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A CLASS OF UNBIASED INTEGER GPS AMBIGUITY ESTIMATORS

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ABSTRACT: This contribution introduces a class of integer ambiguity estimators which are unbiased. The condition for unbiasedness is formulated and it is shown that this condition is satisfied for three ambiguity estimators which are often used in GPS ambiguity resolution. They are the 'rounding' estimator, the 'bootstrapped' estimator and the least-squares estimator. The geometry underlying these three integer estimators is also discussed and compared. .

1. INTRODUCTION

GPS ambiguity resolution is the process of resolving the unknown cycle ambiguities of the double-difference (DD) carrier phase data as integers. Although the GPS research in the last two decades or so resulted in a variety of different methods and proposals for ambiguity resolution, the development of a rigorous theory for integer ambiguity estimation and validation stayed somewhat behind. This can be seen when considering the pitfalls that are present in some of the proposed procedures. For a discussion see [*Teunissen*, 1997].

Fortunately in the last five years or so, good progress has been made in developing rigorous and efficient procedures for ambiguity estimation. There is however still some work to do in the area of qualifying the stochastic properties of the integer ambiguities. The purpose of this contribution is to focuss on one of these properties, namely the expectation of the integer estimator. For that purpose we first introduce the many-to-one map that defines an integer estimator. We then formulate conditions which lead to unbiased integer estimators. This shows that there exists a whole class of integer ambiguity estimators which are unbiased. Each member of this class will thus also produce 'fixed' baselines which are unbiased.

Using the conditions for unbiasedness we also show that three of the more frequently used ambiguity estimators are unbiased. They are the 'rounding' estimator, the 'bootstrapped' estimator and the least-squares estimator. The geometry of these estimators is also identified and compared.

2. GPS AMBIGUITY RESOLUTION

Let the GPS model of linear(ized) observation equations be given as

$$E\{y\} = Aa + Bb\ ,\ D\{y\} = Q_y \tag{1}$$

with $E\{.\}$ the expectation operator and $D\{.\}$ the dispersion operator. The m-vector y is the given data vector, while a and b are the unknown parameter vectors. Matrices A and B are the corresponding design matrices. The data vector will usually consist of the 'observed minus computed' single- or dual frequency double-differenced (DD) phase and/or pseudo range (code) observations, accumulated over all observation epochs. The entries of the n-vector a are the unknown DD carrier phase ambiguities, expressed in units of cycles rather than range. They are known to be integer valued. The entries of the p-vector b consists of the remaining unknown parameters, such as for instance baseline components (coordinates) and possibly atmospheric delay parameters (troposphere, ionosphere).

Within the context of GPS ambiguity resolution, the usual steps in solving (1) are as follows. First the model is solved without taking the integer ambiguity constraints into account. As a result one obtains the so called 'float' solution and corresponding variance-covariance matrix

$$\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix}\ ,\ \begin{bmatrix} Q_{\hat{a}} & Q_{\hat{a}\hat{b}} \\ Q_{\hat{b}\hat{a}} & Q_{\hat{b}} \end{bmatrix} \tag{2}$$

The real-valued ambiguities of this solution are then used in the second step to come up with their corresponding 'most likely' integer values. It is thus in this step that the integer constraints on the ambiguities are taken into account. The solution vector of integer ambiguities will be denoted as $\check{a}$. The idea of the final and third step is to use these integer ambiguities to adjust the 'float' solution $\hat{b}$. That is, the solution of the first step is corrected by means of the residual vector $\hat{a} - \check{a}$. The final 'fixed' solution so obtained reads then

$$\check{b} = \hat{b} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} (\hat{a} - \check{a}) \tag{3}$$

The goal of the present contribution is not to discuss the intricacies of the above computational steps, but to concentrate instead on the mapping from $\hat{a}$ to $\check{a}$. Since $\hat{a}$ is a random vector, the integer vector $\check{a}$ is random too. The probabilistic properties of this integer vector will depend on how the mapping from $\hat{a}$ to $\check{a}$ is defined. This therefore also holds true for its first moment, the expectation. Once we know the expectation of $\check{a}$, we also know the expectation of $\check{b}$. The 'float' estimators, $\hat{a}$ and $\hat{b}$, are namely known to be unbiased. Hence,

$$E\{\check{b}\} = b - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} (a - E\{\check{a}\}) \tag{4}$$

The 'fixed' solution $\check{b}$ is thus unbiased once the integer estimator $\check{a}$ is unbiased. In this contribution we will therefore concentrate on the first moment of $\check{a}$ and show how unbiased estimators of the integer ambiguities can be constructed.

3. CONSTRUCTING UNBIASED INTEGER ESTIMATORS

3.1 The integer distribution

The problem of integer ambiguity estimation is to formulate a mapping from the real-valued ambiguities to their integer-valued counterparts

$$\hat{a} \in R^n \Longrightarrow \check{a} \in Z^n \tag{5}$$

Mapping the continuous space of reals to the discrete space of integers will require a many-to-one map and not a one-to-one map. Thus we could think of partitioning R^n such that each of its subsets is assigned to a grid point of Z^n. The estimation rule would then boil down to choosing that particular grid point for which the float solution lies in its subset. Thus to each grid point of Z^n, say z, we assign a nonempty subset $S_z \subset R^n$ and when $\hat{a}$ lies in this subset then z is chosen as the integer solution. Hence

$$\hat{a} \in S_z \Longleftrightarrow \check{a} = z \tag{6}$$

With this formulation, the choice of integer estimator becomes equivalent to the choice of which subsets S_z to take for the partitioning of R^n. It seems reasonable to choose the subsets such that their overlap is empty. Otherwise one can not assign $\hat{a}$ to a single grid point. Thus

$$S_{z_i} \cap S_{z_j} = \{0\} \;\text{ for }\; z_i \neq z_j \tag{7}$$

Note that technically speaking the subsets are allowed to have common borders as long as the probability that $\hat{a}$ lies on the border is zero. Since $\hat{a} \in R^n$, a second requirement of the subsets is that they should cover R^n completely. The union of subsets should therefore cover the n-dimensional space completely,

$$R^n = \cup_{z \in Z^n} S_z \tag{8}$$

We are now in the position to formulate, at least in a conceptual way, the distribution of the integer estimator $\check{a}$. This distribution is of the *discrete* type and it will be denoted as $P(\check{a} = z)$. It is a probability mass function, having zero masses at nongrid points and nonzero masses at some or all grid points. If we denote the *continuous* probability density function of $\hat{a}$ as $p_{\hat{a}}(x)$, the distribution of $\check{a}$ follows as

$$\boxed{P(\check{a} = z) = \int_{S_z} p_{\hat{a}}(x)dx \;,\; z \in Z^n} \tag{9}$$

This expression holds for an arbitrary distribution of the 'floated' ambiguities $\hat{a}$. In the application of GPS ambiguity resolution however, the vector of observables y is usually assumed to be normally (Gaussian) distributed with the mean and dispersion given in (1). As a result the ambiguities of the 'float' solution are normally distributed too, with mean a and variance-covariance matrix $Q_{\hat{a}}$. The probability density function of $\hat{a}$ reads therefore

$$p_{\hat{a}}(x) = \frac{1}{\sqrt{det(Q_{\hat{a}})}(2\pi)^{\frac{1}{2}n}} \exp\{-\frac{1}{2} \parallel x - a \parallel^2_{Q_{\hat{a}}}\} \tag{10}$$

with the squared weighted norm $\parallel \cdot \parallel^2_{Q_{\hat{a}}} = (.)^T Q_{\hat{a}}^{-1}(.)$. In the following we will make use of some of the properties of this distribution.

3.2 The expectation of the integer estimator

We will now try to formulate a set of conditions, which, when fulfilled by the subsets S_z, will result in an unbiased estimator $\check{a}$. The approach taken is as follows. First we will verify that the symmetry of the probability mass function about the mean of $\hat{a}$ is sufficient for the integer estimator to be unbiased. Then we will show that the probability mass function becomes symmetric about the mean a, when all subsets S_z are reflection-symmetric about this mean.

Let us start with the expectation of $\check{a}$. By definition it is given as

$$E\{\check{a}\} = \sum_{z \in Z^n} z P(\check{a} = z) \tag{11}$$

A sufficient condition for the estimator $\check{a}$ to be unbiased, is the symmetry of its distribution about a. Thus

$$P(\check{a} = a + z) = P(\check{a} = a - z)\ ,\ \forall z \in Z^n \Longrightarrow E\{\check{a}\} = a \tag{12}$$

This can be shown as follows. The expectation may also be written as $E\{\check{a}\} = \sum(a + z)P(\check{a} = a + z)$ and as $E\{\check{a}\} = \sum(a - z)P(\check{a} = a - z)$. Taking the sum of the last two expressions and using the symmetry of the distribution gives $2E\{\check{a}\} = \sum 2aP(\check{a} = a + z) = 2a$, from which the unbiasedness follows.

We will now show that the probability mass function is indeed symmetric about a, when all subsets S_z are *reflection-symmetric* about this same mean. Thus

$$\boxed{\begin{array}{c}\text{Let } R \text{ be a reflection about } a \text{ and } R(S_{a+z}) = S_{a-z}\ ,\ \forall z \in Z^n \\ \text{then } P(\check{a} = a + z) = P(\check{a} = a - z)\ ,\ \forall z \in Z^n\end{array}} \tag{13}$$

In order to prove (13), we will make use of the transformation formula for integrals [*Fleming*, 1997]. When applying transformation $R : y = 2a - x$, we get

$$\int_{S_{a+z}} c.\exp\{-\frac{1}{2} \parallel x - a \parallel^2_{Q_{\hat{a}}} dx = \int_{R(S_{a+z})} c.\exp\{-\frac{1}{2} \parallel a - y \parallel^2_{Q_{\hat{a}}} dy$$

where c is the proportionality constant of the multivariate normal distribution. The left-hand side equals $P(\check{a} = a + z)$, while the right-hand side equals $P(\check{a} = a - z)$ because of the reflection-symmetric property of both the subsets and the multivariate normal distribution. This proves (13).

As an example of an unbiased estimator, consider the one- dimensional, zero-mean case, $n = 1, a = 0$. If we define the subsets as

$$S_z = \begin{cases} [z - x, z + 1 - x) & z > 0 \\ (x - 1, 1 - x) & z = 0 \\ (x - 1 + z, x + z] & z < 0 \end{cases} \tag{14}$$

for any x between zero and one, the corresponding integer estimator is unbiased due to the reflection-symmetric property about the origin.

Although the condition of (13) is sufficient to guarantee unbiasedness, it is not yet a practical condition since the mean a is generally unknown. Hence, in order to make the condition practical we need to strengthen it and require that (13) holds $\forall a \in Z^n$. Thus we require the subsets to be *uniformly* reflection-symmetric. Note that with this extra constraint on the subsets, the subsets automatically become reflection-symmetric about their own grid point. To see this, take $z = 0$ in (13) and observe that the condition is assumed to hold for all $a \in Z^n$. Thus $R(S_a) = S_a$ for all grid points a. Note that the subsets of (14) do not satisfy this strengthened condition of reflection-symmetry, unless $x = \frac{1}{2}$. An example of subsets that do, is

$$S_z = \begin{cases} (z - x, z + x) & z \text{ odd} \\ (z - 1 + x, z + 1 - x) & z \text{ even} \end{cases} \tag{15}$$

for any x between zero and one.

4. THREE UNBIASED ABIBUITY ESTIMATORS

In this section we will consider three different estimators of the integer ambiguities. They are the 'rounding' estimator, the 'bootstrapped' estimator and the least-squares estimator. We will show that all three estimators are unbiased.

4.1 Integer rounding

The simplest way to obtain an integer vector from the real-valued 'float' solution is to round each of the entries of $\hat{a}$ to its nearest integer. This solution follows from minimizing the unweighted norm of $\hat{a} - a$. The solution to

$$\min_{a \in Z^n} \sum_{i=1}^{n} (\hat{a}_1 - a_1)^2 + \ldots + (\hat{a}_n - a_n)^2 \tag{16}$$

reads therefore $\check{a}_R = ([\hat{a}_1], \ldots, [\hat{a}_n])^T$, where '[.]' denotes rounding to the nearest integer. Note that the same solution is obtained when the 2-norm in (16) is replaced by the 1-norm.

Since componentwise rounding implies that each real-valued ambiguity estimate $\hat{a}_i, i = 1, \ldots, n$, is mapped to its nearest integer, the absolute value of the difference between the two is at most $\frac{1}{2}$. The subsets $S_{R,z}$ that belong to this integer estimator are therefore given as

$$\boxed{S_{R,z} = \cap_{i=1}^{n} \{\hat{a} \in R^n \mid \mid \hat{a}_i - z_i \mid \leq \frac{1}{2}\} \ , \ \forall z \in Z^n} \tag{17}$$

The subset $S_{R,z}$ is an n-dimensional cube, centred at the grid point z and which has all sides of length 1. The subsets are clearly reflection-symmetric. We therefore have

$$\boxed{E\{\check{a}_R\} = E\{\hat{a}\} = a \text{ and } E\{\check{b}_R\} = E\{\hat{b}\} = b} \tag{18}$$

The integer estimator $\check{a}_R$ is thus indeed an unbiased estimator.

4.2 Integer bootstrapping

The bootstrapped estimator can be seen as a generalization of the previous estimator. It still makes use of integer rounding, but it also takes some of the correlation between the ambiguities into account. The bootstrapped estimator follows from a sequential conditional least- squares adjustment and it is computed as follows. If n ambiguities are available, one starts with the first ambiguity $\hat{a}_1$, and rounds its value to the nearest integer. Having obtained the integer value of this first ambiguity, the real-valued estimates of all remaining ambiguities are then corrected by virtue of their correlation with the first ambiguity. Then the second, but now corrected, real-valued ambiguity estimate is rounded to its nearest integer. Having obtained the integer value of the second ambiguity, the real-valued estimates of all remaining $n-2$ ambiguities are then again corrected, but now by virtue of their correlation with the second ambiguity. This process is then continued until all ambiguities are taken care of. The components of the bootstrapped estimator $\check{a}_B$ are given as

$$\begin{array}{lllll} \check{a}_{B,1} & = & [\hat{a}_1] & & \\ \check{a}_{B,2} & = & [\hat{a}_{2|1}] & = & [\hat{a}_2 - \sigma_{\hat{a}_2\hat{a}_1}\sigma_{\hat{a}_1}^{-2}(\hat{a}_1 - \check{a}_{B,1})] \\ & \vdots & & & \\ \check{a}_{B,n} & = & [\hat{a}_{n|N}] & = & [\hat{a}_n - \sum_{i=1}^{n-1}\sigma_{\hat{a}_n\hat{a}_{i|I}}\sigma_{\hat{a}_{i|I}}^{-2}(\hat{a}_{i|I} - \check{a}_{B,i})] \end{array} \quad (19)$$

where the shorthand notation $\hat{a}_{i|I}$ stands for the ith least-squares ambiguity obtained through a conditioning on the previous $I = \{1,\ldots,(i-1)\}$ sequentially rounded ambiguities.

As it was shown in [*Teunissen*, 1996], the sequential conditional least-squares solution is closely related to the triangular decomposition of the ambiguity variance-covariance matrix. Let the LDU- decomposition of the variance-covariance matrix be given as $Q_{\hat{a}} = LDL^T$, with L a unit lower triangular matrix and D a diagonal matrix. Then $(\hat{a} - z) = L(\hat{a}' - z)$, where $\hat{a}'$ denotes the conditional least-squares solution obtained from a sequential conditioning on the entries of z. The variance-covariance matrix of $\hat{a}'$ is given by the diagonal matrix D. This shows, when a componentwise rounding is applied to $\hat{a}'$, that z is the integer solution of the bootstrapped method. Hence, if c_i denotes the ith canonical unit vector having a 1 as its ith entry, the subsets $S_{B,z}$ that belong to the bootstrapped estimator follow as

$$\boxed{S_{B,z} = \cap_{i=1}^{n}\{\hat{a} \in R^n \mid \mid c_i^T L^{-1}(\hat{a} - z) \mid \leq \frac{1}{2}\}\ ,\ \forall z \in Z^n} \quad (20)$$

Note that these subsets reduce to the ones of (17) when L becomes diagonal. This is the case when the ambiguity variance- covariance matrix is diagonal. In that case the two integer estimators $\check{a}_R$ and $\check{a}_B$ are identical.

It follows from (20) that the subsets $S_{B,z}$ are reflection-symmetric. Thus also the bootstrapped estimator is an unbiased estimator and we have

$$\boxed{E\{\check{a}_B\} = E\{\hat{a}\} = a \text{ and } E\{\check{b}_B\} = E\{\hat{b}\} = b} \quad (21)$$

4.3 Integer least-squares

Least-squares problems in which some of the parameters are integer-valued are non-standard. This type of least-squares problem was first introduced in [*Teunissen*, 1993] and has been coined with the term '*integer least-squares*'. In the context of GPS ambiguity resolution, the integer least-squares ambiguities follow from solving

$$\min_{z \in Z^n} \| \hat{a} - z \|_{Q_{\hat{a}}}^2 \quad (22)$$

Thus $\check{a}_{LSQ}$ is the integer least-squares solution when $\| \hat{a} - \check{a}_{LSQ} \|^2_{Q_{\hat{a}}} \leq \| \hat{a} - z \|^2_{Q_{\hat{a}}}$, for all $z \in Z^n$. This inequality can also be written as an inequality which is linear in $\hat{a}$, namely as $(z - \check{a}_{LSQ})^T Q_{\hat{a}}^{-1}(\hat{a} - \check{a}_{LSQ}) \leq \frac{1}{2} \| z - \check{a}_{LSQ} \|^2_{Q_{\hat{a}}}$. Hence, the subsets $S_{LSQ,z}$ that belong to the least-squares estimator follow as

$$\boxed{S_{LSQ,z} = \cap_{c_i \in Z^n} \{\hat{a} \in R^n \mid \mid w_i \mid \leq \frac{1}{2} \| c_i \|_{Q_{\hat{a}}}\} \ , \ \forall z \in Z^n} \tag{23}$$

with

$$w_i = \frac{c_i^T Q_{\hat{a}}^{-1}(\hat{a} - z)}{\sqrt{(c_i)^T Q_{\hat{a}}^{-1}(c_i)}}$$

Note that the w_i are the well-known w-test statistics for testing one- dimensional alternative hypotheses [*Baarda*, 1968], [*Teunissen*, 1985]. The absolute values of the w_i are thus required to be not larger than the 'critical values' $\frac{1}{2} \| c_i \|_{Q_{\hat{a}}}$. They equal half the distance of a pair of grid points. Geometrically, the w_i can be interpreted as orthogonal projectors which project $(\hat{a} - z)$ onto the direction vectors $c_i \in Z^n$. As with the distance, orthogonality is hereby measured in the metric of $Q_{\hat{a}}$. One can however also describe the subsets of (23) in the ordinary canonical metric. For that purpose we write $\mid w_i \mid \leq \frac{1}{2} \| c_i \|_{Q_{\hat{a}}}$ as $\mid (\frac{1}{2}c_i)^T Q_{\hat{a}}^{-1} x \mid \leq (\frac{1}{2}c_i)^T Q_{\hat{a}}^{-1}(\frac{1}{2}c_i)$ with $x = \hat{a} - z$. Since the direction of the normal vector of the ellipsoid $x^T Q_{\hat{a}}^{-1} x = \chi^2$ is given by $Q_{\hat{a}}^{-1} x$ at x, it follows, when $\chi^2 = (\frac{1}{2}c_i)^T Q_{\hat{a}}^{-1}(\frac{1}{2}c_i)$, that the pair of hyperplanes $(\frac{1}{2}c_i)^T Q_{\hat{a}}^{-1} x = \pm(\frac{1}{2}c_i)^T Q_{\hat{a}}^{-1}(\frac{1}{2}c_i)$ are *tangent planes* of this ellipsoid at the point $x = \frac{1}{2}c_i$. Hence, the single subsets that contribute to the intersection in (23) each cover a region between two such parallel tangent planes. ¿From this it follows that the subsets $S_{LSQ,z}$ are also reflection-symmetric. Therefore,

$$\boxed{E\{\check{a}_{LSQ}\} = E\{\hat{a}\} = a \text{ and } E\{\check{b}_{LSQ}\} = E\{\hat{b}\} = b} \tag{24}$$

which shows that the integer least-squares estimator is also unbiased. It is interesting to observe that apart from their reflection-symmetry, all three subsets $S_{R,z}, S_{B,z}$ and $S_{LSQ,z}$ are also translation-similar. The condition for being translation-similar, $S_z = S_0 + z, \forall z \in Z^n$, is however not implied by (13) and therefore not required for unbiasedness. Still it makes sense to ask the subsets to be translation-similar, because it means that the shape of the probability mass function $P(\check{a} = z)$ is independent of the location of its point of *symmetry. In particular it implies* $\int_{S_a} c.\exp\{-\frac{1}{2} \| x - a \|^2_{Q_{\hat{a}}}\}dx = constant, \forall a \in Z^n$. This makes sense, because it implies that the integer estimator has a probability of correct estimation, $P(\check{a} = a)$, which is independent of the unknown mean a.

To conclude, we compare (23) with (20). First observe that (20) can also be written as

$$S_{B,z} = \cap_{c_i \in Z^n} \{\hat{a} \in R^n \mid \mid c_i^T D^{-1} L^{-1}(\hat{a} - z) \mid \leq \frac{1}{2} c_i^T D^{-1} c_i\} \tag{25}$$

By also using the LDU-decomposition of $Q_{\hat{a}}$ in (23), we may write

$$S_{LSQ,z} = \cap_{c_i \in L^{-1}(Z^n)} \{\hat{a} \in R^n \mid \mid c_i^T D^{-1} L^{-1}(\hat{a} - z) \mid \leq \frac{1}{2} c_i^T D^{-1} c_i\} \tag{26}$$

This shows that the two subsets only differ in their choice of the c_i-vectors. In the bootstrapped case the intersection $\cap_{c_i \in Z^n}$ is taken, while in the least-squares case it is the intersection $\cap_{c_i \in L^{-1}(Z^n)}$. This shows that the two subsets are identical when the matrix entries of L^{-1} are all integer. This is the case when L is an admissible ambiguity transformation [*Teunissen*, 1995].

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