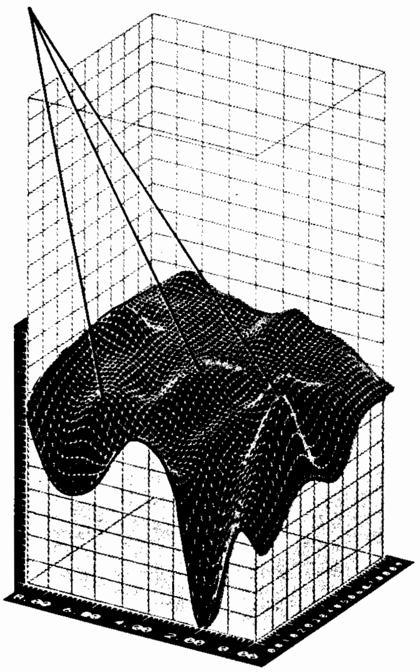

# A New Way to Fix Carrier-Phase Ambiguities

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*Of the two basic GPS observables, the pseudorange and the carrier phase, the carrier phase is by far the more precise. It has, however, an Achilles' heel: the initial measurements of the carrier phases of the signals received by a GPS receiver as it starts tracking the signals are undetermined, or ambiguous, by an integer number of carrier wavelengths. A GPS receiver has no way of distinguishing one carrier cycle from another. The best it can do is measure the fractional phase and then keep track of phase changes. Therefore, the initial unknown ambiguities must be estimated from the GPS data, and the correct estimates must be integers. There lies the rub: what is the best way to determine the correct integer ambiguities? Much research has been performed to find the most efficient, dependable, and accurate way to fix the ambiguities at their correct integer values.*

*In this month's column, we will learn of a new approach for ambiguity fixing: the Least-squares Ambiguity Decorrelation Adjustment method devised by a team of researchers from the Delft Geodetic Computing Centre and the Department of Geodetic Engineering of the Technical University of Delft in The Netherlands. The team members are Dr. Peter Teunissen, professor of geodetic engineering, and research assistants Paul de Jonge and Christian Tiberius.*

*"Innovation" is a regular column in* GPS World *featuring discussions on recent advances in GPS technology and its applications as well as on the fundamentals of GPS positioning. The column is coordinated by Richard Langley and Alfred Kleusberg of the Department of Geodesy and Geomatics Engineering at the University of New Brunswick. We appreciate receiving your comments as well as suggestions of topics for future columns.*

GPS double-difference, carrier-phase measurements are ambiguous by an unknown integer number of cycles. High-precision, relative GPS positions can be obtained from a short time span of data (from seconds to a few minutes) if the integer double-difference ambiguities can be determined efficiently and reliably. We have developed a procedure — the Least-squares Ambiguity Decorrelation Adjustment (Lambda) method — that can quickly and accurately estimate the integer ambiguities. In this article, we review our method and its underlying principles and also present some numerical results illustrating its performance.

## WHY FIX AMBIGUITIES?

High-precision, relative GPS positioning is based on the least-squares adjustment of precise carrier-phase measurements. With short observation time spans, however, if the ambiguities are treated as *real-valued* numbers (or *floating-point* numbers in computer parlance), they are difficult to separate from the receiver baseline components. That is due to the very high-altitude orbits of the GPS satellites, which results in the relative positions of the satellites with respect to the receivers changing very slowly.

A least-squares adjustment that ignores the intrinsic integer nature of the ambiguities would, therefore, produce highly correlated and imprecise ambiguity and baseline estimates. To increase baseline precision, particularly for short observation time spans, we can fix the ambiguities at their integer values; this enables us to treat the carrier-phase measurements as essentially pseudorange measurements. As a result, we can estimate the baseline coordinates with as much precision as the carrier-phase measurements possess.

## INTEGER LEAST SQUARES

To fix the ambiguities at their correct integer values, we first need a criterion that determines which integer values are most likely the correct ones. We generally assume that we have obtained the most-likely real-valued ambiguities from a least-squares adjustment, the result of which is often called the *float* solution. It seems reasonable to consider integers that are *nearest* to the real-valued estimates as most likely being the correct integer values.

As a measure of nearness, we take the weighted sum of squared differences between the real-valued estimates and their integer counterparts. The weighting takes care of the existing correlation and varying precision of the real-valued ambiguity estimates. The mathematical formula for the weighted sum of squares reads:

$$\chi^2(a) = (a - \hat{a})^T Q_{\hat{a}}^{-1}(a - \hat{a}) \quad a \in Z^n \tag{1}$$

in which $\hat{a}$ denotes the vector of the $n$ real-valued, least-squares ambiguities; $Q_{\hat{a}}$ denotes the n-by-n variance–covariance matrix of $\hat{a}$ (giving the uncertainty of each real-valued ambiguity estimate and its correlation with other estimated ambiguities); and $a$ denotes a vector that is allowed to range through the n-dimensional space of integers $Z^n$. The dimension $n$ equals the number of double-difference ambiguities. Hence, when $m$ satellites are tracked at two dual-frequency receiver sites, the total number of double-difference ambiguities, $n$, equals $2(m - 1)$.

In Equation 1, the vector $\hat{a}$ and matrix $Q_{\hat{a}}$ are known; they are obtained from the float solution. The integer vector $a$, however, is unknown. The most-likely integer ambiguity vector is the vector $a$ that minimizes the value of $\chi^2(a)$. We will denote it as $\check{a}$. Because the minimization of $\chi^2(a)$ amounts to a minimization of a sum of squares over the set of integers, we will refer to the solution $\check{a}$ as the *integer least-squares* estimate of the ambiguities.

## AN INEFFICIENT SEARCH

Computing the integer least-squares ambiguities is not easy. Unlike with ordinary (real-valued) least-squares problems, no standard techniques are available for minimizing $\chi^2(a)$; thus, one generally must resort to methods involving a discrete search strategy. As a first step, we can replace the whole space of integers $Z^n$ with a smaller subset that still contains the solution. For the subset, we take all integer vectors $\boldsymbol{a}$ that satisfy the inequality:

$$(\mathbf{a} - \hat{\mathbf{a}})^T \mathbf{Q}_{\hat{a}}^{-1}(\mathbf{a} - \hat{\mathbf{a}}) \leq \chi^2 \quad [2]$$

in which $\chi^2$ is a suitably chosen positive constant that ensures that the subset contains at least one integer vector $\boldsymbol{a}$.

Geometrically, the inequality in Equation 2 describes an n-dimensional hyperellipsoidal region centered on $\hat{\boldsymbol{a}}$. We will refer to this hyperellipsoidal (or just *ellipsoidal* for short) region as the *ambiguity search space.* Its orientation (rotation with respect to the grid axes) and elongation (ratio of the largest axis length to the smallest axis length) are governed by $Q_{\hat{a}}$, and its size is controlled by the value of $\chi^2$. Figure 1 shows a two-dimensional view of the ambiguity search space. As the grid spacing in the figure equals one cycle, the admissible locations for the integer vector $\boldsymbol{a}$ are given by the grid intersections inside the ellipse.

To determine $\check{a}$, we must perform a search through the ellipsoidal region. Different search procedures are possible and have been implemented in analysis software. Unfortunately, they are all inefficient when applied to rotated and extremely elongated search spaces — spaces that are typical for GPS double-differenced, carrier-phase data from short observation sessions. For example, for dual-frequency data collected over a 1-second observation time span, an elongation of the order of $3 \times 10^4$ is not uncommon. Therefore, if the minor axis of the search space is 1 centimeter long, its major axis would be 300 meters long!

**THE IDEAL SITUATION**

To understand how we can lighten the burden of the search, it helps if we first ask ourselves the question, What should the structure of $\chi^2(a)$ be to make the search as efficient as possible? Clearly, the search becomes trivial when all ambiguities are fully decorrelated. In that case, the variance–covariance matrix of the ambiguities, $Q_{\hat{a}}$, is diagonal, and $\chi^2(a)$ reduces to a sum of independent squares. That implies that we can find the minimum of $\chi^2(a)$ by minimizing each of the $n$ individual squares in $\chi^2(a)$ separately. Therefore, the integer least-squares solution follows simply from rounding the individual, real-valued ambiguity estimates to their nearest integers.

A diagonal matrix $Q_{\hat{a}}$ also implies that the axes of the ambiguity search space are aligned with the grid axes. One way we can achieve that alignment is by rotating the search space (see Figure 2). Computationally, that would correspond to what is known as an *eigenvalue decomposition* of $Q_{\hat{a}}$, using the matrix of normalized eigenvectors as the rotation matrix. Unfortunately, such a rotation destroys the integer nature of the transformed ambiguities and cannot be used here.

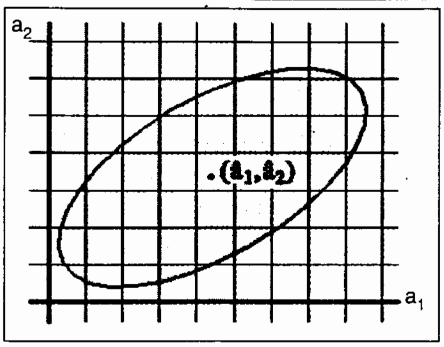



**Figure 1.** In the simplified two-dimensional case, the ambiguity search space is an ellipse centered on the real-valued estimates of the ambiguities $\hat{a}_1$ and $\hat{a}_2$. The grid spacing is one carrier cycle.

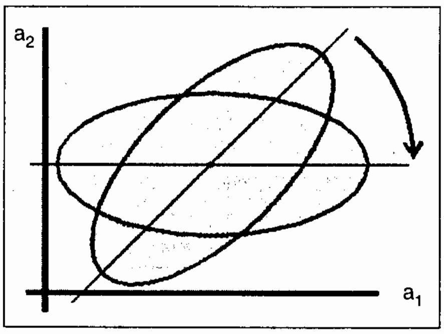



**Figure 2.** If the search space is rotated so that the axes of the search space are parallel to the grid axes, the ambiguities will be fully decorrelated. However, the integer nature of the ambiguities is destroyed in the process.

**DECORRELATED AMBIGUITIES**

Instead of using a rotation of the search space, we can also achieve a full decorrelation of the ambiguities by *squeezing* the search space along the grid axes. Consider the two-dimensional ambiguity search space of original ambiguities $a_1$ and $a_2$. This ellipse will be elongated, and its principal axes will not coincide with the grid axes (see Figure 3a). But by pushing the two horizontal tangents of the ellipse inward, while at the same time keeping the area of the ellipse and its two vertical tangents fixed, we will end up with an ellipse that is perfectly aligned with the grid axes. The transformed ambiguities will then be fully decorrelated, and the transformed ellipse will be less elongated than the original one.

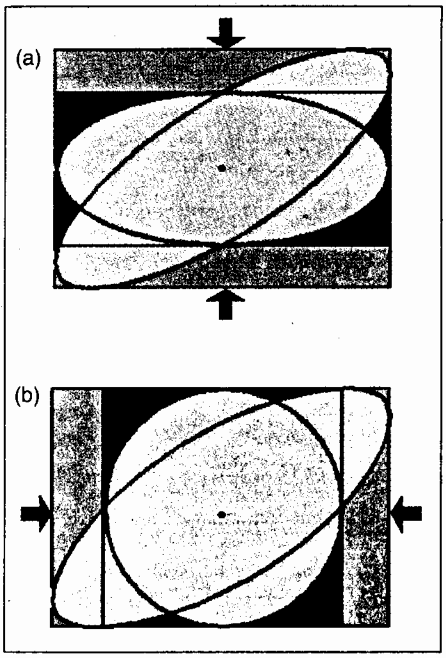


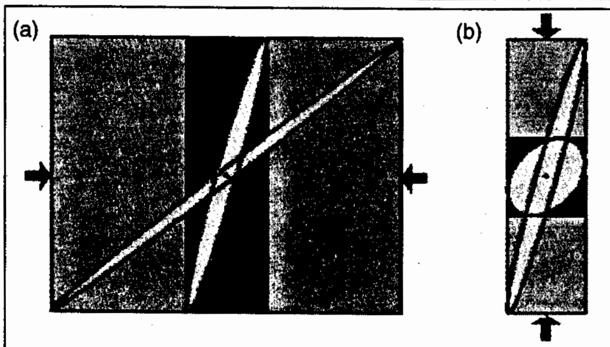

**Figure 3.** Full decorrelation of the ambiguities can also be achieved by *pushing tangents.* In (a), the two horizontal tangents are pushed inward; in (b), the two vertical tangents are pushed inward. This technique does not preserve the integer nature of the ambiguities.

Instead of pushing the two horizontal tangents, we can also work with the two vertical tangents. In that case, the role of the two ambiguities is interchanged. We can obtain full decorrelation of the two ambiguities by pushing the two vertical tangents of the ellipse inward, while keeping the area of the ellipse and the two horizontal tangents fixed (see Figure 3b).

The aforementioned method will generally not preserve the integer nature of the ambiguities, the same failing characteristic of the transformation through rotation. This dilemma indicates the great difficulties in achieving a full decorrelation of the ambiguities, while at the same time preserving their integer nature. Therefore, in practice, we will have to be satisfied with a less-than-perfect result — a nonperfect alignment of the axes of the transformed search space to the grid axes.

Our idea is to use the *integer approximation* of the fully decorrelating transformation. That approach works well, and we can significantly decrease the correlation, although not completely.

The procedure that we follow is based on




| Before | After |
|---|---|
| $Q_{\hat{a}} = \begin{pmatrix} 59.0 & 46.0 \\ 46.0 & 36.0 \end{pmatrix}$ | $Q_{\hat{z}} = \begin{pmatrix} 3.0 & 1.0 \\ 1.0 & 2.9 \end{pmatrix}$ |
| $\rho_{\hat{a}} = 0.998$ | $\rho_{\hat{z}} = 0.350$ |
| $e_{\hat{a}} = 34.4$ | $e_{\hat{z}} = 1.4$ |
| $\sigma_{\hat{a}_1} = 7.7$ | $\sigma_{\hat{z}_1} = 1.7$ |
| $\sigma_{\hat{a}_2} = 6.0$ | $\sigma_{\hat{z}_2} = 1.7$ |

**Figure 4.** A modification of the *pushing tangents* approach guarantees the integer nature of the transformed ambiguities. Shown here are the graphical and numerical results of a simple example with two ambiguities. The tangents are pushed inward in two steps: first the vertical tangents (a), then the horizontal tangents (b).

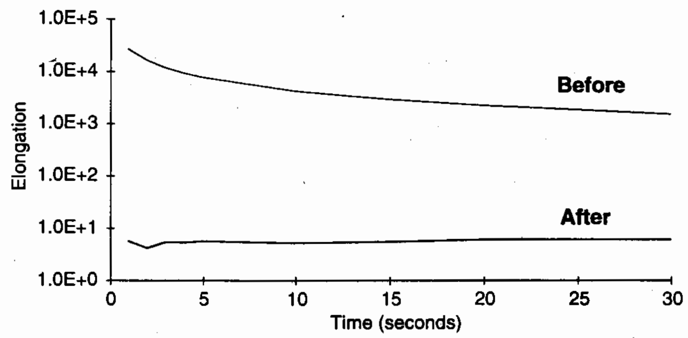



**Figure 5.** Before transformation, the elongation of the ellipsoidal ambiguity search space of a test data set is initially very large and slowly decreases with increasing observation time span. After transformation, the initial elongation is reduced by three orders of magnitude and stays relatively constant with increasing observation time span.

the approach of *pushing tangents,* as depicted in Figure 3. But, to retain the integer nature of the ambiguities, the tangents are not pushed to the limit. Instead, the vertical tangents are pushed inward to a position that guarantees the integer nature of the transformed ambiguities (see Figure 4a). As a result of that transformation, we obtain a less-elongated search space and two transformed ambiguity estimates $\hat{z}_1$ and $\hat{a}_2$ that are less correlated.

Now that $a_1$ has been replaced by $z_1$ through the pushing of vertical tangents, we can continue in an analogous way and replace $a_2$ with $z_2$ by pushing the horizontal tangents (see Figure 4b). As a result, we obtain the transformed ambiguity estimates $\hat{z}_1$ and $\hat{z}_2$, which are much less correlated than the original ambiguity estimates $\hat{a}_1$ and $\hat{a}_2$ and also have an ambiguity search space that is more spherical.

For the two-dimensional example shown in Figure 4, we have also given the variance–covariance matrix of the ambiguities before and after the transformation, $Q_{\hat{a}}$ and $Q_{\hat{z}}$. Note the improvement in precision (given by the standard deviations, σ) and the decrease in both correlation (given by the correlation coefficient, ρ) and elongation (given by the ratio of the lengths of the largest to the smallest axes of the ellipse, *e*). The elongation has been pushed toward its minimum value of 1.0. With the new ambiguity estimates $\hat{z}_1$ and $\hat{z}_2$ and their variance–covariance matrix $Q_{\hat{z}}$, we can now perform the ambiguity search much more efficiently.

### THE N-DIMENSIONAL CASE

When the aforementioned principles are generalized to the n-dimensional case, the decorrelation of the least-squares ambiguities results in the following n-by-n transformation from the original ambiguity vector $\hat{\mathbf{a}}$ to the new ambiguity vector $\hat{\mathbf{z}}$:

$$\hat{\mathbf{z}} = \mathbf{Z}^T\hat{\mathbf{a}} \qquad [3]$$

The variance–covariance matrix of the transformed ambiguities follows from the application of the error-propagation law to Equation 3, resulting in:

$$\mathbf{Q}_{\hat{z}} = \mathbf{Z}^T\,\mathbf{Q}_{\hat{a}}\mathbf{Z} \qquad [4]$$

As a consequence, the original search space represented by Equation 2 is replaced by the transformed search space:

$$(\mathbf{z} - \hat{\mathbf{z}})^T\,\mathbf{Q}_{\hat{z}}^{-1}(\mathbf{z} - \hat{\mathbf{z}}) \leq \chi^2 \qquad [5]$$

We then use the search space represented by Equation 5 to search for the integer least-squares ambiguity vector. $Q_{\hat{z}}$ is much more diagonal than the original variance–covariance matrix $Q_{\hat{a}}$, so this search is much more efficient than the search based on the original search space. Because the decorrelating transformation given by Equation 3 preserves both the volume of the search space and the integer nature of the ambiguities, the original and transformed search spaces contain the same number of grid points. Moreover, the correspondence between the original and transformed ambiguities is one-to-one, making it easy for us to transform the solution back to $\check{a}$ in order to obtain the integer least-squares solution for the original ambiguities.

### TEST RESULTS

We analyzed the performance of our method employing a representative example that utilizes dual-frequency, carrier-phase measurements taken from a seven-satellite configuration. The sampling interval was 1 second. The *a priori* standard deviation of the phase observations was set to 3 millimeters. In Figure 5, the elongation of the ambiguity search space for this data set is given as a function of the observation time span as it ranges from 1 to 30 seconds.

**Reduction in Elongation.** For a 1-second observation time span, the elongation is reduced in the transformation by three orders of magnitude. Also, we can see that the elongation of the transformed search space is nearly independent of the observation time span, whereas the elongation before the transformation decreases with an increase in observation time span. That characteristic is caused by the change in receiver-satellite geometric configuration. Even for a 1-hour observation time span, the elongation before transformation is still more than twice as large as the elongation after transformation.

**Improvement in Precision.** Figure 6 shows the increase in the ambiguities' precision, which occurs from using the transformation of Equation 3. In Figure 6a, the standard deviations, expressed in cycles, are given for the

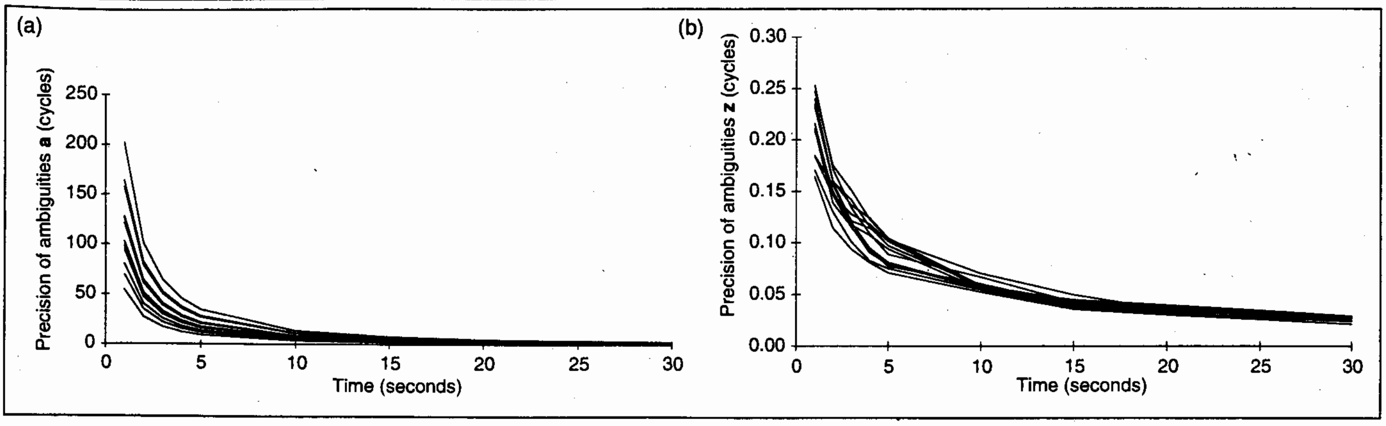



**Figure 6.** The precision of the 12 ambiguities estimated from a test data set before transformation (a) ranges from about 50 to 200 cycles for an observation time span of 1 second. The transformed ambiguities (b) have precisions ranging from 0.1 to 0.25 cycles for the same time span.

12 original double-difference ambiguities. Figure 6b shows the precision of the transformed ambiguities. Note the difference in scale along the vertical axis.

For an observation time span of 1 second, the transformation reduces the standard deviations of the ambiguities from 50–200 cycles to 0.1–0.25 cycles. Again, because of the changing receiver-satellite geometric configuration, the standard deviations decrease as the observation time span increases.

**Efficiency.** The ambiguity search space of the transformed ambiguities $z$, realized by the decorrelating transformation of Equation 3, allowed us to estimate the integer ambiguities very efficiently. To compare the search before and after the decorrelating transformation properly, we searched the full ellipsoid for the 1-second observation time span. It contained two candidate vectors. The computer solved the transformed problem in 10 milliseconds, significantly less time than the more than 3 hours it needed for the original problem.

**A Further Test.** We analyzed another seven-satellite case for its computational aspects. The observation time span was only 1 second. The timing was done on a 33-MHz, 486 personal computer with an optimized search algorithm.

In Table 1, the elongation of the search space, precision of the ambiguity estimates, and the computation times are given for both the before and after transformation cases. The computation time is the time needed for computing the integer least-squares estimates. *Before* transformation refers to the time needed for the search based on the original ambiguities $\boldsymbol{a}$; *after* transformation refers to the search based on the transformed ambiguities $z$. For the latter, the time needed for constructing the transformation matrix $\boldsymbol{Z}$ is included.

**Table 1. Ambiguity-fixing performance characteristics of a seven-satellite test case**

| Characteristic | Before | After |
|---|---|---|
| Elongation | 34,244.0 | 7.5 |
| Precision (cycles): | | |
| minimum | 71.0 | 0.17 |
| maximum | 223.8 | 0.24 |
| Computation time (seconds) | 1,342.8 | 0.064 |

## Further Reading


For a basic introduction to the carrier-phase observable, see

- "The GPS Observables," by R.B. Langley in *GPS World*, Vol. 4, No. 4, April 1993, pp. 52–59.

There is an extensive literature on carrier-phase ambiguity fixing. For an introduction to the subject, see

- "Robust Techniques for Determining GPS Phase Ambiguities," by C. Goad in the *Proceedings of the Sixth International Geodetic Symposium on Satellite Positioning*, held in Columbus, Ohio, in March 1992, pp. 245–254.
- "Instantaneous Ambiguity Resolution," by R. Hatch in *Kinematic Systems in Geodesy, Surveying, and Remote Sensing, Proceedings of IAG Symposium No. 107*, held in Banff, Alberta, Canada, in September 1990, pp. 299–308.

For further details of the authors' Lambda method, see

- "Least-squares Estimation of the Integer GPS Ambiguities," an invited lecture presented by P.J.G. Teunissen in "Section IV, Theory and Methodology," at the International Association of Geodesy General Meeting, held in Beijing, China, in August 1993. (Lecture material available from the author.)
- "A New Method for Fast Carrier Phase Ambiguity Estimation," by P.J.G. Teunissen in the *Proceedings of PLANS '94, the IEEE Position, Location and Navigation Symposium*, held in Las Vegas, Nevada, in April 1994, pp. 562–573.


### CONCLUDING REMARKS

Our Least-squares Ambiguity Decorrelation Adjustment method very quickly estimates integer least-squares ambiguities, particularly for short observation time spans. For example, typical computation times on a 486 personal computer are much less than 1 second. Furthermore, the method can be applied to data obtained from a wide variety of receivers because, in principle, it is independent of whether pseudorange data, in addition to carrier-phase data, are available or whether single- or dual-frequency measurements are used. When dual-frequency data are used, the method can be applied to other types of ambiguities, such as wide-lane ambiguities.

For more details on the Lambda method, readers should consult the references listed in the sidebar. They may also contact the Delft Geodetic Computing Centre by post: Thijsseweg 11, NL-2629 JA Delft, The Netherlands; by fax: 011 +31 (15) 783711; or by e-mail: lgr@tudgv1.tudelft.nl. ■

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