INTEGER LEAST-SQUARES ESTIMATION OF THE GPS PHASE AMBIGUITIES

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ABSTRACT

The purpose of this contribution is to give a brief review of the least-squares ambiguity decorrelation adjustment as it was introduced in [1]. Starting from the principle of integer least-squares estimation, it is first discussed why the search for the traditional integer double-difference (DD) ambiguities performs so poorly in case of short observational time spans using carrier phase data only. This is supported by means of some representative results about the precision, correlation and spectrum of the DD-ambiguities. Then the possibilities are explored for using ambiguities other than the traditional DD-ambiguities. This leads to the admissible class of ambiguity transformations. It is then indicated how a decorrelating ambiguity transformation can be constructed. Its construction is based on an integer approximation of the conditional least-squares transformation. Finally it is demonstrated how the method succeeds in decorrelating the ambiguities, thereby making a highly efficient search for the transformed integer least-squares ambiguities possible.

1. INTRODUCTION

The GPS observables are code-derived pseudorange measurements and carrier phase measurements, which can be available on both of the two frequencies L_1 and L_2. In particular the very low noise behaviour of the carrier phase measurements makes high precision relative positioning possible. However, since the GPS-receivers only provide relative measurements of phase from the start of signal tracking, the carrier phase data are biased by an unknown integer number of wavelengths, known as the phase ambiguities. A pre-requisite for obtaining high precision relative positioning results, based on carrier phase data, is therefore that the phase ambiguities become sufficiently separable from the baseline coordinates. Such a separability is achieved when carrier phase data are used that correspond to sufficiently differing receiver-satellite geometries. But since GPS satellites are in very high altitude orbits, their relative positions with respect to the receiver change slowly, which implies that long time spans between the first and the last collected carrier phase data are necessary so as to ensure separability. A large reduction in the time span is possible however, if one explicitly aims at resolving for the integer-values of the ambiguities. In the last couple of years, important progress has been made in this area of integer ambiguity fixing, see e.g. [2-6]. The inclusion of fast ambiguity resolution algorithms has for instance made rapid static surveying with GPS possible. Nevertheless, it is still expedient to seek ways of improving the efficiency of the various components of the ambiguity fixing process. At the 1993 IAG General Assemblee in Beijing, China, a new method for the fast estimation of the integer double-difference (DD) ambiguities was introduced [1]. This method of the least-squares ambiguity decorrelation adjustment (LAMBDA) has been developed so as to transform the original integer least-squares problem into an equivalent problem, but one that is much easier to solve. The purpose of this contribution is to give a brief review of the method and to demonstrate its capabilities. In section two the relationship between the integer and noninteger least-squares problem is discussed. This relationship is established using the principle of conditional least-squares estimation. In

paper presented at the International Symposium on Kinematic Systems in Geodesy, Geomatics and Navigation. Banff, Alberta, Canada. August 30 - September 2, 1994.

sections three and four, some of the statistical characteristics of the least-squares DD-ambiguities are discussed. They are the precision, correlation and spectrum of the DD-ambiguities. Based on these characteristics it becomes clear why the search for the integer least-squares DD-ambiguity estimates performs so poorly when short observational time spans are used. Sections five and six deal with the method itself. In section five it is shown what type of regular transformations on the DD-ambiguities are admissible. Some examples are given. Finally in section six it is demonstrated how the method succeeds in decorrelating the ambiguities, thereby returning ambiguities with a largely flattened spectrum and a dramatically improved precision. As a result, the search for the transformed integer least-squares ambiguities can be performed in a highly efficient manner.

2. INTEGER AND NONINTEGER LEAST-SQUARES

As starting point of our discussion we take the linear(ized) system of observation equations

$$y = Aa + Bb + e \ , \tag{1}$$

where:

y	:	is the vector of observed minus computed DD carrier phase measurements,
a	:	is the vector of unknown integer DD ambiguities,
b	:	is the vector that contains the increments of the unknown baseline components,
A,B	:	are the design matrices for ambiguity terms and baseline components, and,
e	:	is the vector of measurement noise and unmodelled errors.

Our estimation criterion for the determination of the unknowns a and b will be based on the principle of least-squares. The least-squares criterion for solving the linear(ized) system of observation equations reads

$$\min_{a,b} \| y - Aa - Bb \|^2_{Q_y} \ , \tag{2}$$

where $\| . \|^2_{Q_y} = (.)^* Q_y^{-1} (.)$, Q_y is the variance-covariance matrix of the DD carrier phase observables. Z^n is the n-dimensional space of integer numbers and R^3 is the 3-dimensional space of reals. The minimization problem (2) would be an ordinary unconstrained least-squares problem if all the parameters were allowed to range through the space of reals, i.e. if

$$a \in R^n \text{ and } b \in R^3 \tag{3}$$

would hold. In our case however, we do have the additional information that all the DD-ambiguities are integer-valued. Instead of (3), we therefore have

$$a \in Z^n \text{ and } b \in R^3 \ . \tag{4}$$

The minimization problem (2) together with (4) is referred to as an integer least-squares problem. It is a constrained least-squares problem due to the *integer*-constraint $a \in Z^n$. The solution of the integer least-squares problem will be denoted as $\check{a}$ and $\check{b}$. The solution of the corresponding unconstrained least-squares problem will be denoted as $\hat{a}$ and $\hat{b}$. The estimates $\hat{a}$ and $\hat{b}$ are sometimes also referred to as the "float solution", and $\check{a}$ and $\check{b}$ as the "fixed solution". As it was shown in [1], the objective function of (2) can be decomposed into the following sum of $(n+2)$ squares

$$\| y - Aa - Bb \|^2_{Q_y} = \mathrm{I} + \mathrm{II} + \mathrm{III} \ , \tag{5}$$

with

$$\begin{aligned} \mathrm{I} &= \| \hat{e} \|^2_{Q_y} \ , \\ \mathrm{II} &= \sum_{i=1}^{n} (\hat{a}_{i|I} - a_i)^2 / \sigma_{\hat{a}(i|I,i|I)} \ , \\ \mathrm{III} &= \| \hat{b} | a - b \|^2_{Q_{\hat{b}|a}} \ , \end{aligned}$$

and where

$$\begin{aligned} \hat{e} &= y - A\hat{a} - B\hat{b} \ , \\ \hat{a}_{i|I} &= \hat{a}_i - \sum_{j=1}^{i-1} \sigma_{\hat{a}(i,j|J)} \sigma^{-1}_{\hat{a}(j|J,j|J)} (\hat{a}_{j|J} - a_j) \ , \\ \hat{b}|a &= \hat{b} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} (\hat{a} - a) \ . \end{aligned}$$

In this decomposition, $\hat{e}$ is the unconstrained least-squares residual vector, $\hat{a}_{i|I}$ is the least-squares estimate of the ith-ambiguity conditioned on a_1 up to and including a_{i-1}, with $\sigma_{\hat{a}(i|I,i|I)}$ being its variance, and $\hat{b}|a$ is the least-squares estimate of the baseline vector conditioned on a. Note that the estimates $\hat{a}_{i|I}$ follow from a sequential conditional least-squares adjustment on the ambiguities.

The above decomposition has the advantage that it easily allows us to relate the noninteger least-squares problem to the integer least-squares problem. Let us first consider the *noninteger* least-squares problem. In that case, $a \in R^n$ and $b \in R^3$ need to be chosen such that the quadratic form of (5) is minimized. Varying $b \in R^3$ only has an effect on the third term III in the decomposition. Both I and II are independent of $b \in R^3$. Hence, the value of b that needs to be chosen is the one that minimizes III. The minimum value of III is clearly III = 0 and it is obtained when b is chosen to be equal to $\hat{b}\,|\,a$. Let us now vary $a \in R^n$. Varying $a \in R^n$ only has an effect on the second term II. The first term I is independent of a and the third term III will remain zero when b is taken to be equal to $\hat{b}\,|\,a$. Hence, the value of a that needs to be chosen is the one that minimizes II. A closer look at the n-number of squares of II shows that II is minimized when the a_i are chosen to be equal to the $\hat{a}_i$ for $i = 1,...,n$. With this choice also II reduces to zero. Furthermore, it follows when a is chosen to be equal to $\hat{a}$, that $b = \hat{b}\,|\,a$ reduces to $\hat{b} = \hat{b}\,|\,\hat{a}$. This shows therefore that $\hat{a}$ and $\hat{b}$ are indeed the minimizers of the noninteger least-squares problem. And since II = 0 and III = 0, the corresponding minimum of the objective function follows as

$$\min_{a \in R^n, b \in R^3} \| y - Aa - Bb \|^2_{Q_y} = \| \hat{e} \|^2_{Q_y} . \tag{6}$$

Let us now consider the *integer* least-squares problem. In this case we have $a \in Z^n$ and $b \in R^3$. The decomposition of (5) shows that we can again consider the two squares II and III separately. That is, the minimizer for a follows from minimizing II and the minimizer for b follows from minimizing III. Starting with b we can again make III equal to zero if we choose b as $\hat{b}\,|\,a$. In case of a however, we now have the integer-constraint $a \in Z^n$. This implies, since the sequential conditional least-squares estimates $\hat{a}_{i|I}$ are generally real-valued, that II cannot be made equal to zero. The values of a_i that minimize II are denoted as $\check{a}_i$. With these values the corresponding estimate for b becomes $\check{b} = \hat{b}\,|\,\check{a}$. In case of the integer least-squares problem we thus have III = 0 and for the corresponding minimum of the objective function:

$$\min_{a \in Z^n, b \in R^3} \| y - Aa - Bb \|^2_{Q_y} = \| \hat{e} \|^2_{Q_y} + \sum_{i=1}^{n} (\hat{a}_{i|I} - \check{a}_i)^2 / \sigma_{\hat{a}(i|I,i|I)} . \tag{7}$$

Above, the relationship between the integer least-squares problem and the noninteger least-squares problem was shown. The ordinary least-squares estimates $\hat{a}$ and $\hat{b}$ can be computed using standard procedures. Also the estimate of b, being $\hat{b}\,|\,\check{a}$, can be computed quite easily once $\check{a}$ is known. For the computation of the integer minimizers $\check{a}_i$ however, no standard procedure is available. The bottleneck in finding the solution for the integer least-squares problem (7) is therefore given by

$$\min_{a_i \in Z} \sum_{i=1}^{n} (\hat{a}_{i|I} - a_i)^2 / \sigma_{\hat{a}(i|I,i|I)} . \tag{8}$$

And this is particularly true in case the integer DD-ambiguities a_i need to be estimated for short observational time spans based on carrier phase data only. In order to solve (8), a search for the integer least-squares ambiguities is performed which is based on the following set of bounds [1]

$$(a_i - \hat{a}_{i|I})^2 \leq l_i \sigma_{\hat{a}(i|I,i|I)} \chi^2 \quad \text{for } i = 1,...,n \tag{9}$$

where

$$l_i = (1 - \chi^2_{i-1}/\chi^2) \text{ and } \chi^2_{i-1} = \sum_{j=1}^{i-1} (a_j - \hat{a}_{j|J})^2 / \sigma_{\hat{a}(j|J,j|J)} .$$

It is assumed that the positive constant χ^2 has been chosen such that the region (9) at least contains the the sought for integer least-squares solution. As it was remarked earlier the estimates $\hat{a}_{i|I}$ follow from a *sequential* conditional least-squares adjustment on the ambiguities. It is because of this adjustment that we are in the position to formulate sharp and *successive bounds* on the individual ambiguities. They form the basis of our search for the integer ambiguities [1].

3. ON THE PRECISION OF THE DD-AMBIGUITIES

In order to highlight some of the difficulties one will face when computing the integer least-squares DD-ambiguities, we first will discuss in this and the folowing section some characteristics of the precision, correlation and spectrum of the DD-ambiguities. Our example is based on a 100 metre baseline with a 7 satellite configuration, using dual-frequency carrier phase data only. The skyplot of the satellite configuration is shown in figure 1. The results that will be shown are based on the use of two epochs separated by 1 second. The a priori standard deviation of both the L_1 and L_2 carrier phases was set at the value of $\sigma = 3\,\text{mm}$. Correlation in time and correlation between the channels were assumed to be non-existent.

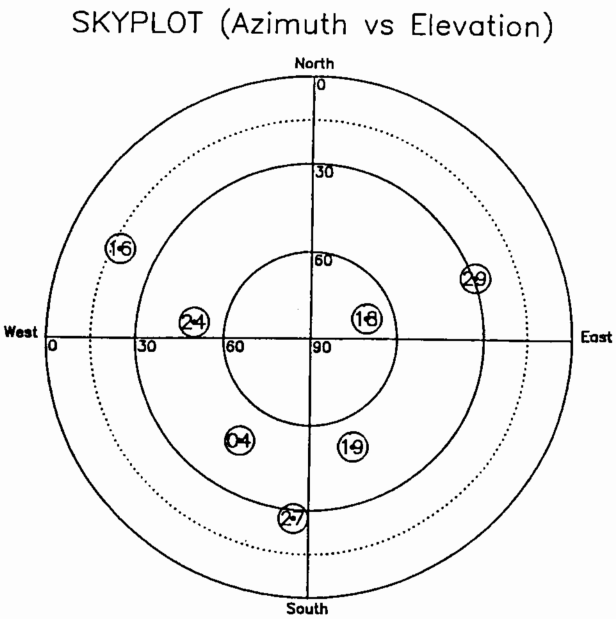

Figure 1: Skyplot of the 7 satellite configuration.

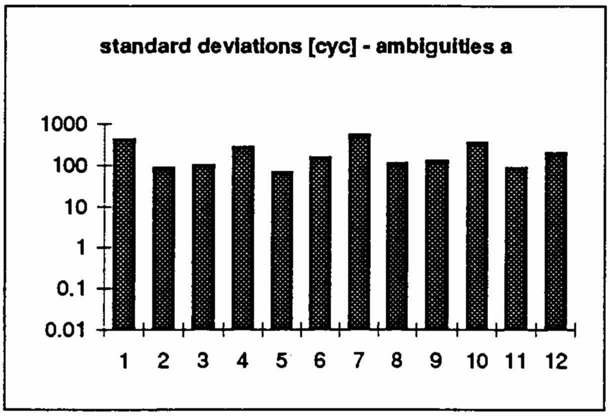

Figure 2: The standard deviations of the twelve least-squares DD-ambiguities.

Since GPS satellites are in very high altitude orbits, their relative positions with respect to the receiver change slowly, which implies in case of short observational time spans, that the DD-ambiguities - when treated as being real-valued - become very poorly separable from the baseline coordinates. As a result, the precision with which both the baseline and DD-ambiguities can be estimated will be rather poor. This is particularly true in the absence of precise *P*-code data. For our example the precision of the twelve DD-ambiguities is shown in figure 2. Note that the precision of the least-squares DD-ambiguities is indeed rather poor since their standard deviations range from 64 cycles to 519 cycles.

One can of course improve the precision of the DD-ambiguities by considering a longer observational time span. The reason for choosing for our example the short observational time span of 1 second using the minimum number of two epochs, is, that we aim at applications in which near real-time results are required. One can of course also improve the precision of the DD-ambiguities through the inclusion of code-observations. In that case the minimum number of epochs required reduces in principle from two to one. For our example however, we have chosen to exclude the precise code-observations. This has been done for two reasons. First of all, it may indeed happen in practical applications that one lacks the availability of the precise code-observations. The second reason for excluding the code-observations in this contribution is to accentuate the performance of the LAMBDA-method when it is based on the minimal amount of information. Hence, the results that will be shown in section 6 are obtained without the help of any code-observations. In [7] however, it is shown and discussed what the extra input of the precise code-observations brings in terms of improving the computational performance of solving the integer least-squares problem.

Based on the above precision results, one may be inclined to conclude that the poor precision of the DD-ambiguities will have a detrimental effect on ones ability to solve for the ambiguity fixing problem. Here however, some remarks of caution are in order. As it was stressed in [8], it is important that one distinguishes between the following two problems of GPS-ambiguity fixing:

1. The ambiguity *estimation* problem, and

2. The ambiguity *validation* problem.

The first problem, which is the topic of the present contribution, is simply concerned with ones ability to solve for the minimization problem (8). Hence, it deals with ones ability to compute the integer least-squares ambiguities. It has therefore nothing to do in principle with the quality of the computed integer least-squares solution. One will namely always be able to compute an integer least-squares solution, whether it is of poor quality or not. The question is however, how efficient this computation can be performed. The second problem of above depends on the outcome of the first and is concerned with the question whether one is willing to accept the computed integer least-squares solution. This second problem is therefore, in contrast to the first, completely dependent on both the quality of the computed solution and the criteria formulated for accepting the solution.

With the above distinction made, one can conclude that a poor precision of the DD-ambiguities will affect the validation-problem, but not necessarily the estimation-problem. To make this clear, consider first the problem in one dimension. A DD-ambiguity of poor precision will generally admit more integer candidates than a very precise DD-ambiguity. Hence, it will be more difficult to validate an integer value stemming from a DD-ambiguity of poor precision, then one stemming from a very precise DD-ambiguity. The effort however, of computing the most likely integer value will be the same for both cases in this one dimensional example.
The fact that the precision of the DD-ambiguities in itself not necessarily affects the estimation-problem can also be carried forward to the n-dimensional case. First consider the case in which the least-squares ambiguities are of a very high precision. Let us for instance assume that their standard deviations are all very much smaller than one cycle. One may then perhaps be inclined to believe that the search for the integer least-squares estimates of the ambiguities can be replaced by a simple rounding of the least-squares ambiguity estimates to their nearest integers. This however, is not true. That is, even if the ambiguities are of a high precision, it is not guaranteed that one will find the integer least-squares solution by means of a simple rounding to the nearest integer.
Secondly we consider the case that the least-squares estimates of the ambiguities are of a very poor precision. In addition we assume that the variance-covariance matrix of the DD-ambiguities is diagonal. In that case all ambiguities are correlation-free and the integer least-squares problem (8) reduces to

$$\min_{a_1,\ldots,a_n \in Z} (a_1 - \hat{a}_1)^2/\sigma_{\hat{a}(1,1)} + \ldots + (a_n - \hat{a}_n)^2/\sigma_{\hat{a}(n,n)} \quad . \quad (10)$$

In this case the objective function has been reduced to a sum of independent squares, implying that we can work with n separate scalar integer least-squares problems. Hence, in this case the integer least-squares solution sought does follow from a simple rounding of $\hat{a}_i$ to the nearest integer. And this effort is clearly independent of the individual values the variances $\sigma_{\hat{a}(i,i)}$ may take.

4. ON THE CORRELATION AND SPECTRUM OF THE DD-AMBIGUITIES

From the above we can conclude that the integer least-squares problem becomes trivial, when all the DD-ambiguities are fully decorrelated. In reality however, the DD-ambiguities are highly correlated. Based on the same data as used above, figure 3 shows a histogram of the absolute values of all the DD-ambiguity correlation coefficients.

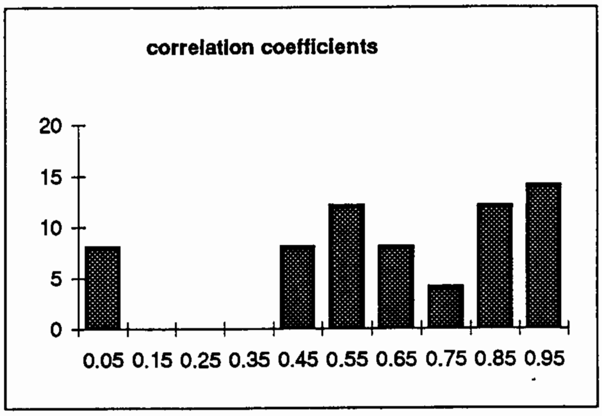

Figure 3: Histogram of the absolute values of the DD-ambiguity correlation coefficients

With 7 satellites and dual-frequency carrier phase data, the number of DD-ambiguities equals twelve. Hence the number of correlation coefficients equals sixty-six. The figure clearly shows that the majority of the correlation coefficients are larger than a half in absolute value. And quite a few of them are even very close to one in absolute value.

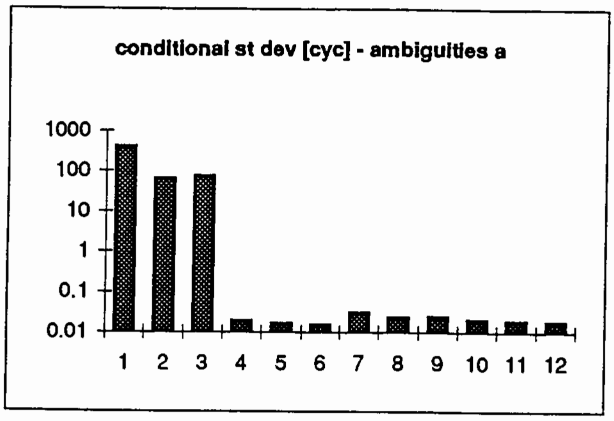

Figure 4: The spectrum of conditional standard deviations of the DD-ambiguities.

A measure of the diagonality of the variance-covariance matrix $Q_{\hat{a}}$ is given by the ambiguity decorrelation number $r_{\hat{a}}$. It was introduced in [1] as $r_{\hat{a}}^2 = \det(R_{\hat{a}})$, with $R_{\hat{a}}$ being the correlation matrix of the DD-ambiguities. In two dimensions we have $r_{\hat{a}}^2 = 1-\rho_{\hat{a}}^2$, with $\rho_{\hat{a}}$ being the ambiguity correlation coefficient. Since the ambiguity decorrelation number $r_{\hat{a}}$ is complementary to the correlation coefficients, the variance-covariance matrix $Q_{\hat{a}}$ is diagonal when $r_{\hat{a}}$ equals one and it is far from diagonality when $r_{\hat{a}}$ is close to zero. Due to the usual high correlation between the DD-ambiguities, the ambiguity decorrelation number $r_{\hat{a}}$ is typically very close to zero. For our example it equals $r_{\hat{a}} = 2.998 * 10^{-36}$. Other numerical examples of typical values that $r_{\hat{a}}$ may take are given in [9].

The ambiguity decorrelation number also provides us with a means of linking the correlation of the DD-ambiguities with the bounds of the intervals of (9) as used in our search for the integer least-squares solution. As it was shown in [1], the ambiguity decorrelation number squared equals the product of all n-number of ratios of conditional variances and corresponding unconditional variances: $r_{\hat{a}}^2 = \prod_{i=1}^{n} \sigma_{\hat{a}(i|I,i|I)} / \sigma_{\hat{a}(i,i)}$ This shows, since the variances are all of about the same order and $r_{\hat{a}}$ is typically very small due to the high correlation, that at least some of the conditional variances $\sigma_{\hat{a}(i|I,i|I)}$ must be very small indeed.

The spectrum of the conditional standard deviations of the DD-ambiguities is shown in figure 4. The figure clearly shows that quite a few of the conditional standard deviations are indeed very small. There are three large conditional standard deviations and nine extremely small ones. The shape of the spectrum as shown in figure 4 is very typical for GPS relative positioning. It is a consequence of the intrinsic structure of the carrier phase model of observation equations and the chosen parametrization in terms of the DD-ambiguities. The implication of the discontinuity in the spectrum for the search of the integer least-squares DD-ambiguities is as follows. Since the first three conditional variances are rather large, the first three bounds ($i = 1,2,3$) of (9) will be rather loose. Hence, quite a number of integer triples will satisfy these first three bounds. The remaining conditional variances however are very small. The corresponding bounds of (9) will therefore be very tight indeed. This implies, when we go from the third to the fourth ambiguity that we have a high likelihood of not being able to find an integer quartet that satisfies the first four bounds. Hence, the potential of halting is very significant when one goes from the third to the fourth ambiguity. As a consequence a large number of trials are required, before one is able to find an integer n-tuple that satisfies all n bounds. This is therefore the reason why in case of short observational time spans with carrier phase data only, the search for the integer least-squares DD-ambiguities performs so poorly.

5. THE CLASS OF ADMISSIBLE AMBIGUITY TRANSFORMATIONS

In the previous section some characteristics of the statistical properties of the DD-ambiguities were discussed in order to illustrate the cause for the poor performance of the search for the integer least-squares estimates of the DD-ambiguities. The LAMBDA-method aims at removing the drawbacks that are associated with the use of the traditional DD-ambiguities. The idea is to reparametrize the original integer least-squares problem (8) such that an equivalent formulation is obtained, but one that is much easier to solve. The reparametrization is achieved through a transformation of the original DD-ambiguities to a new set of ambiguities. In this section the class of admissible ambiguity transformations is briefly reviewed. For more details on the characteristics of the ambiguity transformations, the reader is referred to [10].

Let the transformed ambiguities, their least-squares estimates and corresponding variance-covariance matrix be given as

$$z = Z^{*}a,\ \hat{z} = Z^{*}\hat{a},\ Q_{\hat{z}} = Z^{*}Q_{\hat{a}}Z\ , \tag{11}$$

with Z being an n-by-n matrix of full rank. It follows then that $(\hat{a}-a)^{*}Q_{\hat{a}}^{-1}(\hat{a}-a) = (\hat{z}-z)^{*}Q_{\hat{z}}^{-1}(\hat{z}-z)$. However, in order to be allowed to replace the original integer least-squares problem (8) with the reparametrized integer least-squares problem

$$\min_{z_i \in Z} \sum_{i=1}^{n} (\hat{z}_{i|I} - z_i)^2 / \sigma_{\hat{z}(i|I,i|I)}\ . \tag{12}$$

we need to be sure that the two minimization problems are truely equivalent. As it turns out, this is not the case if we only require Z to be of full rank. What is needed in addition is the guarantee that the integerness of the ambiguities remains preserved under the transformation with matrix Z^{*}. A first consequence of this requirement is that all the entries of the matrix Z need to be integer. The integer entries of Z together with the integerness of the DD-ambiguity vector a guarantees then that all the entries of z are integer as well. The requirement that all entries of Z need to be integer is however still not sufficient. This follows from the fact that the entries of a need not be integer even when all the entries of both z and Z are integer. Such a situation is clearly not acceptable, since it could imply that an integer fixing of the transformed ambiguities based on solving the reparametrized integer least-squares problem (12), corresponds to a fixing of the original DD-ambiguities on *non-integer* values. The second consequence of having to guarantee that the integerness of the ambiguities remains preserved is therefore that also all the entries of the inverse of Z need to be integer.

In [10] it has been proven that the above given restrictions on Z are indeed necessary and sufficient. In order to get some feeling for the ambiguity transformations that are admissible, some examples for both the single-channel and multi-channel case will be given. Let the DD-ambiguity vector a consist of pairs of L_1-and L_2-ambiguities related to the same satellite and let Z be partitioned as $Z = \text{diag.}(Z_{22}, Z_{22}, \ldots, Z_{22})$. Examples of admissible ambiguity transformations on the *single*-channel level are then

$$Z_{22}^{*} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \quad Z_{22}^{*} = \begin{bmatrix} 1 & -1 \\ 4 & -5 \end{bmatrix},$$

$$Z_{22}^{*} = \begin{bmatrix} -60 & 77 \\ -7 & 9 \end{bmatrix} \quad Z_{22}^{*} = \begin{bmatrix} 53 & -68 \\ -7 & 9 \end{bmatrix}.$$

It is easily verified that the entries of the inverses of these matrices are all integer. This shows for instance that it is allowed to pair the wide-lane ambiguity to the L_2-ambiguity. Note however, that it is not allowed to pair the wide-lane ambiguity to the narrow-lane ambiguity. In that case namely, we would have

$$Z_{22}^{*} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \text{ and } \left(Z_{22}^{*}\right)^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix},$$

showing that the entries of the inverse are non-integer. Examples of admissible ambiguity transformations on the *multi*-channel level are for instance the class of all permutation matrices and the class of transformations that change the choice of reference satellite in the DD-ambiguities. Also note, that once certain ambiguity transformations are identified, other ambiguity transformations can be derived from them by performing certain matrix operations, like inversion, transposition and multiplication.

6. THE LEAST-SQUARES AMBIGUITY DECORRELATION ADJUSTMENT

In section 4 we have seen that the search for the integer least-squares solution suffers from the fact that the spectrum of conditional variances of the DD-ambiguities shows a large discontinuity. The aim of the LAMBDA-method is therefore to construct a decorrelating ambiguity transformation Z that flattens the spectrum of conditional variances. In two dimensions the ambiguity transformation is constructed from a sequence of integer approximations of the conditional least-squares transformation. They are of the form

$$Z^{*} = \begin{bmatrix} 1 & 0 \\ -[\sigma_{\hat{a}(2,1)}\sigma^{-1}_{\hat{a}(1,1)}] & 1 \end{bmatrix} \text{ or } Z^{*} = \begin{bmatrix} 1 & -[\sigma_{\hat{a}(1,2)}\sigma^{-1}_{\hat{a}(2,2)}] \\ 0 & 1 \end{bmatrix}, \qquad (13)$$

where [.] stands for "rounding to the nearest integer". In the n-dimensional case a similar line of thought is followed. For further details on the method, the reader is referred to [1].

Application of the method to the data of our example resulted in the following multi-channel ambiguity transformation

$$Z^{*} = \begin{bmatrix} -5 & 3 & 7 & 1 & -1 & -2 & 6 & -6 & -5 & 0 & 3 & 0 \\ -2 & 1 & -6 & 3 & -1 & -1 & -6 & -5 & 8 & 4 & -3 & 2 \\ -2 & 1 & 3 & -3 & 3 & -3 & 9 & -5 & -2 & 2 & 0 & 1 \\ 2 & 4 & 1 & -2 & -6 & -1 & 1 & 2 & -5 & -4 & 8 & 4 \\ 5 & -1 & -2 & -4 & 5 & -2 & 7 & -4 & 3 & 0 & 0 & -1 \\ 0 & -3 & 3 & -1 & -5 & 4 & -9 & 3 & 2 & 7 & 5 & -7 \\ 0 & -6 & 2 & 3 & 3 & -7 & 2 & 1 & 0 & -1 & -2 & 6 \\ 2 & 4 & 4 & -3 & -2 & -4 & -3 & -6 & 2 & 4 & 6 & 1 \\ -1 & -2 & -4 & 2 & -1 & 5 & -4 & 9 & 1 & -3 & -3 & -2 \\ -4 & 5 & -1 & 1 & -1 & -1 & 4 & -2 & 7 & -4 & 4 & -1 \\ 3 & -3 & -3 & 3 & -1 & -1 & 5 & 2 & 1 & -8 & 4 & 1 \\ 0 & 1 & -1 & 2 & 3 & -3 & 1 & -2 & 4 & -3 & -2 & 2 \end{bmatrix}$$

Note that Z^{*} is truely a multi-channel transformation. Every new ambiguity is formed as a linear combination of all original DD-ambiguities. Although the way in which this transformation is constructed guarantees its admissibility, the integerness of all entries of the inverse of Z^{*} is also easily verified numerically. The inverse of Z^{*} reads namely

$$(Z^{*})^{-1} = \begin{bmatrix} -199 & -190 & -353 & -507 & -199 & -489 & 118 & 380 & -190 & 340 & 113 & -1051 \\ 240 & 86 & -154 & 127 & 272 & 41 & 82 & -41 & 195 & -27 & -127 & 36 \\ -675 & -430 & -308 & -1019 & -738 & -856 & 59 & 711 & -557 & 571 & 394 & -1744 \\ 95 & -27 & -267 & -208 & 91 & -258 & 118 & 258 & 136 & 136 & 4 & -562 \\ 136 & 118 & 290 & 186 & 68 & 195 & -86 & 0 & 299 & -240 & 41 & 498 \\ -1078 & -643 & -231 & -1585 & -1250 & -1282 & 5 & 1205 & -729 & 761 & 729 & -2514 \\ -155 & -148 & -275 & -395 & -155 & -381 & 92 & 296 & -148 & 265 & 88 & -819 \\ 187 & 67 & -120 & 99 & 212 & 32 & 64 & -32 & 152 & -21 & -99 & 28 \\ -526 & -335 & -240 & -794 & -575 & -667 & 46 & 554 & -434 & 445 & 307 & -1359 \\ 74 & -21 & -208 & -162 & 71 & -201 & 92 & 201 & 106 & 106 & 3 & -438 \\ 106 & 92 & 226 & 145 & 53 & 152 & -67 & 0 & 233 & -187 & 32 & 388 \\ -840 & -501 & -180 & -1235 & -974 & -999 & 4 & 939 & -568 & 593 & 568 & -1959 \end{bmatrix}$$

In order to illustrate the performance of the above given decorrelating ambiguity transformation, we first consider the correlation of the transformed ambiguities. Based on the same data as used earlier, figure 5 shows the two histograms of the absolute values of the correlation coefficients of the DD-ambiguities $\hat{a}_i$ and the transformed ambiguities $\hat{z}_i$. It follows upon comparing the two histograms that the ambiguity transformation has indeed achieved a large decrease in correlation between the ambiguities. None of the correlation coefficients $\rho_{\hat{z}(i,j)}$ is close to ± 1 and the largest is even smaller than a half in absolute value. The fact that the correlation has been largely reduced is also clear when we consider the decorrelation number of the transformed ambiguities. It reads $r_{\hat{z}} = 0.087$.

Parallel with the decorrelation of the ambiguities also the precision of the ambiguities gets dramatically improved. As it was shown in [1], the square of the ratio of the decorrelation numbers equals the reciprocal of the product of the ratios of corresponding variances. That is, $r^2_{\hat{a}}/r^2_{\hat{z}} = \prod_{i=1}^{n} \sigma_{\hat{z}(i,i)}/\sigma_{\hat{a}(i,i)}$. This shows, since $r_{\hat{a}}/r_{\hat{z}}$ is very small indeed, that the product of transformed variances must be very much smaller than the product of variances of the DD-ambiguities. In fact, since $r_{\hat{a}}$ is of the order 10^{-36} and $r_{\hat{z}}$ of the order 10^{-1}, it follows with $n = 12$ that the improvement of the standard deviations is on the average of the order 0.001. This dramatic improvement can clearly be seen from figure 6.

To accentuate the fact that the transformed ambiguities are indeed of a very high precision, table 1 gives an overview of the least-squares ambiguity estimates themselves, all expressed in cycles. These results are based on the same data as used before. Shown are the ordinary noninteger least-squares estimates of both the original DD-ambiguities, $\hat{a}_i$, as well as the transformed ambiguities, $\hat{z}_i$. Also shown are the corresponding integer least-squares estimates, $\check{a}_i$ and $\check{z}_i$, and the differences between the noninteger and integer solution. The given 'two epochs, one second' integer least-squares solution was validated by means of a comparison with an integer least-squares solution based on thirty minutes of data. The high precision of the transformed ambiguities can clearly be seen from the table. In fact, for this particular case a simple 'rounding to the nearest integer' of the least-squares estimates of the transformed ambiguities already would suffice for finding the correct integer least-squares solution. But as it was remarked earlier, a high precision of the ambiguities is generally no guarantee

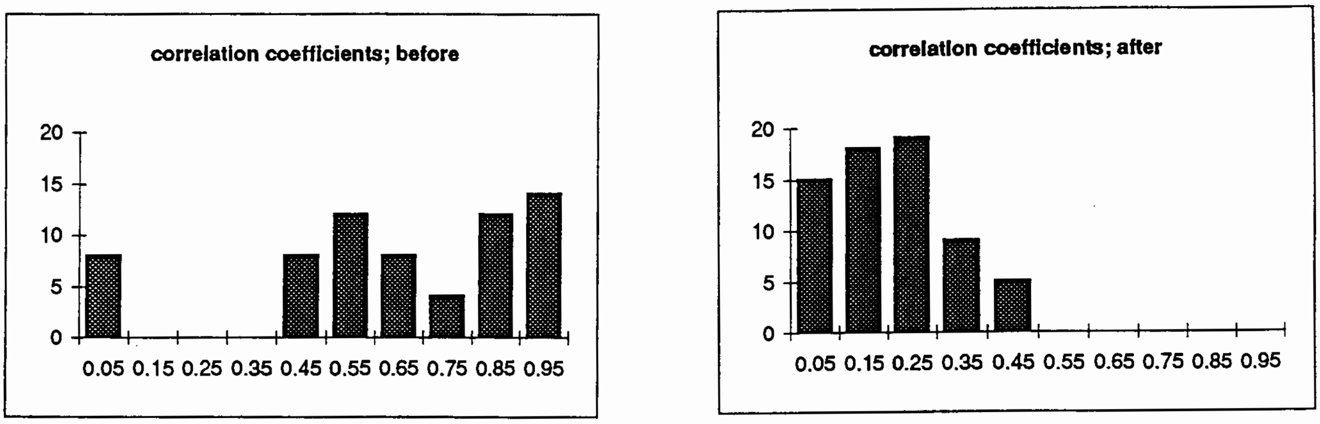

Figure 5: Histograms of $|\rho_{\hat{a}(i,j)}|$ and $|\rho_{\hat{z}(i,j)}|$.

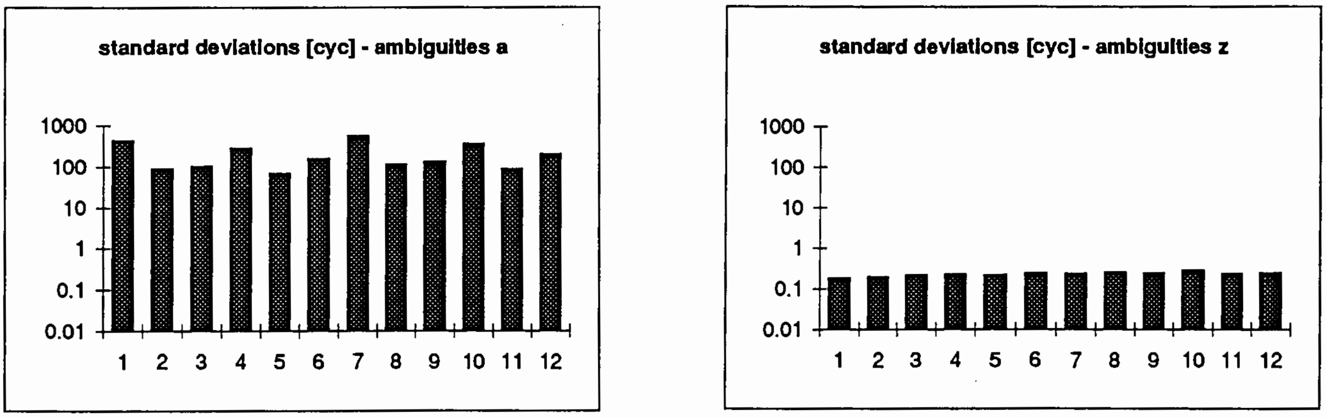

Figure 6: Standard deviations of the original and transformed ambiguities.

$\hat{a}$	$\check{a}$	$\hat{a}-\check{a}$	$\hat{z}$	$\check{z}$	$\hat{z}-\check{z}$
-380.85	-401.00	20.15	384.99	385.00	-0.01
-298.02	-300.00	1.98	-717.90	-718.00	0.10
-197.23	-229.00	31.77	776.09	776.00	0.09
-478.31	-496.00	17.69	-87.03	-87.00	-0.03
307.75	306.00	1.75	1247.04	1247.00	0.04
-221.60	-276.00	54.40	-180.94	-181.00	0.06
-296.30	-312.00	15.70	471.01	471.00	0.01
-231.46	-233.00	1.54	716.14	716.00	0.14
-153.24	-178.00	24.76	-919.09	-919.00	-0.09
-372.19	-386.00	13.81	492.89	493.00	-0.11
240.36	239.00	1.36	491.83	492.00	-0.17
-171.60	-214.00	42.40	376.92	377.00	-0.08

Table 1: The noninteger and integer least-squares estimates of the original and transformed ambiguities.

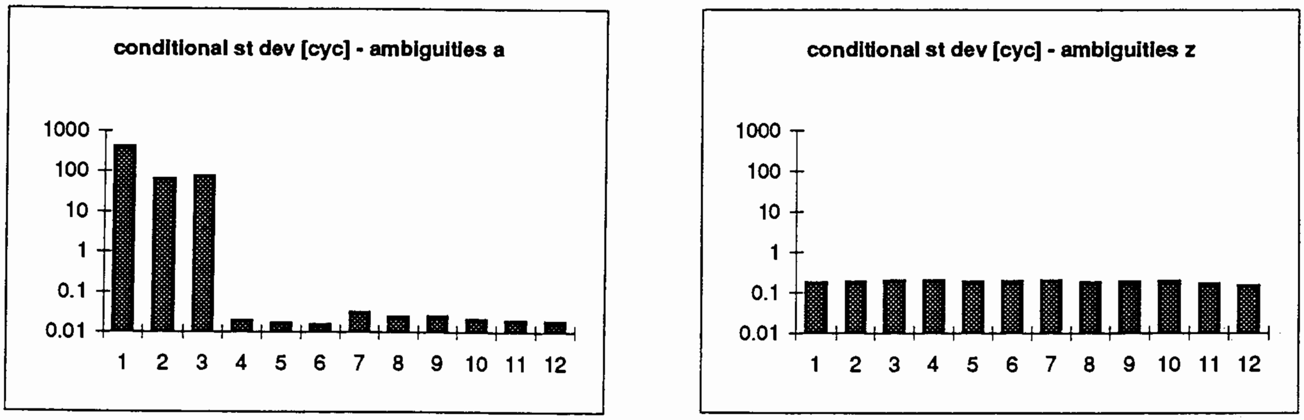

Figure 7: Original and transformed spectrum of conditional standard deviations.

that the integer least-squares solution is found by means of a simple rounding to the nearest integer. Hence, even if the ambiguities are of a high precision, only a search as advocated in section 2 guarantees that the integer least-squares solution is indeed found.

To conclude, we will finally consider the spectrum of conditional standard deviations. As it was shown in section 4, the original spectrum contained a large discontinuity when passing from the third to the fourth conditional standard deviation. The first three conditional standard deviations were rather large, whereas the remaining nine conditional standard deviations were very small indeed. And it was due to this large drop in value of the conditional standard deviations that the search for the integer least-squares ambiguities was hindered by a high likelihood of halting. Figure 7 shows both the original and transformed spectrum of conditional standard deviations.

The improvement in the spectrum is clearly visible from figure 7. The discontinuity has disappeared and all conditional variances are now of the same small order. The original three large conditional standard deviations are pushed down to much smaller values and the original nine small conditional standard deviations have increased somewhat. That the original nine small conditional standard deviations have to increase somewhat in value when the first three are pushed down, is a consequence of the fact that the product of these conditional standard deviations remains invariant under ambiguity transformations [1]. In fact it is the presence of the very small conditional standard deviations in the original spectrum that allows us to significantly decrease the three large values. This therefore makes quite clear what role is played by satellite redundancy and dual frequency data. When both are absent, we only have three DD-ambiguities and their conditional standard deviations will all be large. Hence, in that case the absence of the small conditional variances prohibits us to a high degree from bringing the spectrum down to much smaller values. In case of satellite redundancy and/or dual frequency data however, the presence of the very small conditional variances allows us to flatten the spectrum to a much lower level, as is shown in figure 7. As a result, the search for the transformed integer least-squares ambiguities, based on sequential bounds similar to those of (9), can be executed in a highly efficient manner.

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