ON THE GPS DOUBLE-DIFFERENCE AMBIGUITIES AND THEIR PARTIAL SEARCH SPACES

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ABSTRACT

The search for the integer least-squares estimates of the double-difference ambiguities usually suffers from inefficiency when short observational time spans are used based on carrier phase data only. In the present contribution the cause for this inefficiency will be discussed on the basis of the partial search spaces of the double-difference ambiguities. In particular some of the characteristics of the spectrum of conditional variances of the double-difference ambiguities will be stressed. These characteristics are typical for the GPS double-difference ambiguities and they are directly related to the structure of the carrier phase model of observation equations. As a result the least-squares estimators of the GPS double-difference ambiguities are highly correlated and their confidence ellipsoid extremely elongated. It will be shown how the LAMBDA-method, of which the principles were introduced in [8], allows one to overcome the drawbacks that are connected to the use of the double-difference ambiguities. The method is based on an integer approximation of the conditional least-squares transformation and it replaces the original double-difference ambiguities by new ambiguities that show a dramatic decrease in correlation and improvement in precision.

1. INTRODUCTION

The computation of the integer least-squares estimates of the GPS double-difference ambiguities is a non-trivial problem if one aims at numerical efficiency. This is particularly true in case of very short observational time spans and in the absence of precise P-code data. The topic of ambiguity fixing has therefore been a rich source of GPS-research over the last decade or so, see e.g. [1-7]. For different applications, the research resulted in effective search algorithms and provided important insights into the various intricacies of the ambiguity fixing problem. Nevertheless, at present times, it is still expedient to seek ways of improving the efficiency of the various search methods.

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This is in particular true for real-time or near real-time applications of GPS. In [8] the author introduced a new method that allows for such a very fast integer least-squares estimation of the ambiguities. The method makes use of an ambiguity transformation that enables one to reformulate the original ambiguity estimation problem as a new problem that is much easier to solve. First numerical results of the method were presented in [9] and [10].

In the present contribution some of the typical characteristics of the GPS double-difference ambiguities that can be seen as the cause for the inefficiency in the search, will be highlighted and discussed. Ofcourse, it is well known since the relative positions of the GPS satellites change only very little with respect to the receiver over short observational time spans, that in such cases the least-squares double-difference ambiguities are generally of a very poor precision. But as it will be shown, it is in particular the shape of the spectrum of conditional variances of the double-difference ambiguities that prohibits one from executing an efficient search for these ambiguities. This shape, which is so typical for GPS, at the same time however allows one through a reparametrization of the ambiguities, to overcome the inefficiency in the search.

2. THE DOUBLE-DIFFERENCE AMBIGUITY SEARCH SPACE

As our point of departure we consider the integer least-squares problem

$$\min_{a}.(\hat{a}-a)^{*}Q_{\hat{a}}^{-1}(\hat{a}-a)\ ,\ a\in Z^{n}\ , \tag{1}$$

in which $\hat{a}$ denotes the n-vector of real-valued least-squares estimates of the double-difference ambiguities and $Q_{\hat{a}}$ the corresponding variance-covariance matrix. Due to the presence of the integer-constraint $a\in Z^{n}$ in (1), there are unfortunately in general no standard techniques available for solving this least-squares problem as they are available for solving ordinary least-squares problems. As a consequence one has to resort to methods that in one way or another make use of a discrete search strategy for finding the integer least-squares solution. The idea is to first restrict the solution space by replacing the space of integers, Z^{n}, by a smaller subset that can be enumerated. This smaller subset is referred to as the double-difference ambiguity search space or simply, ambiguity search space. The ambiguity search space is defined as the set of all $a\in Z^{n}$ that satisfy the ellipsoidal inequality

$$(\hat{a}-a)^{*}Q_{\hat{a}}^{-1}(\hat{a}-a)\leq\chi^{2}\ , \tag{2}$$

in which χ^2 is a suitably chosen positive constant. This ellipsoidal region is centred at $\hat{a}\in R^{n}$ and its orientation and elongation are governed by the ambiguity variance-covariance matrix $Q_{\hat{a}}$. The size of the search space can be controlled through the choice of the positive constant χ^2. It will be assumed that χ^2 has been chosen such that the search space at least contains the sought for integer least-squares solution.

In order to set up a search strategy that makes use of sharp bounds, the quadratic form of (2) is first written as a sum of squares. In order to do so the *sequential conditional least-*

squares principle is applied. It follows [8], if we denote the least-squares estimate of a_i conditioned on the first $(i-1)$-number of ambiguities as $\hat{a}_{i|I}$ and its variance as $\sigma_{\hat{a}_{(i|I,i|I)}}$, that the ellipsoidal inequality (2) can also be written as

$$\sum_{i=1}^{n} (\hat{a}_{i|I} - a_i)^2 / \sigma_{\hat{a}_{(i|I,i|I)}} \leq \chi^2 \; . \tag{3}$$

The sum-of-squares structure in this inequality now allows us to come up with a set of n scalar inequalities that also can be used to characterize the ambiguity search space. This set of inequalities is given by

$$\begin{cases} (\hat{a}_1 - a_1)^2 & \leq \; \sigma_{\hat{a}(1,1)} \chi^2 \\ (\hat{a}_{2|1} - a_2)^2 & \leq \; \sigma_{\hat{a}_{(2|1,2|1)}} [\chi^2 - (\hat{a}_1 - a_1)^2 / \sigma_{\hat{a}_{(1,1)}}] \\ & \vdots \\ (\hat{a}_{n|N} - a_n)^2 & \leq \; \sigma_{\hat{a}_{(n|N,n|N)}} [\chi^2 - \sum_{j=1}^{n-1} (\hat{a}_{j|J} - a_j)^2 / \sigma_{\hat{a}_{(j|J,j|J)}}] \; . \end{cases} \tag{4}$$

Note that this set of inequalities consists of scalar bounds on the *individual* ambiguities. Hence, it can be used to formulate a search strategy for obtaining the sought for integer least-squares solution [8].

3. ON PARTIAL SEARCH SPACES AND THEIR NUMBER OF INTEGER CANDIDATES

The complete set of n scalar inequalities of (4) characterizes the ambiguity search space for all the n-number of ambiguities. But, each one of the first j-number of inequalities of (4), with j varying from 1 up to and including n, may also be seen as characterizing a search space. In this case however, they can be interpreted as characterizing a partial search space. In order to see this consider the following.

If we relax the constraints of (1) by replacing $a \in Z^n$ by the constraints $a_i \in Z$ for $i = 1,\ldots,j$ and $a_i \in R$ for $i = j+1,\ldots,n$, then the integer least-squares problem (1) clearly reduces to

$$\min_{a_{(J)}} . (\hat{a}_{(J)} - a_{(J)})^* Q_{\hat{a}_{(J)}}^{-1} (\hat{a}_{(J)} - a_{(J)}) \; , \; a_{(J)} \in Z^j \; , \tag{5}$$

in which $\hat{a}_{(J)}$ stands for the j-vector $\hat{a}_{(J)} = (\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_j)^*$ and $Q_{\hat{a}_{(J)}}$ stands for its variance-covariance matrix. Hence, instead of solving for the integer least-squares estimates of all the ambiguities we are now solving for the integer least-squares estimates of only the first j-number of ambiguities. It now follows from the sequential conditional least-squares principle that we again can make use of the bounds of (4) to solve for the partial integer least-squares problem (5). That is, the set of bounds consisting of the first j-number of bounds of (4) characterizes the partial ambiguity search space

$$(\hat{a}_{(J)} - a_{(J)})^* Q_{\hat{a}_{(J)}}^{-1} (\hat{a}_{(J)} - a_{(J)}) \leq \chi^2 \ . \qquad (6)$$

Thus, a search based on these first j-number of bounds will allow us to solve for the partial integer least-squares problem (5).

Now that we have defined the concept of the partial ambiguity search space, it is for the purpose of this contribution of interest to count the number of integer ambiguity vectors (or number of integer candidates) that are contained in these n-number of partial ambiguity search spaces. The relevance of knowing how the number of candidates behaves as function of the level j, with j running from 1 up to and including n, is that it gives an indication on how well the search for the integer least-squares solution will perform. For instance, a decreasing function shows that the number of candidates decreases with an increase in level. This implies then, that for a number of integer candidates $a_{(J)} = (a_1, a_2, ..., a_j)^*$ of level j, no integer a_{j+1} can be found such that $a_{(J+1)} = (a_{(J)}^*, a_{j+1})^*$ is an integer candidate of level $j+1$. As a result, the search will exhibit the property of halting [8]. And the likelihood that the search halts will be more pronounced the steeper the decrease in function value is.

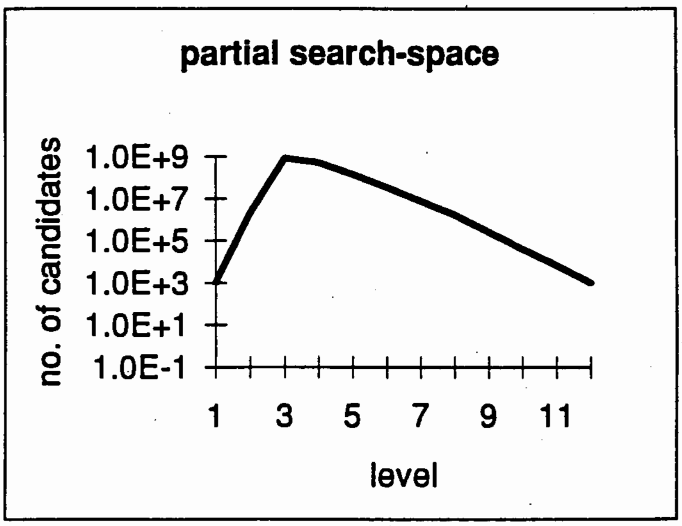

Figure 1: The number of candidates $a_{(J)} = (a_1, a_2, ..., a_j)^*$ per partial search space $j = 1,2,..,12$.

Figure 1 shows a representative plot of the number of candidates as function of the level j, with j running from 1 up to and including $n = 12$. Note the logarithmic vertical scaling. The plot is based on dual-frequency data for a single baseline using carrier-phases only, observing 7 satellites over an observational time span of 1 second, with the a priori standard deviation of the L_1 and L_2 data chosen as $\sigma = 3\text{mm}$ and the χ^2 set at the value of 100. We observe that the function increases from level 1, that it reaches its maximum at level 3 and that it from then on strictly decreases. This behaviour of the function is very typical for the GPS double-difference ambiguities. The fact that the function reaches its maximum at level 3 is because only a single three dimensional baseline is solved for. Due to the sharp decrease in function value when going from level 3 to level 4, the search for the integer least-squares solution of the double-difference ambiguities will unfortunately suffer from a high likelihood of halting. As a result, one will experience in practice that the computational efficiency of finding the integer least-squares estimates of the double-difference ambiguities will be rather poor.

4. WHY IS THE SEARCH FOR THE DOUBLE-DIFFERENCE AMBIGUITIES SO INEFFICIENT?

In the previous section it was shown that in case of GPS the halting problem of the search for the integer least-squares ambiguities is indeed a very serious one. This is particularly true, when the double-difference ambiguity estimates are based on carrier phase data only, collected over a short observational time span. The inefficiency of the search was illustrated by means of the typical behaviour of the function describing the number of integer candidates per partial search space.

Now in order to understand the reason for the inefficiency of the search, we need to understand why the function shown in figure 1 exhibits this typical behaviour. Consider therefore again the scalar inequalities of (4). It will be clear that each one of these inequalities will admit less integer candidates the smaller their respective bounds are. It follows from (4) that these bounds depend on the chosen constant χ^2, on the conditional least-squares estimates $\hat{a}_{i|I}$, on their conditional variances $\sigma_{\hat{a}_{i|I}}$ and on the previously chosen integer candidates. Since the general behaviour of the function shown in figure 1 is typical for all GPS single-baseline solutions, this general behaviour cannot be data-driven but has to be model-driven. Hence, the typical behaviour of the function shown in figure 1 has to be due to the characteristics of the conditional variances $\sigma_{\hat{a}_{i|I}}$, $i=1,\ldots,n$. And indeed, this can be clearly recognized when one considers the spectrum of the conditional variances of the double-difference ambiguities.

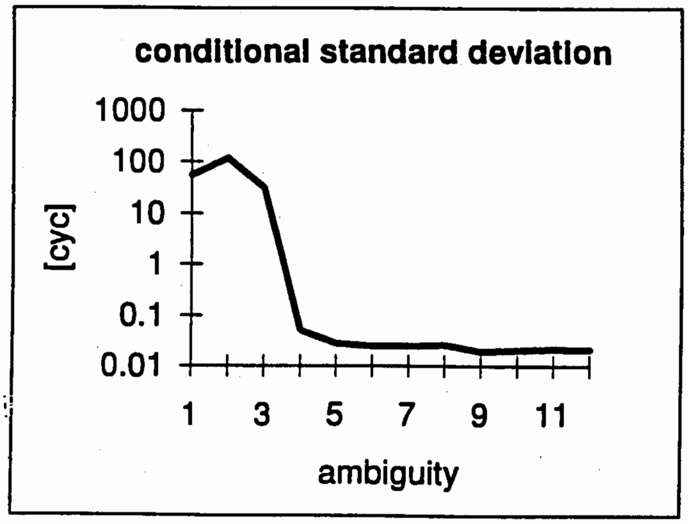

Figure 2: The spectrum of conditional standard deviations of the double-difference ambiguities.

For the same configuration as used in figure 1, figure 2 shows the spectrum of conditional standard deviations of the twelve double-difference ambiguities. The standard deviations are expressed in cycles. Again note the logarithmic scaling along the vertical axis. The figure clearly shows a tremendous drop in value when passing from the third to the fourth standard deviation. There are three large conditional standard deviations and nine extremely small ones. The first three bounds of (4) will therefore be rather loose, while the remaining bounds will be very tight due to the discontinuity in the spectrum. As a consequence the potential of halting will indeed be significant when one passes from the third bound to the fourth bound.

The discontinuity shown in the above spectrum of conditional standard deviations is typical for the GPS *double-difference* ambiguities. It is intrinsically related to the structure of the GPS carrier phase model of observation equations and the chosen parametrization in terms of the double-difference ambiguities.

5. THE LAMBDA-METHOD

In the previous section it was explained why the search for the integer least-squares estimates of the double-difference ambiguities performs so poorly. The method introduced in [8] of the Least-squares AMBiguity Decorrelation Adjustment (LAMBDA), allows for a dramatic improvement in the computational efficiency of estimating the integer ambiguities. In this section the underlying principles of the method are briefly discussed and some numerical results illustrating its performance are presented.

As indicated in the previous section, it is the large discontinuity in the spectrum of conditional variances that prohibits an efficient search for the integer least-squares estimates. The search would therefore improve considerably in efficiency if we would be able to eliminate the discontinuity in the spectrum and lower the values of the large conditional variances. One trivial way of flattening the spectrum would of course be to include more information in the model. This can be reached either through the use of a longer observational time span or through the use of additional GPS observables such as the code observations. It will be shown however, that also without the use of any additional data, a very significant improvement in the spectrum can be reached.

The basic idea that lies at the root of the method is, that integer least-squares estimation of the ambiguities becomes trivial once all the least-squares ambiguities are fully decorrelated. In case of GPS however, the least-squares ambiguities are usually highly correlated and their ambiguity search space is usually extremely elongated. This is particularly true in case of short observational time spans and in the absence of precise *P*-code data. As a consequence the spectrum of conditional standard deviations of the double-difference ambiguities will exhibit a large discontinuity as illustrated in figure 2. The essence of the LAMBDA-method is therefore to aim at a *decorrelation* of the least-squares ambiguities such that the large discontinuity of the spectrum is removed. As a result the original integer least-squares problem is reparametrized such that an equivalent formulation is obtained, but one that is much easier to solve.

In order to explain the underlying principles of the method, we first need to know the admissible class of ambiguity transformations. This is an important issue, since it must be guaranteed with each and any of the reparametrizations applied, that the integerness of the ambiguities remains preserved. In [11] the class of admissible ambiguity transformations was identified. It was shown that ambiguity transformations are admissible if and only if they are *volume preserving* and have entries which all are *integer*.

With the admissible ambiguity transformations identified, we can concentrate on the property of decorrelation. Consider therefore the two-dimensional ambiguity search space as shown in figure 3. The search space is elongated and its principal axes are not aligned with the grid axes. A full decorrelation of the two ambiguities can be reached if we replace the second ambiguity estimate $\hat{a}_2$ by the conditional least-squares estimate $\hat{a}_{2|1}$. Geometrically this can be realized, if we push the two horizontal tangents of the ellipse from their original level inwards to the $\pm(\sigma_{\hat{a}_{2|1,2|1}}\chi^2)^{\frac{1}{2}}$ level. While doing this, we keep the area of the ellipse constant and also the location of the two vertical tangents constant. The ellipse so obtained is less elongated and it has its principal axes aligned to the grid axes.

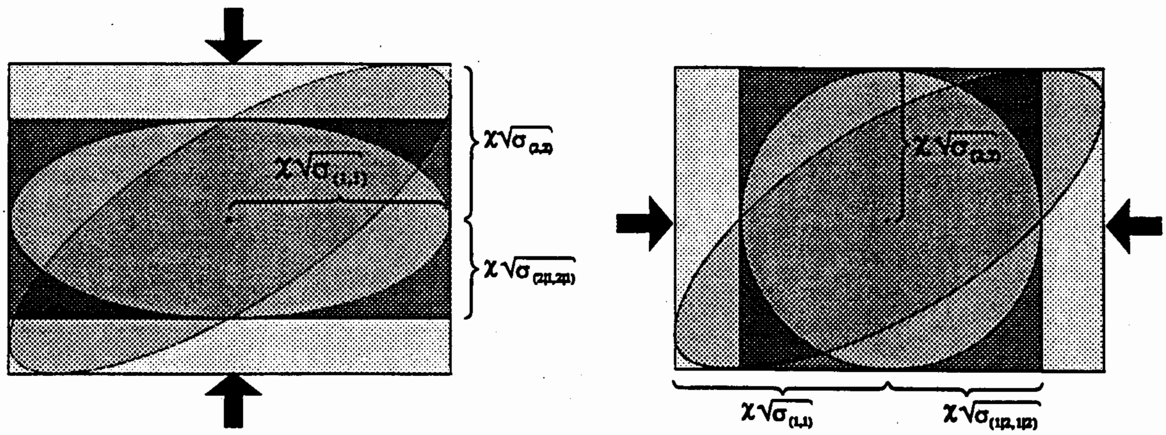

Figure 3: Decorrelating ambiguities by pushing tangents

Although the above transformation guarantees a full decorrelation, it is unfortunately not admissible. The estimate $\hat{a}_{2|1}$ can namely not be interpreted as an unbiased estimate of an integer. Fortunately we can repair this situation quite easily. The idea is therefore to use, as an admissible transformation, the *integer approximation* of the above fully decorrelating transformation. The price we have to pay for the guarantee of integerness is of course that the full decorrelation property is not retained anymore. But although a full decorrelation is now out of the question, one can proof that a significant decrease in correlation can still be achieved [8]. This is possible through the use of a sequence of the above transformation in which each time the role of the two ambiguities is interchanged.

When the above principles are generalized to the n-dimensional case, the decorrelation of the least-squares ambiguities results in a transformation from the original ambiguity vector $\hat{a}$ and its variance-covariance matrix $Q_{\hat{a}}$ to the new ambiguity vector $\hat{z}$ and its variance-covariance matrix $Q_{\hat{z}}$. The transformed ambiguities will then be much less correlated and the variance-covariance matrix $Q_{\hat{z}}$ will be much closer to diagonality than the original variance-covariance matrix $Q_{\hat{a}}$. And as a consequence of the volume-preserving property together with the decrease in correlation, the spectrum of conditional variances of the transformed ambiguities will be largely flattened.

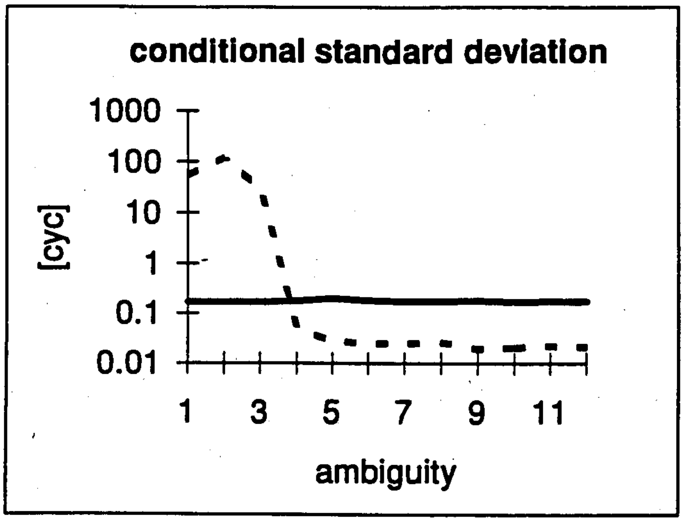

Figure 4: The original and transformed spectrum of conditional standard deviations.

Based on the same data as used before, figure 4 shows both the original and the transformed spectrum of conditional standard deviations. The dramatic improvement in the spectrum is clearly shown. The large discontinuity has been removed and the transformed conditional standard deviations are all of about the same small order.

Now that the original ambiguity vector $\hat{a}$ has been transformed to the new ambiguity vector $\hat{z}$, we can again apply the principle of sequential conditional least-squares estimation and parametrize the ellipsoidal inequality (3) in terms of the transformed ambiguities as

$$\sum_{i=1}^{n} (\hat{z}_{i|I} - z_i)^2 / \sigma_{\hat{z}_{(i|I,i|I)}} \le \chi^2 \; . \tag{7}$$

The corresponding set of n scalar inequalities which is used for the search reads then

$$\begin{cases} (\hat{z}_1 - z_1)^2 & \le \; \sigma_{\hat{z}_{(1,1)}} \chi^2 \\ (\hat{z}_{2|1} - z_2)^2 & \le \; \sigma_{\hat{z}_{(2|1,2|1)}} [\chi^2 - (\hat{z}_1 - z_1)^2 / \sigma_{\hat{z}_{(1,1)}}] \\ & \vdots \\ (\hat{z}_{n|N} - z_n)^2 & \le \; \sigma_{\hat{z}_{(n|N,n|N)}} [\chi^2 - \sum_{j=1}^{n-1} (\hat{z}_{j|J} - z_j)^2 / \sigma_{\hat{z}_{(j|J,j|J)}}] \; . \end{cases} \tag{8}$$

Due to the lowered and flattened spectrum of conditional variances, the search for the integer least-squares solution based on (8) can now be executed in a highly efficient manner. To illustrate how the method has succeeded in reducing the halting problem and thereby improving the efficiency of the search, the number of transformed integer candidates are shown in figure 5 as function of the level j. The corresponding numerical values are given in table 1.

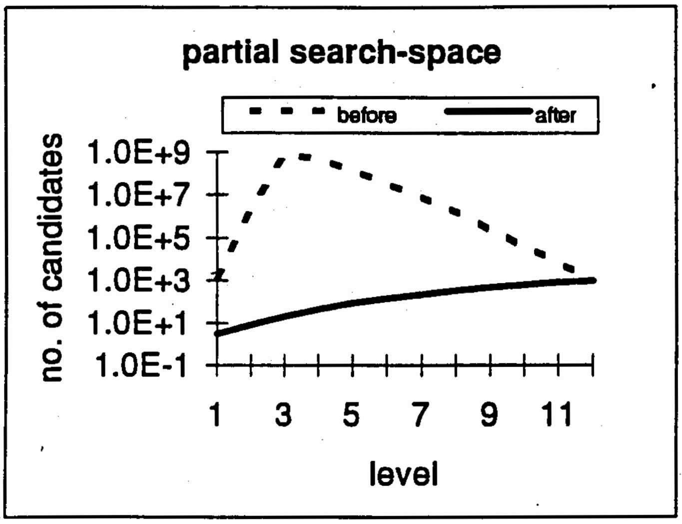

Figure 5: The number of transformed integer candidates $z_j = (z_1, z_2, ..., z_j)^*$ per partial search space $j = 1,2,...,12$ (solid line).

level j	# candidates a_j	# candidates z_j
1	1082	3
2	1984219	8
3	$8.23 * 10^8$	19
4	$4.77 * 10^8$	43
5	$1.4 * 10^8$	84
6	33376533	146
7	7516578	223
8	1674159	331
9	261118	469
10	40999	616
11	6641	803
12	966	966

Table 1: The number of integer candidates in the original and transformed partial search spaces.

Based on these results, the following observations can be made. First we note that the original and transformed ambiguity search spaces both have an identical number of integer candidates, namely 966. This is as it should be and is a consequence of the volume preserving property of the ambiguity transformation. Secondly, we observe that for the first eleven levels the numbers of integer candidates in the transformed partial search spaces are very much smaller than the corresponding numbers of integer candidates in the original partial search spaces. For instance, at the first level we only have 3 integer candidates for z_1, as opposed to the 1082 integer candidates for a_1. This is due to the lowering of the spectrum of conditional variances which results in a drastic improvement in precision of the ambiguities. And finally we note that the number of transformed integer candidates is strictly increasing as function of the level. This is due to the removal of the discontinuity from the original spectrum of conditional variances. As a result of the above properties, the search for the transformed integer least-squares ambiguities commences with tight bounds and is largely freed from the potential problem of halting, thus assuring that the solution can indeed be found in a highly efficient manner.

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