A Proof of Nielsen's Conjecture on the Relationship Between Dilution of Precision for Point Positioning and for Relative Positioning with GPS

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Abstract

The dilution of precision terms for relative positioning as defined in [1], are bounded from above by the corresponding dilution of precision terms for point positioning. In [1], this result is proven for the case of four satellites and conjectured to be valid for the case of more than four satellites. A proof of this conjecture is given. We also extend the result by giving two different lower bounds for the dilution of precision terms. The first lower bound depends on the receiver-satellite geometry, whereas the second does not. The proof of the bounds is based on the solution of a generalized eigenvalue problem.

1 Introduction

Double-difference processing of the NAVSTAR/Global Positioning System (GPS) satellite signals has been employed by the surveying and geodetic community for some time [2]. In analogy with HDOP and VDOP, the horizontal and vertical dilution of precision terms of point positioning, Nielsen [1] introduces corresponding DOP-terms for relative positioning using double differences and demonstrates for the four-satellite case that his DOP-values for relative positioning are bounded from above by the corresponding DOP-values of point positioning. In this contribution we extend Nielsen's result to an arbitrary number of satellites. We also show how the relevant DOP-values are bounded from below. This enables us to identify the condition for which the two types of DOP-values coincide. Section 2 summarizes Nielsen's result and conjecture, while Section 3 gives the solution of a generalized eigenvalue problem. It forms the basis of our main result, which is stated and proven in Section 4.

2 Nielsen's Conjecture

Let $(x_i, y_i, z_i)^T$ be the unit direction vector between the ith satellite and the approximate receiver location, and define the two matrices

$$A_m = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ \vdots & \vdots & \vdots \\ x_m & y_m & z_m \end{bmatrix} \quad , \quad D_m = \begin{bmatrix} -I_{m-1} \\ e_{m-1}^T \end{bmatrix} \tag{1}$$

where I_{m-1} denotes the unit matrix of order $m-1$ and e_{m-1} denotes the vector which has all its $m-1$ entries equal to one. The $m \times 3$ matrix A_m captures the receiver-satellite geometry of the m satellites and the $(m-1) \times m$ matrix D_m^T equals the differencing operator that transforms single-difference measurements to double-difference measurements having satellite m as reference. Note that the double-difference measurements are referenced to a common satellite. For Nielsen's result to be valid, the need for such a common reference was already shown in [1].

The design matrix for point positioning is denoted as H_m and its counterpart for relative positioning based on double-difference measurements is denoted as G_m. These two matrices are given as

$$H_m = [A_m, e_m] \quad , \quad G_m = D_m^T A_m \tag{2}$$

Since both matrices are assumed to be of full rank, we must have $m \geq 4$. For $m = 4$, H_m corresponds with [1, eq.(4)] and G_m with [1, eq.(10)].

The dilution of precision terms for point positioning are defined as

$$\begin{aligned} \text{HDOP}_m &= \sqrt{[H_m^T H_m]_{1,1}^{-1} + [H_m^T H_m]_{2,2}^{-1}} \\ \text{VDOP}_m &= \sqrt{[H_m^T H_m]_{3,3}^{-1}} \end{aligned} \tag{3}$$

where $[H_m^T H_m]_{i,i}^{-1}$ is the ith element on the main diagonal of $[H_m^T H_m]^{-1}$. The corresponding dilution of precision terms for relative positioning are defined in [1] as

$$\begin{aligned} \text{HDOP}_{m,DD} &= \sqrt{[G_m^T G_m]_{1,1}^{-1} + [G_m^T G_m]_{2,2}^{-1}} \\ \text{VDOP}_{m,DD} &= \sqrt{[G_m^T G_m]_{3,3}^{-1}} \end{aligned} \tag{4}$$

where $[G_m^T G_m]_{i,i}^{-1}$ is the ith element on the main diagonal of $[G_m^T G_m]^{-1}$.

The main result of [1] is the proof of the two inequalities

$$\begin{aligned} &\text{HDOP}_{m,DD} \leq \text{HDOP}_m \ \text{ for } m = 4 \\ &\text{VDOP}_{m,DD} \leq \text{VDOP}_m \ \text{ for } m = 4 \end{aligned} \tag{5}$$

For $m > 4$ however, the two inequalities are conjectured to be true. In order to proof this conjecture, we need to compare the two matrices $[H_m^T H_m]^{-1}$ and $[G_m^T G_m]^{-1}$. This will be done in the next section by means of a generalized eigenvalue problem.

3 A Generalized Eigenvalue Problem

Since the two matrices $[H_m^T H_m]^{-1}$ and $[G_m^T G_m]^{-1}$ are of a different order, respectively 4 and 3, we first need to find an expression for the first three rows and columns of matrix $[H_m^T H_m]^{-1}$. It is easily verified that the inverse of

$$[H_m^T H_m] = \begin{bmatrix} A_m^T A_m & A_m^T e_m \\ e_m^T A_m & m \end{bmatrix} \tag{6}$$

is given as

$$[H_m^T H_m]^{-1} = \begin{bmatrix} [F_m^T F_m]^{-1} & -\frac{1}{m}[F_m^T F_m]^{-1} A_m^T e_m \\ -\frac{1}{m} e_m^T A_m [F_m^T F_m]^{-1} & \frac{1}{m} + \frac{1}{m^2} e_m^T A_m [F_m^T F_m]^{-1} A_m^T e_m \end{bmatrix} \tag{7}$$

where $F_m = P_m A_m$, with the orthogonal projector $P_m = I_m - \frac{1}{m} e_m e_m^T$. Since the first three rows and columns of $[H_m^T H_m]^{-1}$ are captured by the matrix $[F_m^T F_m]^{-1}$, the two dilution of precision terms of (3) can be expressed in matrix F_m as

$$\begin{aligned} \mathrm{HDOP}_m &= \sqrt{[F_m^T F_m]^{-1}_{1,1} + [F_m^T F_m]^{-1}_{2,2}} \\ \mathrm{VDOP}_m &= \sqrt{[F_m^T F_m]^{-1}_{3,3}} \end{aligned} \tag{8}$$

Thus in order to compare the dilution of precision terms, we need to compare the two matrices $[F_m^T F_m]^{-1}$ and $[G_m^T G_m]^{-1}$. This comparison can be based on the following generalized eigenvalue problem.

Theorem

Let λ_i and f_i, $i = 1, 2, 3$, be the eigenvalues resp. eigenvectors of the generalized eigenvalue problem

$$[G_m^T G_m]^{-1} f = \lambda [F_m^T F_m]^{-1} f \tag{9}$$

Then

$$\begin{cases} \lambda_1 = 1 - \frac{1}{m} e_{m-1}^T P_{G_m} e_{m-1} & \text{with } \; f_1 = G_m^T e_{m-1} \\ \\ \lambda_2 = \lambda_3 = 1 & \text{with } \; f_2, f_3 \perp [G_m^T G_m]^{-1} G_m^T e_{m-1} \end{cases} \tag{10}$$

where $P_{G_m} = G_m [G_m^T G_m]^{-1} G_m^T$ is the orthogonal projector that projects onto the range space of G_m and along the null space of G_m^T.

Proof

Since P_m projects along e_m and onto the orthogonal complement of e_m, which is the range space of D_m, the projector can be represented in the following two ways

$$P_m = I_m - \frac{1}{m} e_m e_m^T = D_m [D_m^T D_m]^{-1} D_m^T \tag{11}$$

This shows, since $F_m = P_m A_m$ and $G_m = D_m^T A_m$, that

$$F_m^T F_m = G_m^T [D_m^T D_m]^{-1} G_m \tag{12}$$

From (1) it follows that $D_m^T D_m = I_{m-1} + e_{m-1} e_{m-1}^T$ and thus

$$[D_m^T D_m]^{-1} = I_{m-1} - \frac{1}{m} e_{m-1} e_{m-1}^T \tag{13}$$

Substitution of (13) into (12) gives $F_m^T F_m = G_m^T G_m - \frac{1}{m} G_m^T e_{m-1} e_{m-1}^T G_m$ and after inversion

$$[F_m^T F_m]^{-1} = [G_m^T G_m]^{-1} + \frac{[G_m^T G_m]^{-1} G_m^T e_{m-1} e_{m-1}^T G_m [G_m^T G_m]^{-1}}{m - e_{m-1}^T P_{G_m} e_{m-1}} \tag{14}$$

From substituting (14) into (9), the result (10) is now easily verified.

4 The Main Result

We are now in a position to proof the conjecture of Nielsen and to give an extention by including lower bounds on the dilution of precision terms as well. As a direct consequence of the above theorem we have the following bounds for the Raleigh quotient

$$(1 - \frac{1}{m} e_{m-1}^T P_{G_m} e_{m-1}) \leq \frac{f^T [G_m^T G_m]^{-1} f}{f^T [F_m^T F_m]^{-1} f} \leq 1 \tag{15}$$

for all non-null f and $m \geq 4$. By choosing f respectively as $f = (1,0,0)^T$, $f = (0,1,0)^T$ and $f = (0,0,1)^T$, it follows that

$$\begin{aligned} (1 - \tfrac{1}{m} e_{m-1}^T P_{G_m} e_{m-1}) \textstyle\sum_{i=1}^2 [F_m^T F_m]_{i,i}^{-1} \leq \sum_{i=1}^2 [G_m^T G_m]_{i,i}^{-1} \leq \sum_{i=1}^2 [F_m^T F_m]_{i,i}^{-1} \\ (1 - \tfrac{1}{m} e_{m-1}^T P_{G_m} e_{m-1}) [F_m^T F_m]_{3,3}^{-1} \leq [G_m^T G_m]_{3,3}^{-1} \leq [F_m^T F_m]_{3,3}^{-1} \end{aligned} \tag{16}$$

By taking the square-roots, the corresponding bounds for the horizontal and vertical dilution of precision terms follow as

$$\begin{array}{c} \sqrt{1-\frac{1}{m}e_{m-1}^T P_{G_m} e_{m-1}}\ \mathrm{HDOP}_m \leq \mathrm{HDOP}_{m,DD} \leq \mathrm{HDOP}_m \\ \sqrt{1-\frac{1}{m}e_{m-1}^T P_{G_m} e_{m-1}}\ \mathrm{VDOP}_m \leq \mathrm{VDOP}_{m,DD} \leq \mathrm{VDOP}_m \end{array} \tag{17}$$

This result extends (5) in two ways. Apart from the upper bounds, lower bounds are now included as well. Moreover, these bounds are not only valid for $m = 4$, but also for $m > 4$.

Note that $\mathrm{HDOP}_{m,DD} = \mathrm{HDOP}_m$ and $\mathrm{VDOP}_{m,DD} = \mathrm{VDOP}_m$, when $G_m^T e_{m-1} = 0$. From (1) and (2) it follows that this happens when

$$x_m = \frac{1}{m-1}\sum_{i=1}^{m-1} x_i\ ,\ y_m = \frac{1}{m-1}\sum_{i=1}^{m-1} y_i\ ,\ z_m = \frac{1}{m-1}\sum_{i=1}^{m-1} z_i \tag{18}$$

That is, when one of the m satellites is located at the 'center of gravity' of the receiver-satellite configuration.

The two lower bounds of (17) depend on the receiver-satellite geometry through the matrix G_m. Lower bounds that are independent of this geometry can be given as well. Since the eigenvalues of a projector are either 0 or 1, it follows that

$$0 \leq \frac{e_{m-1}^T P_{G_m} e_{m-1}}{e_{m-1}^T e_{m-1}} \leq 1 \tag{19}$$

With $e_{m-1}^T P_{G_m} e_{m-1} \leq m-1$, the geometry independent bounds follow from (17) as

$$\begin{array}{c} \frac{1}{\sqrt{m}}\ \mathrm{HDOP}_m \leq \mathrm{HDOP}_{m,DD} \leq \mathrm{HDOP}_m \\ \frac{1}{\sqrt{m}}\ \mathrm{VDOP}_m \leq \mathrm{VDOP}_{m,DD} \leq \mathrm{VDOP}_m \end{array} \tag{20}$$

5 Conclusion

We have proven Nielsen's conjecture by showing that his dilution of precision terms for relative positioning are also bounded from above by the corresponding dilution of precision terms for point positioning when more than four satellites are tracked. This result was extended by giving lower bounds as well. It was also shown that the two types of dilution of precision coincide when one of the satellites is located at the 'center of gravity' of the receiver-satellite configuration.

6 References

[1] Nielsen, R.O. (1997): Relationship Between Dilution of Precision for Point Positioning and for Relative Positioning with GPS. *IEEE Transactions on Aerospace and Electronic Systems*, Vol. 33, No. 1, pp. 333-337.

[2] Kleusberg, A., P.J.G. Teunissen (Eds.) (1996): *GPS for Geodesy*, Lecture Notes in Earth Sciences, Vol. 60, Springer-Verlag, Heidelberg, New York.