THE GPS PHASE-ADJUSTED PSEUDORANGE

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Abstract

In [1] it was shown that the test statistic as used in the DIA-procedure for the purpose of detecting and identifying GPS phase ambiguity slips is definitely superior in terms of power when compared to the method based on single-channel monitoring of estimated phase ambiguities. Instead of single-channel testing the present contribution is concerned with a commonly applied method of single-channel data processing for (relative) kinematic GPS positioning based on both pseudoranges and carrier phases. This recursive estimation method is generally referred to as the phase-smoothed pseudorange algorithm. It will be shown that the recursive phase-smoothed pseudorange estimator is not a recursive least-squares estimator in a strict sense. Instead it is a recursive pseudo least-squares estimator. Its precision is shown to be very close to optimal in case satellite redundancy is absent. This however may not be the case when satellite redundancy is present. For this more general case an alternative to the phase-smoothed pseudorange algorithm is presented in the paper. It is referred to as the phase-adjusted pseudorange algorithm.

I. INTRODUCTION

Every integrated navigation system should contain real-time quality control procedures for validating the models that underlie the navigation system. The DIA-procedure for the detection, identification and adaptation of model errors is particularly suited for this task, see e.g. [2]-[4]. The DIA-procedure is completely recursive and avoids the explicit use of a parallel bank of augmented navigation filters. Testing in the DIA-procedure is based on the use of uniformly-most-powerful-invariant test statistics. In [1] the power of the test statistics as used in the DIA-procedure was illustrated for a particular partially constant state-space model. This partially constant state-space model is often used when (relative) kinematic GPS positioning is based on the measurement of both pseudoranges and carrier phases. In order to illustrate the power of the test statistics as used in the DIA-procedure two different test statistics for GPS phase ambiguity slips were given in [1]. The first test statistic which was given is the one that is used in the DIA-procedure and which follows when the requirements of uniformly most powerfulness and invariance are imposed. The second test statistic which was given for GPS phase ambiguity slips is the one that follows when monitoring of the estimated phase ambiguities is done on a single-channel basis. It is based on a simple standardized scalar difference of the estimated phase ambiguities. The power of the two test statistics was compared through the ratio of their respective MDB's. And one of the important results of [1] is that the power of the first test statistic largely exceeds that of the single-channel test statistic. Hence the conclusion reads that single-channel monitoring of estimated phase ambiguities for the purpose of detecting and identifying GPS phase ambiguity slips is definitely inferior when compared to the corre-

sponding test statistic as used in the DIA-procedure.
Was one of the objectives of [1] to analyse the power of single-channel testing for GPS phase ambiguity slips, one of the objectives of the present contribution is to analyse the optimality in terms of precision of a commonly used single-channel recursive estimator for (relative) kinematic GPS positioning. This recursive estimator is generally referred to as the phase-smoothed pseudorange estimator. The estimator was first proposed in [5] and since then successfully used in a number of kinematic GPS positioning applications. Experiences with the phase-smoothed pseudorange estimator as reported in the geodetic literature can for instance be found in [6]-[12].
In section 2 a recapitulation of the phase-smoothed pseudorange algorithm will be given. The recapitulation is based on [5] and [6]. It is argued in section 2 that two basic assumptions underlie the phase-smoothed pseudorange algorithm. The first assumption is that the variance of the precise carrier phase observables may be set to zero in the derivation of the estimator. Hence, the phase-smoothed pseudorange estimator is not a recursive least-squares estimator in a strict sense. Instead it should be interpreted as a pseudo least-squares estimator. The second assumption concerns the absence of satellite redundancy. The single-channel processing of the phase-smoothed pseudorange algorithm and its independence of receiver-satellite geometry is strictly speaking only valid as long as satellite redundancy is absent.
In section 3 an alternative recursive algorithm for the processing of both pseudoranges and carrier phases is presented. This phase-adjusted pseudorange algorithm provides for a strict least-squares estimator and incorporates the information stemming from any satellite redundancy.
Finally in section 4 a comparison is made between the phase-smoothed pseudorange estimator and the phase-adjusted pseudorange estimator. It is shown that under certain conditions also the phase-adjusted pseudorange navigation solution can be computed on the basis of single-channel processing. The conditions are time-invariance of the receiver-satellite geometry or an absence of satellite redundancy. It is also shown if in this case the variance of the carrier phases is neglected that then the phase-adjusted pseudorange algorithm reduces to that of the phase-smoothed pseudorange algorithm. In that case the phase-smoothed pseudorange navigation solution can be shown to be close to optimal.

II. THE PHASE-SMOOTHED PSEUDORANGE ALGORITHM

The phase-smoothed pseudorange algorithm reads in recursive form as [5],[6]:

$$(1) \quad \begin{cases} a) \; \underline{\hat{p}}_{k|k-1} = \underline{\hat{p}}_{k-1|k-1} + [\underline{\bar{p}}_k - \underline{\bar{p}}_{k-1}] \\ b) \; \underline{\hat{p}}_{k|k} = \underline{\hat{p}}_{k|k-1} + \frac{1}{k}[\underline{p}_k - \underline{\hat{p}}_{k|k-1}] \end{cases}$$

with:

- $\underline{p}_k$: the pseudorange observable at epoch t_k
- $\underline{\bar{p}}_k$: the carrier phase observable at epoch t_k
- $\underline{\hat{p}}_{k|k-1}$: the predicted pseudorange at epoch t_k
- $\underline{\hat{p}}_{k|k}$: the filtered (or phase-smoothed) pseudorange at epoch t_k

The algorithm is initialized with $\hat{\underline{p}}_{1|1} = \underline{p}_1$. A single equation for the recursion is obtained when (1a) is substituted into (1b):

$$\hat{\underline{p}}_{k|k} = \frac{1}{k}\underline{p}_k + \frac{k-1}{k}[\hat{\underline{p}}_{k-1|k-1} + (\underline{\bar{p}}_k - \underline{\bar{p}}_{k-1})] \tag{2}$$

This equation shows that the filtered pseudorange at epoch t_k is a linear combination of the pseudorange at epoch t_k with weight $1/k$ and the predicted pseudorange at epoch t_k with weight $(k-1)/k$, where prediction is based on the phase difference between t_k and t_{k-1}.
If we assume the dispersion of the pseudoranges and carrier phases to be

$$\sigma_{p_k p_l} = \sigma^2 \delta_{kl} \ , \ \sigma_{\bar{p}_k \bar{p}_l} = \bar{\sigma}^2 \delta_{kl} \ , \ \sigma_{p_k \bar{p}_l} = 0, \tag{3}$$

an application of the error propagation law to (1) gives:

$$\begin{cases} \sigma^2_{\hat{p}_{k|k-1}} & = & \dfrac{k\bar{\sigma}^2 + \sigma^2}{k-1} \\[2ex] \sigma^2_{\hat{p}_{k|k}} & = & \dfrac{(k-1)\bar{\sigma}^2 + \sigma^2}{k}. \end{cases} \tag{4}$$

This result shows that the filtered (or phase-smoothed) pseudorange becomes increasingly more precise. The minimum value of its variance is obtained for the limit $k \to \infty$ as

$$\lim_{k\to\infty} \sigma^2_{\hat{p}_{k|k}} = \bar{\sigma}^2. \tag{5}$$

In order to understand the basic assumptions that underlie the phase-smoothed pseudorange algorithm (1), it is expedient to find out whether the algorithm can be interpreted in a recursive least-squares sense. In order to do so, we start from the following model of observation equations:

$$E\{\begin{pmatrix} \underline{p}_1 \\ \underline{\bar{p}}_2 - \underline{\bar{p}}_1 \\ \underline{p}_2 \\ \underline{\bar{p}}_3 - \underline{\bar{p}}_2 \\ \vdots \\ \underline{\bar{p}}_k - \underline{\bar{p}}_{k-1} \\ \underline{p}_k \end{pmatrix}\} = \begin{pmatrix} 1 & & & & \\ -1 & 1 & & & \\ & 1 & & & \\ & -1 & 1 & & \\ & & & \ddots & \\ & & & -1 & 1 \\ & & & & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_{k-1} \\ p_k \end{pmatrix}, \tag{6}$$

where $E\{.\}$ stands for the expectation operator.
Note that the variance-matrix of the vector of observables of (6) is non-diagonal because of the nonzero covariance between $(\underline{\bar{p}}_{j+1} - \underline{\bar{p}}_j)$ and $(\underline{\bar{p}}_j - \underline{\bar{p}}_{j-1})$. It is this non-diagonal structure of the variance matrix that prohibits one to solve (6) in a recursive least-squares sense. The conclusion must therefore be that the phase-smoothed pseudorange algorithm (1) is *not* a recursive least-squares algorithm in a strict sense. It is possible, however, to

interpret the phase-smoothed pseudorange algorithm (1) in a *pseudo* least-squares sense. The phase-smoothed pseudorange algorithm follows namely as the solution of (6) when (6) is solved in a recursive least-squares manner under the assumption that $\bar{\sigma}^2 = 0$. The approximation involved due to this assumption will probably be acceptable for all practical purposes since $\bar{\sigma}^2 \ll \sigma^2$.
A second assumption that underlies the phase-smoothed pseudorange algorithm (1) is the absence of satellite redundancy. Note that the filtered (or phase-smoothed) pseudorange $\hat{p}_{k|k}$ of (1) is computed on a single-channel basis and that it is independent of the receiver-satellite geometry. This however is strictly speaking only possible in the absence of any satellite redundancy. The implication is therefore that an alternative recursive algorithm for the filtered pseudorange needs to be devised in case satellite redundancy is present. The derivation of this algorithm will be given in the next section.

III. THE PHASE-ADJUSTED PSEUDORANGE ALGORITHM

Instead of starting from model (6) we will start from an equivalent model that explicitly contains the necessary parameters for the navigation solution. This model of observation equations reads in its linearized form as:

$$(7) \qquad E\left\{\begin{pmatrix} \underline{y}_1 \\ \underline{\bar{y}}_1 \\ \underline{y}_2 \\ \underline{\bar{y}}_2 \\ \vdots \\ \underline{y}_k \\ \underline{\bar{y}}_k \end{pmatrix}\right\} = \begin{pmatrix} A_1 & & & & 0 \\ A_1 & & & & I \\ & A_2 & & & 0 \\ & A_2 & & & I \\ & & \ddots & & \vdots \\ & & & A_k & 0 \\ & & & A_k & I \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ \nabla \end{pmatrix},$$

with:

- $\underline{y}_i$: the mx1 vector of linearized (observed minus computed) pseudoranges at epoch t_i,
- $\underline{\bar{y}}_i$: the mx1 vector of linearized carrier phases at epoch t_i,
- x_i: the nx1 (n=4) vector of unknown parameters at epoch t_i containing the increments of the position coordinates of the receiver and the relative receiver clock error,
- A_i : the mxn linearized design matrix, consisting of the line-of-sight unit vectors to the satellites with 1's in the fourth column corresponding to the unknown (relative) receiver clock error,
- ∇: the mx1 vector of unknown ambiguities which are assumed to be constant in time in the absence of cycleslips,

and where the variance matrix is block-diagonal with blocks $Q_{y_i} = \sigma^2 I$ and $Q_{\bar{y}_i} = \bar{\sigma}^2 I$ respectively. The above model is considered valid for the mobile receiver in a kinematic relative positioning or DGPS set-up. GPS system errors such as orbital biases and satellite clock errors are assumed to be corrected for by employing a stationary GPS reference station. These errors are therefore not modelled in (7). It is also assumed that a model is employed if necessary that takes care of the tropospheric delay through observation

corrections. And finally it is assumed that also the ionospheric delay need not be modelled explicitly. The ionospheric delay is either been taken care of at the mobile receiver by employing dual-frequency observations and/or corrections for it are provided through the employment of a stationary GPS reference station.
Since the matrices A_i are assumed to be of full rank, the overall redundancy of model (7) equals $k(2m-n)-m$. At every epoch we have a redundancy of $m-n$ due to the presence of satellite redundancy. This makes for a redundancy of $k(m-n)$ after k number of epochs. In (7) it is also assumed that the phase ambiguities remain constant. This takes care of the additional redundancy of $(k-1)m$.
The first step in solving model (7) recursively consists of an initialization at epoch t_1:

$$(8) \qquad E\{\begin{pmatrix} \underline{y}_1 \\ \underline{\bar{y}}_1 \end{pmatrix}\} = \begin{pmatrix} A_1 & 0 \\ A_1 & I \end{pmatrix} \begin{pmatrix} x_1 \\ \nabla \end{pmatrix} \; ; \; D\{\begin{pmatrix} \underline{y}_1 \\ \underline{\bar{y}}_1 \end{pmatrix}\} = \begin{pmatrix} Q_{y_1} & 0 \\ 0 & Q_{\bar{y}_1} \end{pmatrix}.$$

The least-squares solution of this model reads:

$$(9) \qquad \begin{cases} \underline{\hat{x}}_1 &= (A_1^* Q_{y_1}^{-1} A_1)^{-1} A_1^* Q_{y_1}^{-1} \underline{y}_1 \quad ; \quad Q_{\hat{x}_1} = (A_1^* Q_{y_1}^{-1} A_1)^{-1} \\ \underline{\hat{\nabla}}_1 &= \underline{\bar{y}}_1 - A_1 \underline{\hat{x}}_1 \qquad\qquad\qquad ; \quad Q_{\hat{\nabla}_1} = Q_{\bar{y}_1} + A_1 Q_{\hat{x}_1} A_1^* \end{cases} .$$

Note that the variance matrix $Q_{\hat{\nabla}_1}$ of the estimated ambiguities is generally nondiagonal if satellite redundancy is present $(m > n)$. With the initialization established a start can be made with the recursion. The following model of observation equations holds at epoch t_k:

$$(10) \quad E\{\begin{pmatrix} \underline{\hat{\nabla}}_{k-1} \\ \underline{y}_k \\ \underline{\bar{y}}_k \end{pmatrix}\} = \begin{pmatrix} 0 & I \\ A_k & 0 \\ A_k & I \end{pmatrix} \begin{pmatrix} x_k \\ \nabla \end{pmatrix} \; ; \; D\{\begin{pmatrix} \underline{\hat{\nabla}}_{k-1} \\ \underline{y}_k \\ \underline{\bar{y}}_k \end{pmatrix}\} = \begin{pmatrix} Q_{\hat{\nabla}_{k-1}} & & \\ & Q_{y_k} & \\ & & Q_{\bar{y}_k} \end{pmatrix}.$$

For the purpose of our derivation we rewrite model (10) as:

$$(11) \quad \begin{aligned} & E\{\begin{pmatrix} \underline{\hat{\nabla}}_{k-1} \\ \underline{y}_k \\ \underline{\bar{y}}_k - \underline{\hat{\nabla}}_{k-1} \end{pmatrix}\} = \begin{pmatrix} 0 & I \\ A_k & 0 \\ A_k & 0 \end{pmatrix} \begin{pmatrix} x_k \\ \nabla \end{pmatrix} \; ; \\ & D\{\begin{pmatrix} \underline{\hat{\nabla}}_{k-1} \\ \underline{y}_k \\ \underline{\bar{y}}_k - \underline{\hat{\nabla}}_{k-1} \end{pmatrix}\} = \begin{pmatrix} Q_{\hat{\nabla}_{k-1}} & 0 & -Q_{\hat{\nabla}_{k-1}} \\ 0 & Q_{y_k} & 0 \\ -Q_{\hat{\nabla}_{k-1}} & 0 & Q_{\bar{y}_k} + Q_{\hat{\nabla}_{k-1}} \end{pmatrix} \end{aligned}$$

Note the two types of redundancy involved. There is a redundancy of m since $E\{\underline{y}_k\} = E\{\underline{\bar{y}}_k - \underline{\hat{\nabla}}_{k-1}\}$. This redundancy follows from the assumed time-invariance of the phase ambiguities. And there is a redundancy of $m-n$ since $E\{\underline{y}_k\}$ must lie in the orthogonal complement of the range space of A_k. This redundancy exists only in the presence of satellite redundancy. From (11) the least-squares navigation solution directly follows as:

$$\begin{cases} \underline{\hat{x}}_k &= [A_k^*(Q_{y_k}^{-1} + [Q_{\bar{y}_k} + Q_{\hat{\nabla}_{k-1}}]^{-1})A_k]^{-1} A_k^*(Q_{y_k}^{-1}\underline{y}_k + [Q_{\bar{y}_k} + Q_{\hat{\nabla}_{k-1}}]^{-1}[\underline{\bar{y}}_k - \underline{\hat{\nabla}}_{k-1}]) \\ Q_{\hat{x}_k} &= [A_k^*(Q_{y_k}^{-1} + [Q_{\bar{y}_k} + Q_{\hat{\nabla}_{k-1}}]^{-1})A_k]^{-1} \end{cases}$$

(12)

In order to complete the recursion we still need a tractable expression for the estimator $\underline{\hat{\nabla}}_k$. Since the least-squares residual of $\underline{\hat{\nabla}}_{k-1}$ can be written in terms of the least-squares residual of $\underline{\bar{y}}_k - \underline{\hat{\nabla}}_{k-1}$ as $\underline{\hat{e}}_{\hat{\nabla}_{k-1}} = Q_{\hat{\nabla}_{k-1}(\bar{y}_k - \hat{\nabla}_{k-1})} Q^{-1}_{(\bar{y}_k - \hat{\nabla}_{k-1})} \underline{\hat{e}}_{(\bar{y}_k - \hat{\nabla}_{k-1})}$, the estimator $\underline{\hat{\nabla}}_k = \underline{\hat{\nabla}}_{k-1} - \underline{\hat{e}}_{\hat{\nabla}_{k-1}}$ and its variance matrix follow as:

$$(13) \qquad \begin{cases} \underline{\hat{\nabla}}_k &= \underline{\hat{\nabla}}_{k-1} + Q_{\hat{\nabla}_{k-1}}[Q_{\bar{y}_k} + Q_{\hat{\nabla}_{k-1}}]^{-1}[\underline{\bar{y}}_k - \underline{\hat{\nabla}}_{k-1} - A_k \underline{\hat{x}}_k] \\ Q_{\hat{\nabla}_k} &= Q_{\hat{\nabla}_{k-1}} - Q_{\hat{\nabla}_{k-1}}[Q_{\bar{y}_k} + Q_{\hat{\nabla}_{k-1}}]^{-1} Q_{\hat{\nabla}_{k-1}} \\ & \quad + Q_{\hat{\nabla}_{k-1}}[Q_{\bar{y}_k} + Q_{\hat{\nabla}_{k-1}}]^{-1} A_k Q_{\hat{x}_k} A_k^* [Q_{\bar{y}_k} + Q_{\hat{\nabla}_{k-1}}]^{-1} Q_{\hat{\nabla}_{k-1}} \end{cases}$$

The two sets of equations (12) and (13) constitute the least-squares solution of (10) and together they form the recursive phase-adjusted pseudorange algorithm.

IV. COMPARISON WITH THE PHASE-SMOOTHED PSEUDORANGE

In the previous section the recursive least-squares phase-adjusted pseudorange algorithm was given. In this section this algorithm will be compared with the recursive phase-smoothed pseudorange algorithm of section 2. In order to do so we first note that the navigation solution of (12) can be written as

$$(14) \qquad \underline{\hat{x}}_k = [A_k^* Q_{\tilde{y}_k}^{-1} A_k]^{-1} A_k^* Q_{\tilde{y}_k}^{-1} \underline{\tilde{y}}_k \; ; \; Q_{\hat{x}_k} = [A_k^* Q_{\tilde{y}_k}^{-1} A_k]^{-1},$$

where

$$(15) \qquad \begin{cases} \underline{\tilde{y}}_k &= \underline{y}_k - Q_{y_k}[Q_{y_k} + Q_{\bar{y}_k} + Q_{\hat{\nabla}_{k-1}}]^{-1}[\underline{y}_k - (\underline{\bar{y}}_k - \underline{\hat{\nabla}}_{k-1})] \\ Q_{\tilde{y}_k} &= [Q_{y_k}^{-1} + (Q_{\bar{y}_k} + Q_{\hat{\nabla}_{k-1}})^{-1}]^{-1} \end{cases} .$$

The estimator $\underline{\tilde{y}}_k$ is the least-squares pseudorange estimator which follows when at epoch t_k only the redundancy stemming from the assumed time-invariance of the phase ambiguities is taken into account. As such the estimator $\underline{\tilde{y}}_k$ can be directly compared to the phase-smoothed pseudorange estimator of section 2. The pseudorange estimator $\underline{\tilde{y}}_k$ can be improved by taking the satellite redundancy at epoch t_k into account. If this is done one obtains the least-squares pseudorange estimator $\underline{\hat{y}}_k = A_k \underline{\hat{x}}_k$. The two pseudorange estimators $\underline{\tilde{y}}_k$ and $\underline{\hat{y}}_k$ are therefore only identical in case satellite redundancy is absent. If satellite redundancy is absent then the variance matrix $Q_{\hat{\nabla}_{k-1}}$ becomes diagonal and both estimators of (13) and (15) can be processed on a single-channel basis. Thus like with the phase-smoothed pseudorange algorithm, also the phase-adjusted pseudorange algorithm

performs on a single-channel basis if satellite redundancy is absent.
It turns out however, that this last condition can even be relaxed somewhat. It follows namely that the phase-adjusted pseudorange algorithm can also be processed on a single-channel basis under certain conditions if $m > n$. The conditions referred to are that the design matrices A_i need to be time-invariant, i.e. $A_i = A$ for all i. In order to show this we first note that (13) and (15) reduce to

$$(13') \quad \begin{cases} \underline{\hat{\nabla}}_k = \underline{\hat{\nabla}}_{k-1} + [\frac{\bar{\sigma}^2+\sigma^2}{k\bar{\sigma}^2+\sigma^2}I - \frac{(k-1)\sigma^2}{k(k\bar{\sigma}^2+\sigma^2)}P_B][\underline{\bar{y}}_k - \underline{\hat{\nabla}}_{k-1} - A\underline{\hat{x}}_k] \\ Q_{\hat{\nabla}_k} = \frac{\bar{\sigma}^2+\sigma^2}{k}I - \frac{\sigma^2}{k}P_B, \end{cases}$$

and

$$(15') \quad \begin{cases} \underline{\tilde{y}}_k = \underline{y}_k - \frac{(k-1)\sigma^2}{k(\sigma^2+\bar{\sigma}^2)}[I + \frac{\sigma^2}{(k-1)\sigma^2+k\bar{\sigma}^2}P_B][\underline{y}_k - (\underline{\bar{y}}_k - \underline{\hat{\nabla}}_{k-1})] \\ Q_{\tilde{y}_k} = [\frac{k(\sigma^2+\bar{\sigma}^2)}{\sigma^2(k\bar{\sigma}^2+\sigma^2)}I + \frac{(k-1)\sigma^2/\bar{\sigma}^2}{k(k\bar{\sigma}^2+\sigma^2)}P_B]^{-1}, \end{cases}$$

if $A_i = A$ for all i. The matrix P_B in (13′) and (15′) is the orthogonal projector $P_B = B(B^*B)^{-1}B^*$ that projects orthogonally onto the orthogonal complement of the range space of A. The projector P_B is nondiagonal in case $m > n$ and it vanishes identically in case $m = n$. In case $m > n$ the nondiagonal structure of P_B in (13′) and (15′) prohibits the single-channel processing of $\tilde{y}_k$ and $\hat{\nabla}_k$ respectively. However, since $P_BA = 0$ it follows from (13′), (14) and (15′) that the navigation solution $\underline{\hat{x}}_k$ itself is *invariant* for the P_B-terms of (13′) and (15′). Hence, the least-squares navigation solution $\underline{\hat{x}}_k$ also follows when $\underline{\tilde{y}}_k$ and $Q_{\tilde{y}_k}$ in (14) are replaced by

$$(15'') \quad \begin{cases} \underline{\tilde{y}}'_k = \underline{y}_k - \frac{k-1}{k}\frac{\sigma^2}{\sigma^2+\bar{\sigma}^2}[\underline{y}_k - (\underline{\bar{y}}_k - \underline{\hat{\nabla}}'_{k-1})] \\ Q_{\tilde{y}'_k} = \frac{\sigma^2(k\bar{\sigma}^2+\sigma^2)}{k(\sigma^2+\bar{\sigma}^2)}I, \end{cases}$$

where

$$(13'') \quad \begin{cases} \underline{\hat{\nabla}}'_k = \underline{\hat{\nabla}}'_{k-1} + \frac{\bar{\sigma}^2+\sigma^2}{k\bar{\sigma}^2+\sigma^2}[\underline{\bar{y}}_k - \underline{\hat{\nabla}}'_{k-1} - \underline{\tilde{y}}'_k] \\ Q_{\hat{\nabla}'_k} = \frac{\bar{\sigma}^2+\sigma^2}{k}I, \end{cases}$$

with $\hat{\underline{v}}'_1 = \bar{\underline{y}}_1 - \underline{y}_1$. This result shows that the phase-adjusted pseudorange navigation solution can also be obtained on a single-channel basis if $m > n$ provided that the design matrices A_i are time-invariant. In reality of course the design matrices A_i will never be time-invariant due to the time dependency of the receiver-satellite geometry. But if the time-span considered is such that the relatively slow change of the GPS receiver-satellite geometry can be neglected then the results obtained from a single-channel processing based on (13″) and (15″) are close to the exact phase-adjusted pseudorange navigation solution.
Let us now assume that satellite redundancy is absent and compare the phase-adjusted pseudorange estimator with the phase-smoothed pseudorange estimator. If satellite redundancy is absent then $\hat{\underline{y}}_k = A_k\hat{\underline{x}} = \tilde{\underline{y}}_k = \tilde{\underline{y}}'_k$ and the phase-adjusted pseudorange algorithm is formed by (13″), (14) and (15″) with $\tilde{\underline{y}}_k$ and $Q_{\tilde{y}_k}$ in (14) replaced by $\tilde{\underline{y}}'_k$ and $Q_{\tilde{y}'_k}$. And the phase-smoothed pseudorange algorithm of section 2 follows then from (13″) and (15″) if one *neglects* the small term $\bar{\sigma}^2$. With $\bar{\sigma}^2 = 0$, the first equation of (13″) can namely be written for epoch t_{k-1} as $\hat{\underline{v}}'_{k-1} = \bar{\underline{y}}_{k-1} - \tilde{\underline{y}}'_{k-1}$. This gives $\bar{\underline{y}}_k - \hat{\underline{v}}'_{k-1} = \tilde{\underline{y}}'_{k-1} + (\bar{\underline{y}}_k - \bar{\underline{y}}_{k-1})$ which corresponds to the first equation of (1). The second equation of (1) follows then from the first equation of (15″) for $\bar{\sigma}^2 = 0$.
It will be clear that the phase-adjusted pseudorange estimator $\tilde{\underline{y}}'_k$ and the corresponding phase-smoothed pseudorange estimator are both linear unbiased estimators. The difference between the two estimators lies in their precision. And indeed the variance $\sigma^2_{\tilde{y}'_k}$ of the phase-adjusted pseudorange estimator [see (15″)] differs from the variance $\sigma^2_{\hat{p}_{k|k}}$ of the phase-smoothed pseudorange estimator [see (4)]. Also the phase-adjusted pseudorange estimator becomes increasingly more precise. It follows from (15″) that the minimum value of its variance is obtained for the limit $k \to \infty$ as

$$(16) \qquad \lim_{k\to\infty} \sigma^2_{\tilde{y}'_k} = \frac{\bar{\sigma}^2}{1 + \bar{\sigma}^2/\sigma^2}.$$

Compare this result with that of (5).
The phase-adjusted pseudorange estimator $\tilde{\underline{y}}'_k$, being a strict least-squares estimator, is a best linear unbiased estimator. Hence, of all possible linear unbiased estimators the phase-adjusted pseudorange estimator has the smallest variance. Its precision is therefore better than that of the phase-smoothed pseudorange estimator. The difference in precision at epoch t_k between the phase-smoothed and phase-adjusted pseudorange estimator follows from (4) and (15″) as

$$(17) \qquad \sigma^2_{\hat{p}_{k|k}} - \sigma^2_{\tilde{y}'_k} = \frac{(k-1)\bar{\sigma}^4}{k(\sigma^2 + \bar{\sigma}^2)}.$$

This difference is zero at the initialization epoch t_1 and nonzero but very small ($\bar{\sigma}^2 \ll \sigma^2$) for the other epochs. The conclusion must therefore be that although the phase-smoothed pseudorange estimator is theoretically nonoptimal it is a very close to optimal estimator in case satellite redundancy is absent.

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