ON THE INTEGER NORMAL DISTRIBUTION OF THE GPS AMBIGUITIES

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ABSTRACT

Carrier phase ambiguity resolution is the key to fast and high precision GPS kinematic positioning. Critical in the application of ambiguity resolution is the quality of the computed integer ambiguities. Unsuccessful ambiguity resolution, when passed unnoticed, will too often lead to unacceptable errors in the positioning results. In order to describe the quality of the integer ambiguities, their distributional properties need to be known. This contribution introduces the probability mass function of the integer least-squares ambiguities. This integer normal distribution is needed in order to infer objectively whether or not ambiguity resolution can expected to be successful. Some of its properties are discussed. Attention is given in particular to the probability of correct integer estimation. Various diagnostic measures are presented for evaluating this probability.

Keywords: ambiguity resolution, integer normal distribution, probability of correct integer ambiguity estimation

1 Introduction

GPS ambiguity resolution is the process of resolving the unknown cycle ambiguities of the double-difference (DD) carrier phase data as integers [*Hofmann-Wellenhof et al.*, 1997], [*Kleusberg and Teunissen*, 1996], [*Leick*, 1995], [*Strang and Borre*, 1997]. Once resolved, one usually keeps the ambiguities fixed at their computed integer estimates. That is, all the results that depend on the ambiguity resolution process are usually evaluated as if the integer ambiguities were deterministic constants. From a *theoretical* point of view this is not correct. The estimated ambiguities, although integer, are still stochastic variates. They have been computed from the data and since the vector of observables is assumed to be random, also the integer ambiguity estimator is a random vector. Conceptually we may write $\check{a} = F(y)$, where $\check{a}$ denotes the integer ambiguity vector, y the vector of observables, and $F(.)$ the mapping from the continuous vector of observables to the integer vector of ambiguities. Thus when y is random, $\check{a}$ is random as well. As far as their randomness is concerned, the marked difference between y and $\check{a}$ is that the probability distribution of y is continuous, whereas that of $\check{a}$ is of the discrete type. Thus $\check{a}$ has a probability *mass* function attached to it. This point was emphasized in [*Teunissen*, 1990] and highlighted again in [*Teunissen*, 1997a] as one of the pitfalls in the more classical approaches to ambiguity resolution.

Although theoretically not correct, it is possible that a treatment of the ambiguities as if they were deterministic constants is acceptable from a *practical* point of view. But in order to be able to judge whether this is feasible or not, one first needs to know more about the stochastic characteristics of the integer ambiguities. That is, one needs to know the probability mass function of the ambiguities and in addition, have means available of evaluating this distribution.

In this contribution we will assume that the data are normally distributed and we will use the integer least-squares criterion as estimation principle. This combination assures that a maximum likelihood solution is obtained. In addition, the solution is admissible and minimax as well. But other principles can also be used. One of the earliest is the ambiguity function method [Counselman and Gourevitch, 1981]. A Bayesian approach is used in [*Betti et al.*, 1993], and [*Blewitt*, 1989] and [*Dong and Bock*, 1989] used the bootstrapped estimator. The distribution of the bootstrapped estimator was studied in [*Teunissen*, 1998a].

This contribution is organized as follows. In Sect. 2 a brief review is given of the conceptual steps in solving the integer least-squares problem. In Sect. 3 we present the probability mass function of the integer least-squares ambiguities. This distribution is coined the integer normal distribution. In Sect. 4 we discuss some properties of this distribution. We show that it is symmetric and that its maximum equals the probability of correct integer estimation. We also show that the integer least- squares ambiguities are unbiased and that their precision bounds the probability of correct integer estimation. In Sect. 5 finally, we present some approaches for evaluating the integer normal distribution. Although an exact evaluation is rather difficult, approximations and bounds are given for the probability of correct integer estimation. Some of these approaches were already discussed in [*Teunissen*, 1997a-c]. The first approach is based on simulating the probability mass function. This is possible, since the shape of distribution is independent of the unknown integer ambiguities. For the second approach we make use of the bootstrapped ambiguity estimator. Its probability of correct integer estimation is easily computed. In the third approach we use eigenvalues to bound the variance matrix of the integer least- squares ambiguities. Finally in the last approach, the minimum grid point distance, as measured by the metric of the ambiguity variance matrix, is used to obtain bounds on the region of integration.

2 Integer least-squares estimation

As our point of the departure we will take the following system of linear(ized) observation equations

$$y = Aa + Bb + e \tag{1}$$

where y is the given data vector, a and b are the unknown parameter vectors and e is the noise vector. In principle all the GPS models can be cast in this frame of observation equations. The data vector will then usually consist of the 'observed minus computed' single- or dual-frequency double-differenced (DD) phase and/or pseudo range (code) observations, accumulated over all observation epochs. The entries of the n-vector a are the DD carrier phase ambiguities, expressed in units of cycles rather than range. They are known to be integer valued. The entries of the p-vector b consist of the remaining unknown parameters, such as for instance baseline components (coordinates) and possibly atmospheric delay parameters (troposphere, ionosphere).

When using the least-squares principle, the above system of observation equations can be solved by means of the minimization problem

$$\min_{a,b}(y - Aa - Bb)^T Q_y^{-1}(y - Aa - Bb) \quad, a \in Z^n \ , b \in R^p \tag{2}$$

with Q_y the variance-covariance matrix of the observables, Z^n the n-dimensional space of integers and R^p the p-dimensional space of real numbers. This type of least-squares problem was first introduced in [*Teunissen*, 1993] and has been coined with the term '*integer least-squares*'. It is a nonstandard least-squares problem due to the integer constraints $a \in Z^n$.

Conceptually one can divide the computation of the solution to (2) into three different steps. In the first step one simply disregards the integer constraints on the ambiguities and performs a standard least- squares adjustment. As a result one obtains the (real-valued) least- squares estimates of a and b, together with their variance- covariance matrix

$$\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix} , \begin{bmatrix} Q_{\hat{a}} & Q_{\hat{a}\hat{b}} \\ Q_{\hat{b}\hat{a}} & Q_{\hat{b}} \end{bmatrix} \tag{3}$$

This solution is often referred to as the 'float' solution. In the second step the 'float' ambiguity estimate $\hat{a}$ and its variance- covariance matrix are used to compute the corresponding integer ambiguity estimate. This implies solving the minimization problem

$$\min_{a \in Z^n} (\hat{a} - a)^T Q_{\hat{a}}^{-1} (\hat{a} - a) \tag{4}$$

Its solution will be denoted as $\check{a}$. Finally in the third step, the integer ambiguities are used to correct the 'float' estimate $\hat{b}$. As a result one obtains the 'fixed' solution

$$\check{b} = \hat{b} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} (\hat{a} - \check{a}) \tag{5}$$

Thus in summary, the 'float' solutions $\hat{a}$ and $\hat{b}$ follow from solving (2) *without* the integer constraints, while the 'fixed' solutions $\check{a}$ and $\check{b}$ follow from solving (2) *with* the integer constraints included.

From a computational point of view, the most difficult part in the above three steps is the solution of (4). The difficulty lies in the fact that most GPS least-squares ambiguities are highly correlated. This is due to the short observation time spans used and the fact, that in case of GPS, the relative receiver-satellite geometry changes only slowly with time. As a consequence of the high correlation, the search needed to solve (4) becomes very time consuming. To remedy this situation, the least-squares ambiguity decorrelation adjustment (LAMBDA) was introduced

[*Teunissen*, 1993], [*Jonge de and Tiberius*, 1996]. By means of the decorrelation process of this method the original DD ambiguities are transformed to new ambiguities which have the property of being far more precise than the original ambiguities, see also the textbooks [*Kleusberg and Teunissen*, 1996], [*Hofmann-Wellenhof et al.*, 1997], [*Strang and Borre*, 1997]. Examples of practical results can be found in e.g. [*Tiberius and de Jonge*, 1995], [*Jonge de and Tiberius*, 1996], [*Jonkman*, 1998].

The goal of the present contribution is not to discuss the intricacies of the above computational steps, but instead to present some of the distributional properties of the integer least-squares estimator $\check{a}$. This is in particular of relevance for GPS ambiguity resolution. It is namely only through the distributional properties of $\check{a}$ that one will be able to objectively decide whether or not a successful 'fixing' of the integer ambiguities is likely to happen.

3 The distribution of the integer least-squares ambiguities

It will be assumed that our vector of observables is normally (Gaussian) distributed with mean $E\{y\} = Aa + Bb$ and dispersion $D\{y\} = Q_y$. As a result the (real-valued) least-squares ambiguities will be normally distributed too, with mean a and variance-covariance matrix $Q_{\hat{a}}$. Thus

$$y \sim N(Aa + Bb, Q_y) \implies \hat{a} \sim N(a, Q_{\hat{a}}) \tag{6}$$

Note that the mean of the real-valued least-squares estimator is an integer vector. The multivariate normal probability density function of $\hat{a}$ reads therefore

$$p_{\hat{a}}(a) = \frac{1}{\sqrt{det(Q_{\hat{a}})}(2\pi)^{\frac{1}{2}n}} \exp\{-\frac{1}{2} \parallel \hat{a} - a \parallel^2_{Q_{\hat{a}}}\} \tag{7}$$

with the squared weighted norm $\parallel . \parallel^2_{Q_{\hat{a}}} = (.)^T Q_{\hat{a}}^{-1}(.)$ and $a \in Z^n$.

Since the *integer* least-squares ambiguities follow from solving (4), they are functions of the stochastic *real valued* least-squares ambiguities and therefore stochastic variates themselves as well. Thus both $\hat{a}$ and $\check{a}$ are random vectors, although their distributions differ. The distribution of $\hat{a}$ is a continuous one, whereas the distribution of $\check{a}$ is of the discrete type. The mapping from the continuous random vector $\hat{a}$ to the discrete random vector $\check{a}$ is namely a many-to-one map. To see this, consider the following subset of R^n

$$S_z = \{x \in R^n \mid \parallel x - z \parallel^2_{Q_{\hat{a}}} \leq \parallel x - u \parallel^2_{Q_{\hat{a}}}, \forall u \in Z^n\} \tag{8}$$

It contains all values of $\hat{a}$ which are mapped to the single integer grid point $z \in Z^n$ when solving the integer least- squares problem (4). Thus the integer least-squares solution equals z when $\hat{a}$ lies in S_z and vice versa. Hence

$$\hat{a} \in S_z \iff \check{a} = z \tag{9}$$

Thus the subset S_z acts as a *pull-in-region* for $\hat{a}$. That is, whenever $\hat{a}$ lies in the subset S_z, it is pulled to z, being the centre grid point of the set. These pull-in-regions were studied by [*Jonkman*, 1998] for the geometry-free GPS model. Note that each integer grid point $z \in Z^n$ has such a pull-in-region assigned to it. Also note that these subsets are disjoint and that they together cover R^n, i.e. $S_{z_i} \cap S_{z_j} = \{0\}$ for $i \neq j$ and $R^n = \cup_{z \in Z^n} S_z$. With this information it is now possible to formulate the probability *mass* function of $\check{a}$. It reads

$$\boxed{P(\check{a} = z) = P(\hat{a} \in S_z) = \int_{S_z} \frac{1}{\sqrt{det(Q_{\hat{a}})}(2\pi)^{\frac{1}{2}n}} \exp\{-\frac{1}{2} \parallel x - a \parallel^2_{Q_{\hat{a}}}\}dx \ , \ a, z \in Z^n} \tag{10}$$

The discrete distribution of the integer least-squares ambiguities follows thus from mapping the volume of the normal distribution over the subsets S_z to each of their centre grid points z. Since this distribution does not appear to have a name yet, see e.g. [*Johnson et al.*, 1994], we will coin it the *integer normal distribution*. In the following section we will present some of its properties.

4 Properties of the integer normal distribution

In this section we will focus on a few properties of the integer normal distribution. They are related to the mode and shape of the distribution.

4.1 The probability of correct integer estimation is largest

Of the infinite number of probability masses of (10), there is one which is particularly of importance for GPS ambiguity resolution. It is $P(\check{a} = a)$, the probability of correct integer estimation. This is the probability that $\check{a}$ coincides with the true but unknown integer mean a. In order for the integer least-squares principle to make sense, the least we can ask of this principle is that the resulting probability of correct integer estimation is always larger than any of the probabilities of wrong integer estimation. We will now show that this indeed holds true. Thus

$$\boxed{\max_{z \in Z^n} P(\check{a} = z) = P(\check{a} = a)} \tag{11}$$

This will be proven by showing that

$$\int_{S_a} c\exp\{-\frac{1}{2} \parallel x - a \parallel^2_{Q_{\hat{a}}}\}dx \geq \int_{S_a} c\exp\{-\frac{1}{2} \parallel x - z \parallel^2_{Q_{\hat{a}}}\}dx = \int_{S_z} c\exp\{-\frac{1}{2} \parallel x - a \parallel^2_{Q_{\hat{a}}}\}dx$$

where c is the proportionality factor of the multivariate normal distribution. Note that the left-hand side equals $P(\check{a} = a)$ and the right-hand side $P(\check{a} = z)$. The inequality follows by noting that $\hat{a} \in S_a$ implies $\exp\{-\frac{1}{2} \parallel \hat{a} - a \parallel^2_{Q_{\hat{a}}}\} \geq \exp\{-\frac{1}{2} \parallel \hat{a} - z \parallel^2_{Q_{\hat{a}}}\}$, $\forall z \neq a$. In order to prove the above integral equality, we apply the general transformation formula for integrals [*Fleming*, 1977]. The following change of variable transformation is applied to the second integral, $T : x = -y + a + z$. Note that the absolute value of the Jacobian equals one and that the region of integration transforms from S_a to S_z, since $T^{-1}(S_a) = \{y \in R^n \mid \parallel y - z \parallel^2_{Q_{\hat{a}}} \leq \parallel y - a - z + u \parallel^2_{Q_{\hat{a}}} , \forall u \in Z^n\} = S_z$. This shows that the above integral equality holds true.

Although it is of course comforting to know that the probability of correct integer estimation is the largest of all nonzero probabilities, it is not sufficient for ambiguity resolution to be successful. For that to be the case, one still has to evaluate the probability of correct integer estimation and check whether its value is sufficiently close to one. This issue will be taken up in Sect. 5.

4.2 The integer normal distribution is symmetric

Apart from knowing the maximum of the probability mass function, it is also of interest to know how the probability masses are distributed about this maximum. This concerns the shape of the distribution. We know that the multivariate normal distribution (7) is symmetric about a. We will now show that the corresponding integer normal distribution is symmetric about a as well. We have

$$\boxed{P(\check{a} = a - z) = P(\check{a} = a + z)\ ,\ \forall z \in Z^n} \tag{12}$$

In order to show this, we have to prove

$$\int_{S_{a-z}} c\exp\{-\frac{1}{2} \parallel x - a \parallel^2_{Q_{\hat{a}}}\}dx = \int_{S_{a+z}} c\exp\{-\frac{1}{2} \parallel x - a \parallel^2_{Q_{\hat{a}}}\}dx$$

This result follows from applying the change of variable transformation, $T : x = 2a - y$ and noting that $T^{-1}(S_{a-z}) = \{y \in R^n \mid \parallel y - a - z \parallel^2_{Q_{\hat{a}}} \leq \parallel y - 2a + u \parallel^2_{Q_{\hat{a}}}\ ,\ \forall u \in Z^n\} = S_{a+z}$.

4.3 The 'float' and 'fixed' solutions are unbiased

A third property of the integer least-squares estimator is its unbiasedness. We are thus in the happy situation that not only the real-valued least-squares ambiguities are unbiased, but their integer least-squares counterparts as well. Hence,

$$\boxed{E\{\check{a}\} = E\{\hat{a}\} = a} \tag{13}$$

This property of unbiasedness is a direct consequence of the symmetry of the distribution. To see this, recall the definition of the expectation of $\check{a}$. It reads $E\{\check{a}\} = \sum zP(\check{a} = z)$, with the sum taken over all grid points of Z^n. This may also be written as $E\{\check{a}\} = \sum(z+a)P(\check{a} = a+z)$ and as $E\{\check{a}\} = \sum(a-z)P(\check{a} = a-z)$. Taking the sum of these last two expressions and noting that $P(\check{a} = a+z) = P(\check{a} = a-z)$, gives $2E\{\check{a}\} = \sum 2aP(\check{a} = a+z) = 2a$ from which the unbiasedness follows.

From the unbiasedness of the integer least-squares ambiguities it also follows that the 'fixed' solution $\check{b}$ is unbiased. For the 'float' solution we have $E\{\hat{a}\} = a$ and $E\{\hat{b}\} = b$. This together with $E\{\check{a}\} = a$, shows that the expectation of (5) is given as

$$\boxed{E\{\check{b}\} = E\{\hat{b}\} = b} \tag{14}$$

With the above results we have proven that the inclusion of the integer constraints does not introduce any biases when using the least- squares principle. That is, the 'fixed' solutions are unbiased whenever the 'float' solutions are. But biases may still be introduced of course, when models are used that are misspecified (e.g. due to cycle slips or outliers in the data).

Finally we note that the property of unbiasedness is not restricted to the integer least-squares estimator. Certain other integer estimators can be shown to be unbiased as well. Such a class of unbiased integer estimators was introduced in [*Teunissen*, 1998b].

4.4 Precision and probability of the integer ambiguities

Although it is comforting to know that the integer least-squares ambiguities are unbiased, this is not enough for ambiguity resolution to be successful. For that to be the case, we also need a sufficiently small variability of the integer least-squares ambiguities about their integer means. We will now show how the ambiguity precision is related to the probability of correct integer estimation.

By definition, the variance-covariance matrix of the integer least-squares ambiguities is given as

$$Q_{\check{a}} = \sum_{z \in Z^n} (z-a)(z-a)^T P(\check{a}=z) \tag{15}$$

where, as before, a denotes the integer mean of $\check{a}$. The sum is taken over all grid points in Z^n. Note the absence of $P(\check{a}=a)$ in the above expression. This shows that the expected variability of $\check{a}$ is due to the probabilities of wrong integer estimation. Hence, the integer ambiguities will have a poor precision if the probabilities of wrong integer estimation are not negligible. The reverse of this statement is true as well. That is, if the variance of the integer ambiguities is sufficiently small, then the probability of correct integer estimation is sufficiently large. To see this, consider the jth diagonal entry of (15). It reads

$$c_j^T Q_{\check{a}} c_j = \sum_{z \in Z^n} [c_j^T(z-a)]^2 P(\check{a}=z)$$

where c_j denotes the jth canonical unit vector. The left-hand side equals the variance of the jth integer ambiguity. In the sum on the right-hand-side, the contribution for $z=a$ is absent and the non-zero minimum of $[c_j^T(z-a)]^2$ equals 1. This gives the inequality $\sigma^2_{\check{a}_j} \geq \sum_{z \in Z^n \setminus \{a\}} P(\check{a}=z)$, or $P(\check{a}=a) \geq 1-\sigma^2_{\check{a}_j}$. Since such an inequality holds for all diagonal entries of the ambiguity variance matrix, we finally get

$$\boxed{P(\check{a}=a) \geq 1 - \frac{1}{n}\text{trace } Q_{\check{a}}} \tag{16}$$

This shows that the probabiliy of correct integer estimation is bounded from below by one minus the average variance of the integer least- squares ambiguities. Hence, smaller variances will push the probability of correct integer estimation closer to one. In fact, this probability equals one already when one of the variances of the integer least-squares ambiguities vanishes.

5 Evaluation of the integer normal distribution

In this section we will present different approaches for evaluating the integer normal distribution. Particular attention will be given to the probability of correct integer estimation. Although these approaches differ in the way they try to approximate this probability, they all make use of the variance matrix of the least-squares ambiguities. The first approach is based on simulating the probability of correct integer estimation. The second approach uses the probability of correct integer estimation of a less optimal integer estimator, namely the bootstrapped estimator. The third approach uses bounds on the ambiguity variance-covariance matrix to obtain corresponding bounds on the probability of correct integer estimation. Finally, in the last approach such bounds are obtained by bounding the region of integration.

5.1 Simulating the probability mass function

In general it is very difficult to evaluate the integer normal distribution (10) exactly. This is due to the rather complicated geometry of the integration region S_z. The method of simulation can however be used to obtain approximations of the probabilities $P(\check{a} = z)$. This goes as follows. We know that the 'float' solution is distributed as $\hat{a} \sim N(a, Q_{\hat{a}})$. We also know that the integer normal distribution is symmetric about the mean a. Hence, in order to obtain the required probability masses we may shift the distribution over a and restrict our attention to $N(0, Q_{\hat{a}})$, draw samples from it and use these samples to obtain the corresponding integer samples by means of solving (4). Repeating this procedure a sufficient number of times, allows us then to built up the required frequency table. The probability of correct integer estimation is then given as $P(\check{a} = 0)$.

Thus first one starts generating, using a random generator, n independent samples from the univariate standard normal distribution, say $s_1, \ldots, s_n$ from $N(0, 1)$. These samples are then collected in the vector $s = (s_1, \ldots, s_n)^T$ and transformed by means of $\hat{a} = Gs$, where matrix G equals the Cholesky factor of the ambiguity variance-covariance matrix $Q_{\hat{a}}$, i.e. $Q_{\hat{a}} = GG^T$. Hence, $\hat{a}$ is now a sample from $N(0, Q_{\hat{a}})$. Using this sample to solve (4) results in the corresponding integer least-squares sample. By repeating this process an N-number of times, one obtains a collection of N integer vectors. Of this collection one can now infer how often a particular grid point, say z, is visited. This gives the frequency N_z. An approximation to the required probability masses follows then from the relative frequencies. Thus

$$\boxed{P(\check{a} = z) \approx \frac{N_z}{N}} \tag{17}$$

Successful ambiguity resolution can now be expected feasible when the probability $P(\check{a} = 0)$ is sufficiently close to one. Note that this procedure requires that problem (4) has to be solved N-times. For large N, this becomes a very time consuming task if not an efficient search is in place for solving the integer least-squares problem. This shows that the simulation should not be based on the original DD ambiguities, but instead on the transformed ambiguities obtained by means of the LAMBDA method [*Teunissen*, 1993], [*Jonge de and Tiberius*, 1996]. The integer normal distribution of the transformed ambiguities differs of course from the one of the DD ambiguities. For the DD ambiguities for instance, the nonzero probabilities $P(\check{a} = z)$, with $z \neq 0$, will be more spread out. However, since there is a one-to-one correspondence between the two distributions, the probability of correct integer estimation will be the same for both distributions.

In order to get an idea of how large N should be taken in the simulation, we consider the probability that N_0 out of N integer vectors equal the zero vector. If the N samples are drawn independently from the normal distribution $N(0, Q_{\hat{a}})$, then this probability is governed by the binomial distribution and is given as

$$P(N_0) = \frac{N!}{(N - N_0)!N_0!} P_0^{N_0} (1 - P_0)^{N - N_0}$$

where we made use of the abbreviation $P_0 = P(\check{a} = 0)$. The mean (expectation) and variance (dispersion) of the relative frequency N_0/N follow therefore as

$$E\{N_0/N\} = P_0 \quad \text{and} \quad D\{N_0/N\} = P_0(1-P_0)/N$$

Note that the first expression is in fact the motivation for using the relative frequency as an estimator for P_0, the probability of correct integer estimation. The second expression gives the precision of this estimator. It depends on both P_0 and N.

Using the above mean and variance we may now apply the Chebyshev inequality to obtain an upperbound on the probability that the relative frequency N_0/N differs more than ϵ from P_0. The corresponding Chebyshev inequality reads

$$P(\mid \frac{N_0}{N} - P_0 \mid \geq \epsilon) \leq \frac{P_0(1-P_0)}{N\epsilon^2} \tag{18}$$

The required number of samples N can be obtained by setting both ϵ and the upperbound to a small enough value. For instance, when the probability of correct integer estimation equals $P_0 = 1 - 10^{-3}$, an upperbound of one percent and a deviation of $\epsilon = 10^{-3}$ leads to a required number of samples of $N = 10^5$. This shows that in general a large number of samples are needed to get a sufficiently precise estimate of the probability of correct integer estimation. Instead of using the above Chebyshev inequality, one may also use the Gaussian approximation for the binomial distribution to obtain an estimate of the required number of samples, when N is large. This will usually give a somewhat less conservative estimate of N.

Instead of using a random generator to obtain samples of the integer normal distribution, one may of course apply the same idea to actual 'real-world' experiments. In that case the experiment will consist of repeatedly estimating the integer ambiguities, while keeping count of the success rates. Such a study was performed in [*Jonkman*, 1998] for the geometry-free model and in [*Tiberius and de Jonge*, 1995] for the geometry-based GPS model.

5.2 Probability of the bootstrapped estimator

Instead of using simulation, one may also try to formulate bounds on the probability of correct integer least-squares estimation. One such bound is obtained if we consider the probability of a less optimal integer estimator, the bootstrapped estimator.

As it was remarked earlier, it is difficult in general to evaluate the integer normal distribution (10) exactly. The evaluation becomes relatively simple though when $\hat{a}$ is a scalar or when $Q_{\hat{a}}$ is diagonal. In the scalar case the integer least- squares estimator coincides with the operation 'round to the nearest integer'. For this case the integer normal distribution takes the form

$$P(\check{a} = i) = \int_{(i-a)-\frac{1}{2}}^{(i-a)+\frac{1}{2}} \frac{1}{\sigma_{\hat{a}}\sqrt{2\pi}} \exp\{-\frac{1}{2}(x/\sigma_{\hat{a}})^2\} dx \tag{19}$$

where i ranges over the set of integers. For the purpose of ambiguity resolution we are particularly interested in the probability of correct integer estimation. This probability is given as $P(\check{a} = a) = P(\mid \hat{a} - a \mid \leq \frac{1}{2})$, which can be evaluated by means of (19). The probability of correct integer estimation becomes then

$$P(\check{a} = a) = 2\Phi(\frac{1}{2\sigma_{\hat{a}}}) - 1 \;\; \text{with} \;\; \Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}z^2\} dz \tag{20}$$

In the multivariate case one can still use the 'rounding operation' to obtain the integer least-squares solution, provided the variance- covariance matrix is diagonal. In that case the problem decouples into n scalar problems of the above type. Hence, the probability of correct integer estimation becomes then

$$P(\check{a} = a) = \Pi_{i=1}^{n}[2\Phi(\frac{1}{2\sigma_{\hat{a}_i}}) - 1] \tag{21}$$

In the actual practice of GPS, the ambiguity variance-covariance matrix is of course fully populated and therefore nondiagonal. Expression (21) does therefore not apply. However, a similar expression can be obtained if we consider the so-called bootstrapped estimator. This integer estimator follows from a sequential conditional least- squares adjustment and is computed as follows. If n ambiguities are available, one starts with the first ambiguity $\hat{a}_1$, and rounds its value to the nearest integer. Having obtained the integer value of this first ambiguity, the real-valued estimates of all remaining ambiguities are then corrected by virtue of their correlation with the first ambiguity. Then the second, but now corrected, real-valued ambiguity estimate is rounded to its nearest integer. Having obtained the integer value of the second ambiguity, the real-valued estimates of all remaining $n-2$ ambiguities are then again corrected, but now by virtue of their correlation with the second ambiguity. This process is continued until all ambiguities are taken care of. Since the components of this bootstrapped estimator are conditionally independent, it follows that its probability of correct integer estimation takes the form of (21), but now with the unconditional standard deviations replaced by their sequential conditional counterparts. Thus if we denote the bootstrapped estimator as $\check{a}_B$, we have

$$\boxed{P(\check{a}_B = a) = \Pi_{i=1}^{n}[2\Phi(\frac{1}{2\sigma_{\hat{a}_{i|(i-1),\ldots,1}}}) - 1]} \tag{22}$$

where $\sigma_{\hat{a}_{i|(i-1),\ldots,1}}$ denotes the *conditional* standard deviation. It can be shown that this probability provides an upperbound on the probability of correct integer estimation based on the simple 'componentwise rounding' mechanism, i.e. $P(\check{a}_B = a) \geq P(\cap_{i=1}^{n} \mid \hat{a}_i - a_i \mid \leq \frac{1}{2})$, see [*Teunissen*, 1998a]. The bootstrapped solution will thus more often lead to the correct integer ambiguities than the solution based on the 'componentwise rounding'. In fact, when (22) is sufficiently close to one, the bootstrapped estimator becomes a viable alternative to the integer least-squares estimator. Note however that (22) depends on the conditional standard deviations of the individual ambiguities. Since the first three conditional standard deviations of the DD ambiguities are known to be large in general [*Teunissen*, 1996], the probability $P(\check{a}_B = a)$ can expected to be small when using DD ambiguities. This shows that one should first apply the decorrelation process of the LAMBDA method, before using (22) to evaluate the probability of correct integer estimation.

5.3 Bounding the ambiguity variance matrix

Another way to get a grip on the probability of correct integer estimation, while at the same time avoiding the complicated integration of (10), is to make use of scaled unit matrices as variance matrices. We know that the computation of the probability of correct integer estimation becomes straightforward when the ambiguity variance matrix is diagonal. That is, when the least-squares ambiguities are fully decorrelated. This suggests that we bound the actual ambiguity variance matrix from above and below by diagonal matrices, and compute the probability of correct integer estimation that would belong to these diagonal matrices. The simplest way of bounding the actual ambiguity variance matrix from above and below, is to make use of its maximum and minimum eigenvalue. This gives

$$\lambda_{min} I_n \leq Q_{\hat{a}} \leq \lambda_{max} I_n$$

Using these bounds, the corresponding bounds for the probability of correct integer estimation reads

$$\boxed{[2\Phi(\frac{1}{2\sqrt{\lambda_{max}}}) - 1]^n \leq P(\check{a} = a) \leq [2\Phi(\frac{1}{2\sqrt{\lambda_{min}}}) - 1]^n} \tag{23}$$

Note that the two bounds coincide when the two extreme eigenvalues coincide. This is the case when the ambiguity variance matrix itself is a scaled unit matrix. In the actual practice of GPS this will not happen. In fact, the two extreme eigenvalues will differ considerably when the variance matrix of the DD ambiguities is used. In that case the above two bounds would become too loose to be useful. When using the decorrelated ambiguities as produced by the LAMBDA method, the elongation of the ambiguity search space is considerably reduced and the ratio of the two extreme eigenvalues is pushed towards its minimum of one. Hence, the above bounds are much sharper when using the eigenvalues of the transformed ambiguity variance matrix, than when using the eigenvalues of the original DD ambiguity variance matrix.

5.4 Bounding the region of integration

Another way to get a grip on the probability of correct integer estimation is to replace the original region of integration S_a by a subset $L_a \subset S_a$ and by an enclosing set $U_a \supset S_a$. In that case the probability of correct integer estimation lies in the interval

$$\boxed{P(\hat{a} \in L_a) \leq P(\check{a} = a) = P(\hat{a} \in S_a) \leq P(\hat{a} \in U_a)} \tag{24}$$

Both regions of integration should of course be chosen such that the corresponding probabilities are easily evaluated in practice. The probability $P(\hat{a} \in L_a)$ can then be used to infer whether ambiguity resolution can expected to be successful, while the probability $P(\hat{a} \in U_a)$ will show when one can expect successful ambiguity resolution to fail. Thus there is enough confidence that ambiguity resolution will be successful when the lowerbound is sufficiently close to one, while no such confidence exists when the upperbound turns out to be too small.

In the following we will use the minimum grid point distance, or minimum norm, to bound the region of integration [*Teunissen*, 1997b]. The smallest distance between two integer grid points, as measured in the metric of the ambiguity variance matrix, will correspond with the situation that one has the greatest difficulty in discriminating between two grid points. Hence, if this distance is large enough one can expect ambiguity resolution to be succesful. On the other hand, succesful ambiguity resolution will become problematic if this distance is too small.

5.4.1 The probability $P(\hat{a} \in U_a)$

Let us first consider the geometry of the pull-in-region S_a. Since

$$\| \hat{a} - a \|^2_{Q_{\hat{a}}} \leq \| \hat{a} - z \|^2_{Q_{\hat{a}}} \Longleftrightarrow (z-a)^T Q_{\hat{a}}^{-1}(\hat{a}-a) \leq \frac{1}{2} \| z - a \|^2_{Q_{\hat{a}}} \quad , \forall z \in Z^n$$

it follows that

$$S_a = \{\hat{a} \in R^n \mid \mid w_i \mid \leq \frac{1}{2} \| c_i \|_{Q_{\hat{a}}}, \forall c_i \in Z^n\} \text{ with } w_i = \frac{c_i^T Q_{\hat{a}}^{-1}(\hat{a}-a)}{\sqrt{(c_i)^T Q_{\hat{a}}^{-1}(c_i)}} \tag{25}$$

Note that w_i is the well-known w-test statistic for testing one- dimensional alternative hypotheses [*Baarda*, 1968], [*Teunissen*, 1985]. It is distributed as $w_i \sim N(0,1)$. Geometrically, w_i can be interpreted as an orthogonal projector which projects $(\hat{a}-a)$ onto the direction vector c_i. Hence, S_a is the intersection of the *banded* subsets $\{\hat{a} \in R^n \mid \mid w_i \mid \leq \frac{1}{2} \| c_i \|_{Q_{\hat{a}}}\}$, all centred at a and each having a width $\| c_i \|_{Q_{\hat{a}}}$. Since any *finite* intersection of these banded subsets encloses S_a, the following choice for U_a is suggested

$$U_a = \{\hat{a} \in R^n \mid \mid w_i \mid \leq \frac{1}{2} \| c_i \|_{Q_{\hat{a}}}, i = 1, \ldots, p\} \supset S_a \tag{26}$$

Thus U_a is taken as the intersection of p such bands. Note that the choice for p and for the grid vectors c_i is still left open. The simplest choice would be $p = 1$. In that case the probability is easily evaluated due to the standard normal distribution of w_1. For c_1 one could still take any one of the grid vectors. The best choice in this case would be the nonzero grid vector having shortest length. The probability reads then

$$P(\hat{a} \in U_a) = 2\Phi(\frac{1}{2} \min_{z \in Z^n \setminus \{0\}} \| z \|_{Q_{\hat{a}}}) - 1 \tag{27}$$

This probability can now be used as upperbound for the probability of correct integer estimation. If a sharper upperbound is needed, one will have to choose p larger than 1. The situation becomes then more complicated due to the fact that the w_i are correlated. In order to tackle this case, we first define the p-vector

$$v = (v_1, \ldots, v_p)^T \text{ with } v_i = \frac{w_i}{\| c_i \|_{Q_{\hat{a}}}} \tag{28}$$

Then $U_a = \{\hat{a} \in R^n \mid \cap_{i=1}^p \mid v_i \mid \leq \frac{1}{2}\}$. The probability $P(\hat{a} \in U_a)$ equals therefore the probability that the 'componentwise rounding' of the vector $v = (v_1, \ldots, v_p)^T$ produces the zero vector. Hence, we can now make use of the bootstrapped results of the previous subsection. That is, $P(\hat{a} \in U_a)$ and thus also $P(\check{a} = a)$ will be bounded from above by the probability that the 'sequential rounding' of the entries of v produces the zero vector. Hence,

$$\boxed{P(\check{a} = a) \leq \Pi_{i=1}^p [2\Phi(\frac{1}{2\sigma_{v_{i|(i-1),\ldots,1}}}) - 1]} \tag{29}$$

where $\sigma_{v_{i|(i-1),\ldots,1}}$ denotes the conditional standard deviation of v_i. These sequential conditional standard deviations of v follow from applying an LDL^T-decomposition to the variance-covariance matrix of v. The entries of this variance-covariance matrix Q_v are given as

$$\sigma_{v_i v_j} = \frac{c_i^T Q_{\hat{a}}^{-1} c_j}{\| c_i \|_{Q_{\hat{a}}}^2 \| c_j \|_{Q_{\hat{a}}}^2} \tag{30}$$

After applying the LDL^T-decomposition to this variance-covariance matrix, the sequential conditional variances follow as the diagonal entries of the diagonal matrix D. Note that the variance-covariance matrix needs to be of full rank in order to avoid that some of the conditional variances of v become zero. This implies that the grid vectors c_i, $i = 1, \ldots, p \leq n$, need to be linear independent.

5.4.2 The probability $P(\hat{a} \in L_a)$

As with the choice $U_a \supset S_a$, also a choice for the subset $L_a \subset S_a$ can be made by considering the geometry of the pull-in-region S_a. Note that the widths of the bands that make up the intersection of (25) vary in length. This shows that a subset of S_a can be obtained by replacing all these varying widths by one single width, namely the smallest one. Hence, $\{\hat{a} \in R^n \mid \mid w_i \mid \leq \frac{1}{2} \min_{z \in Z^n \backslash \{0\}} \| z \|_{Q_{\hat{a}}}, \forall z_i \in Z^n\} \subset S_a$. Moreover, since $\mid w_i \mid = \frac{1}{2} \min_{z \in Z^n \backslash \{0\}} \| z \|_{Q_{\hat{a}}}$ describes a pair of opposite *planes of support* of the ellipsoid $\| \hat{a} - a \|_{Q_{\hat{a}}}^2 = (\frac{1}{2} \min_{z \in Z^n \backslash \{0\}} \| z \|_{Q_{\hat{a}}})^2$ [*Teunissen*, 1996], it follows that the ellipsoid

$$L_a = \{\hat{a} \in R^n \mid \| \hat{a} - a \|_{Q_{\hat{a}}}^2 \leq (\frac{1}{2} \min_{z \in Z^n \backslash \{0\}} \| z \|_{Q_{\hat{a}}})^2\} \tag{31}$$

is also a subset of S_a. We therefore have the lowerbound

$$\boxed{P(\check{a} = a) \geq P(\chi^2(n,0) \leq (\frac{1}{2} \min_{z \in Z^n \backslash \{0\}} \| z \|_{Q_{\hat{a}}})^2)} \tag{32}$$

with $\chi^2(n,0)$ the central Chi-square distribution with n degrees of freedom. These probabilities can be computed using [*Johnson et al.*, 1994]

$$P(\chi^2(n,0) \leq x) = \begin{cases} 1 - e^{-x/2} \sum_{i=0}^{\frac{1}{2}(n-2)} (\frac{x}{2})^i / i! & n = \text{even} \\ (2\Phi(\sqrt{x}) - 1) - e^{-x/2} \sum_{i=0}^{\frac{1}{2}(n-3)} (\frac{x}{2})^{i+\frac{1}{2}} /, (i + \frac{3}{2}) & n = \text{odd} \end{cases}$$

Note that for $n = 1$, we have $\chi^2(1,0) = N^2(0,1)$ and $\min_{z \in Z^n \backslash \{0\}} \| z \|_{Q_{\hat{a}}}^2 = 1/\sigma_{\hat{a}}^2$. This shows that for the one-dimensional case, (32) reduces to (20). The upperbound (29) and lowerbound (32) are then also identical.

Note that the minimum grid point distance is used in both the upperbound (27) and lowerbound (32). Computation of this integer least-squares problem can be avoided if one settles for less sharp bounds, using eigenvalues and/or (conditional) variances. The following four bounds can then be used instead,

$$\begin{cases} \frac{1}{\lambda_{max}} & \leq & \min_{z \in Z^n \backslash \{0\}} \| z \|_{Q_{\hat{a}}}^2 & \leq & \frac{1}{4\sigma_{\hat{a}_i}^2}(e + \frac{1}{e})^2 \text{ with } e^2 = \frac{\lambda_{max}}{\lambda_{min}}, i = 1, \ldots, n \\ \frac{1}{\sigma_{\hat{a}_{i|I}}^2} & \leq & \min_{z \in Z^n \backslash \{0\}} \| z \|_{Q_{\hat{a}}}^2 & \leq & \frac{1}{\sigma_{\hat{a}_{n|N}}^2} \text{ when } z_i \neq 0, z_j = 0 \text{ for } j < i \end{cases} \tag{33}$$

with $\sigma^2_{\hat{a}_i}$ and $\sigma^2_{\hat{a}_{i|I}}$ being the ith unconditional and conditional ambiguity variance respectively, and with λ_{min} and λ_{max} being the two extreme eigenvalues of the ambiguity variance matrix. Note, although the minimum norm is invariant for the choice of ambiguity parametrization, that the above bounds depend on the type of ambiguities used.

The first lowerbound follows from the definition of the maximum eigenvalue. The second lowerbound, sharper than the first, follows when the square of the grid point distance is given a sum-of-squares form using the LDU-decomposition and noting that this sum is never smaller than the first nonzero entry in the sum. To use this lowerbound, one still needs to check whether $z_i \neq 0, z_j = 0$ for $j < i$. This can be avoided by taking as lowerbound the smallest reciprocal (conditional) ambiguity variance. For the unconditional variances this can be understood as follows. Since $z \neq 0$, at least one of its entries is nonzero, say z_i. Thus the reciprocal value of $\sigma^2_{\hat{a}_i}$ can be taken as lowerbound. But this shows that certainly the minimum of all reciprocal ambiguity variances can be used as lowerbound. For the conditional variances a similar reasoning applies. The first upperbound follows from applying Kantovorich inequality, see e.g. [*Rao*, 1973], with z chosen as the canonical unit vector having 1 as its ith entry. Finally, the second upperbound follows from taking the length of $z = (1, 0, \ldots, 0)^T$ as upperbound.

6 Summary

In this contribution we introduced the discrete distribution of the integer least-squares ambiguities and called it the *integer normal distribution.* This probability mass function was shown to be symmetric and centred at the integer mean of the real-valued least-squares ambiguity vector $\hat{a}$. It was also shown that the 'fixed' solutions $\check{a}$ and $\check{b}$ are unbiased. Hence, no biases are introduced when collapsing the pull-in-regions S_z to their respective grid points. The maximum of the integer normal distribution coincides with the probability of correct integer estimation. Since this maximum is particularly of relevance for GPS ambiguity resolution, we presented different approaches for evaluating this maximum. It was pointed out that an exact evaluation is rather difficult in general, due to the complicated geometry of the pull-in-regions. The first approach was based on reconstructing the probability mass function by means of a simulation. The second approach made use of the integer bootstrapped estimator. Due to the sequential conditioning on which this estimator is based, the probability of correct integer estimation is easily evaluated for the bootstrapped estimator. The third approach was based on bounding the ambiguity variance matrix. Finally, the last approach was based on geometric bounds for the pull-in-regions, where use was made of the minimum grid point distance.

When comparing the different approaches, a few remarks can be made. The advantage of the first approach, although rather computational intensive, is that the approximations of the probabilities can be obtained in principle with any desired level of accuracy. This is not the case with the approaches that use bounds, although they are often much easier to evaluate. Here one generally depends on the precision of the least-squares ambiguities. The more precise the ambiguities are, the sharper the bounds generally become. The only influence one can exercise on the bounds is by using a proper ambiguity parametrization. That is, the noninvariant bounds can be made sharper by using decorrelated ambiguities instead of DD ambiguities. Note however that not all bounds converge to the exact probability of correct integer estimation in case the ambiguity variance matrix becomes a diagonal matrix. This is only the case for the second and third approach, but not for the last approach.

We emphasize in conclusion that the integer normal distribution is completely specified by the variance-covariance matrix of the real- valued least-squares ambiguities. Hence, all diagnostic measures presented can be computed once this matrix is known. This implies that it is possible to compute the probability of correct integer estimation before the actual measurements are carried out. Only the design matrix and variance-covariance matrix of the GPS observables need to be known. The theory presented in this contribution can therefore be used to analyse different measurement scenarios as to their strength in resolving the integer ambiguities successfully.

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