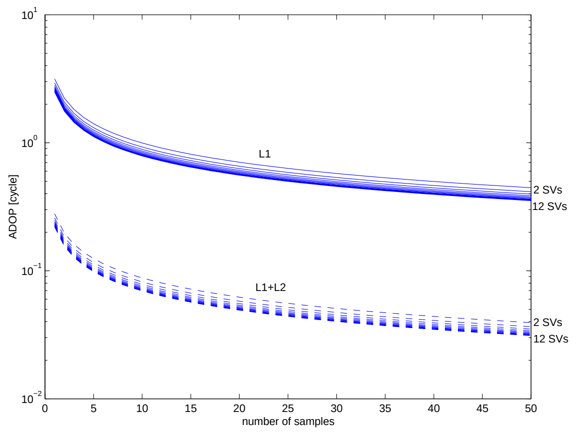

Figure 1: L1 and L1+L2 *ADOP*s for the short baseline geometry-free GPS model

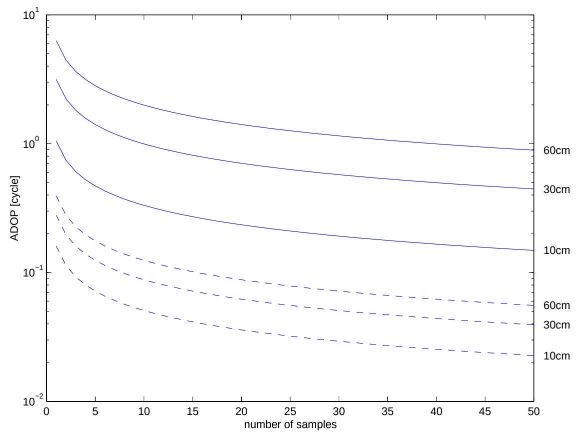

Figure 2: L1 and L1+L2 *ADOP*s for a varying pseudorange precision

It will be clear that the *ADOP* depends on the observation precision. Changes in the standard deviations will change the *ADOP*. Significant changes in the precision of the carrier phase measurements are not likely. Over the years though, the precision of the pseudoranges did change. Due to technological advances, the precision of the receiver-outputted pseudoranges has improved.

Figure 2 shows for the two-satellite case, what a change in the pseudorange precision does to the *ADOP*s. The standard deviations were set at the values 60cm, 30cm and 10cm. The full curves correspond to the single-frequency case and the dashed curves to the dual-frequency case. The figure shows that single-frequency ambiguity resolution becomes feasible when the pseudorange precision reaches the level of 10cm or better.

Example 2: long baseline geometry-free model

In case of long baselines the ionospheric delays can not be neglected anymore. These delays will have to be included as unknowns into the observation equations. As a consequence, dual-frequency data are needed per se to be able to solve the model. Figure 3 shows the corresponding *ADOP*s. Note that these dual-frequency *ADOP*s are of about the same magnitude as the single-frequency *ADOP*s of the short baseline model. Since the values of the *ADOP*s stay well above the 0.3 cycle level, one can not expect to have a successful ambiguity resolution.

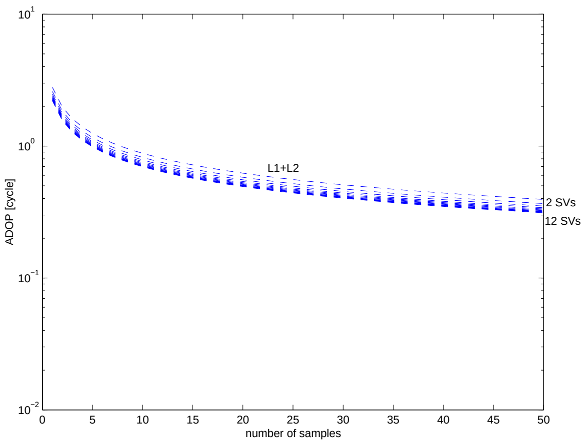

Figure 3: *ADOP*s for the long baseline geometry-free GPS model

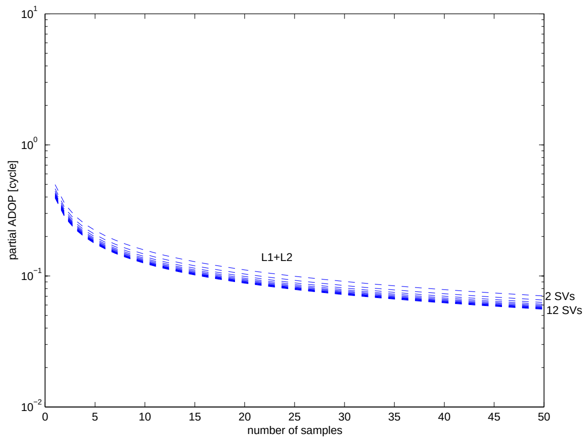

Figure 4: *ADOP*s for the most precise half of the number of ambiguities

It is of course quite disappointing that one can not expect to have a successful ambiguity resolution for the long baselines. One should keep in mind however that the above results are based on the assumption that all ambiguities need to be resolved. The success rate of ambiguity resolution improves if one settles for resolving less ambiguities, for instance only the most precise half of the number of ambiguities. This is shown in figure 4.

Example 3: ambiguity resolution for attitude determination

So far we considered the single baseline, geometry-free model. The *ADOP*s may also be computed for other type of GPS models. Consider a four-receiver network used for attitude determination. In this case we have a geometry-based model, since the relative receiver-satellite geometry is included in the observation equations. The network consists of three independent baselines. Since these baselines are usually extremely short, the atmospheric delays are neglected.

Figure 5 shows the corresponding *ADOP*-profile for a period of 24 hours at the location Plymouth on 7 January 1999. This *ADOP*-profile is based on the 4-receiver GPS attitude model

using single epoch, dual-frequency data. Hence, it corresponds to the case of instantaneous attitude determination. The cut-off elevation was set at 20 degrees.
Since the relative receiver-satellite geometry is now included in the model, one can expect the *ADOP*s to be smaller than the dual-frequency, single-epoch *ADOP*s of figure 1. And indeed as figure 5 shows, most of the *ADOP*-values are about 0.1 cycle or even much smaller. The figure shows also however, that there are still two periods for which the *ADOP*s are larger than 0.15 cycles. Hence, there are still two periods in the day where successful ambiguity resolution for instantaneous attitude determination may turn out to be problematic. Of course, the *ADOP*s will improve if one assumes that a lower cut-off elevation is permitted. For the present case, all *ADOP*s will be smaller than 0.07 cycles if a cut-off elevation of 10 degrees is used.

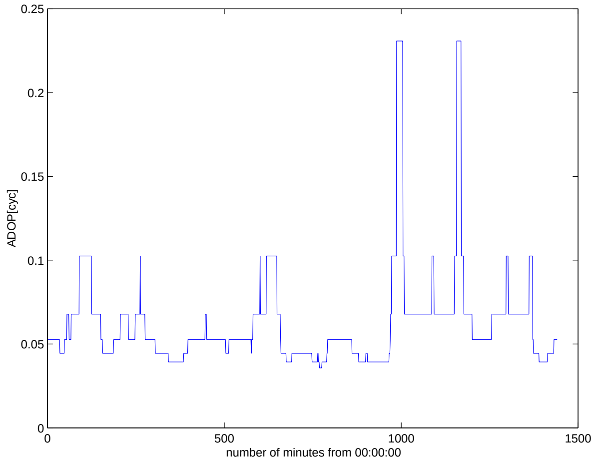

Figure 5: 24-hour ADOP-profile for instantaneous attitude determination, based on dual-frequency data, using a cut-off elevation of 20 degrees

Example 4: short baseline geometry-based model
So far we have plotted the *ADOP*s. In this last example we will use the *ADOP*s to plot the upperbound of the ambiguity success rates. These upperbounds are shown in figure 6 for the single epoch, short baseline, geometry-based model, for a period of 24 hours at the location Plymouth on 7 January 1999. The single-frequency case (cut-off elevation of 10 degrees) is shown in figure 6a and the dual-frequency case (cut-off elevation of 20 degrees) is shown in figure 6b. From figure 6a we learn that there are periods in which one should not rely too much on the results of instantaneous ambiguity resolution using L1 data only. The success rates in these periods are simply too small. A similar result can be seen for the dual-frequency case in figure 6b. Note however that the cut-off elevation has now been set at the value of 20 degrees.

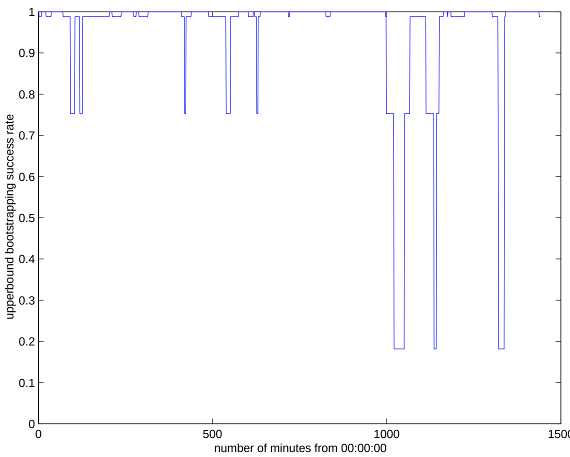

Figure 6a: Single-frequency case (cut-off 10 degrees)

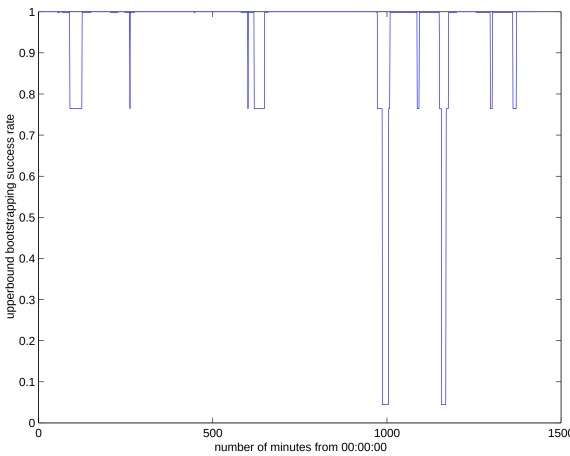

Figure 6b: Dual-frequency case (cut-off 20 degrees)

7. References

Baarda, W. (1968): *A testing procedure for use in geodetic networks*. Netherlands Geodetic Commission, Publications on Geodesy, New Series, Vol. 2, No. 5.

Boon, F., B. Ambrosius (1997): Results of real-time application of the LAMBDA method in GPS based aircraft landings. *Proc KIS97*, pp. 339-345.

Boon, F., P.J. de Jonge, C.C.J.M. Tiberius (1997): Precise aircraft positioning by fast ambiguity resolution using improved tropospheric modelling. *Proc ION GPS-97*, Vol. 2, pp. 1877-1884.

Cross, P. A. (1994): Quality measures for differential GPS positioning. *The Hydro. J.*, No. 72, pp. 17-22.

Dedes, G., C. Goad (1994): Real-time cm-level GPS positioning of cutting blade and earth movement equipment. In*: Proc. 1994 Nat. Tech. Meeting ION*, San Diego, California, pp. 587-594.

de Jong, C.D. (1998): A modular approach to precise GPS positioning. *GPS Solutions*, Vol. 2, No. 3.

de Jonge P.J., C.C.J.M. Tiberius (1996): The LAMBDA method for integer ambiguity estimation: implementation aspects. Delft Geodetic Computing Centre *LGR Series* No. 12, Delft University of Technology.

Han, S. (1995): Ambiguity resolution techniques using integer least-squares estimation for rapid static or kinematic positioning. *Symp. Satellite Navigation Technology: 1995 and beyond*, 10 p.

Hatch, R. (1982): The synergism of GPS code and carrier measurements. In: *Proc. 3rd Int. Geod. Symp. Satellite Positioning*. Las Cruces, New Mexico, 8-12 February, Vol. 2, pp. 1213-1231.

Hofmann-Wellenhof, B., H. Lichtenegger, J. Collins (1997): *Global Positioning System: Theory and Practice*. 4th edition. Springer Verlag.

Jonkman, N.F. (1998): Integer GPS ambiguity estimation without the receiver-satellite geometry. *LGR-Series* No. 18, Delft Geodetic Computing Centre, 95 pp.

Kleusberg, A., P.J.G. Teunissen (eds) (1996): *GPS for Geodesy*, Lecture Notes in Earth Sciences, vol. 60. Springer Heidelberg New York.

Lachapelle, G., G. Lu, R. Loncarevic (1994): Precise shipborne attitude determination using wide antenna spacing. *Proc KIS94*, pp. 323-330.

Leick, A. (1995): *GPS Satellite Surveying*, 2nd edition. John Wiley, New York.

Parkinson, B., J.J. Spilker (eds) (1996): *GPS: Theory and Applications*, Vols 1 and 2, AIAA, Washington DC.

Strang, G., K. Borre (1997): *Linear Algebra, Geodesy, and GPS*. Wellesley-Cambridge Press.

Teunissen, P.J.G. (1985): Quality control in geodetic networks. Chapter 17 in *Optimization and Design of Geodetic Networks*, E. Grafarend and F. Sanso (eds), Springer Verlag.

Teunissen, P.J.G. (1993): Least-squares estimation of the integer GPS ambiguities. Invited Lecture, Section IV Theory and Methodology, IAG General Meeting, Beijing, China. Also in: Delft Geodetic Computing Centre *LGR Series* No. 6, Delft University of Technology.

Teunissen, P.J.G. (1996): An analytical study of ambiguity decorrelation using dual-frequency code and carrier phase. *Journal of Geodesy.*, 70:515-528.

Teunissen, P.J.G. (1997a): Some remarks on GPS ambiguity resolution. *Art. Satellites*, Vol. 32, No. 3.

Teunissen, P.J.G. (1997b): A canonical theory for short GPS baselines. *Journal of Geodesy*, Vol. 71. Part I: the baseline precision, No. 6, pp. 320-336. Part II: the ambiguity precision and correlation, No. 7, pp. 389-401. Part III: the geometry of the ambiguity search space, No. 8, pp. 486-501. Part IV: precision versus reliability, No. 9, pp. 513-525.

Teunissen, P.J.G. (1998a): A class of unbiased integer GPS ambiguity estimators. *Art. Satellites*, Vol. 33, No. 1.

Teunissen, P.J.G. (1998b): Success probability of integer GPS ambiguity rounding and bootstrapping. *Journal of Geodesy*, in print.

Teunissen, P.J.G., D. Odijk (1997): Ambiguity dilution of precision: definition, properties and application. *Proc. ION GPS-97*, Kansas City, USA, 16-19 September, pp. 891-899.

Teunissen, P.J.G., P. Joosten, D. Odijk (1998a): The reliability of GPS ambiguity resolution. *GPS Solutions*, Vol. 2, No. 3.

Teunissen, P.J.G., D. Odijk, P. Joosten (1998b): A probabilistic evaluation of correct GPS ambiguity resolution. *Paper presented at ION GPS-98,* Nashville, USA, 15-18 September.

Tiberius, C.C.J.M., P.J. de Jonge (1995): Fast positioning using the LAMBDA method. In*: Proc. DSNS-95*, Bergen, Norway, April 24-28, Paper No. 30.

Zinn, N., P.J.V. Rapatz (1995): Reliability analysis in marine seismic networks. *The Hydro. J.*, No. 76, pp. 11-18.