Quality Control in Integrated Navigation Systems

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ABSTRACT

Real-time estimation of parameters in dynamic systems becomes increasingly important in the field of high precision navigation. The real-time estimation inevitably requires real-time testing of the models underlying the navigation system. This paper presents: 1. A real-time recursive testing procedure that can be used in conjunction with the well-known Kalman filter algorithm; 2. Diagnostic tools for inferring the detectability of particular model errors. The testing procedure consists of three steps: detection, identification and adaptation. It can accommodate model errors in both the measurement model and dynamic model of the integrated navigation system. The tests proposed are optimal in the uniformly-most-powerful-invariant sense.

CONTENTS

1. Introduction
2. Recursive Filtering under H_o
3. Recursive Testing of H_o against H_a
4. A Testing Procedure
5. On Minimal Detectable Biases (MDB)
6. Conclusion
7. References

INTRODUCTION

It is well-known that the real-time Kalman filter produces optimal estimates with well defined statistical properties. The estimates are unbiased and they have minimum variance within the class of linear unbiased estimates. The quality of the estimates is however only guaranteed as long as the assumptions underlying the mathematical model hold. Misspecifications in the model will invalidate the results of filtering and thus also any conclusion based on them. It is therefore of importance to have ways to verify the validity of the assumed mathematical model.

This paper presents a general procedure for the real-time validation of the measurement—and dynamic model of an integrated navigation system. The results present are based on the *quality control theory* as developed at the Delft Geodetic Computing Centre (1-5).

The contents of the paper is as follows. First we briefly discuss the model underlying linear(ized) dynamic systems and introduce the recursive Kalman filter algorithm. In section 3 we introduce a teststatistic T that is optimal in the uniformly-most-powerful-invariant sense for testing for the presence of bias in the predicted residuals. Our recursive testing procedure of section 4 is based on the teststatistic T. The testing procedure consists of three steps:

1. Detection
2. Identification
3. Adaptation

An overall model test is presented to detect unspecified model errors in the nullhypothesis H_o. After detection of a model error, identification of the potential source of the model error is needed. The identification step consists of a search among all candidates for the most likely alternative hypothesis H_a and most likely starting time l. Finally, after identifying the most likely model error, adaptation of the recursive filter is carried out to eliminate statevector biases. Our testing procedure is recursive and can be computed in real-time through a scheme that closely parallels the nominal Kalman filter algorithm. It does not need explicitly a bank of parallel running Kalman filters. The material of section 5 is based on the concept of the power of a statistical test. Since one is generally in practical applications less interested in the power probability than in the bias that generated it, it is proposed in section 5 to use the inverted power function. In this way boundary values of the biases

Based on a paper presented at PLANS '90.
0885/8985/90/0700/0035 $1.00 © 1990 IEEE

In: IEEE Aerospace and Electronic Systems Magazine, Vol. 5, No. 7, 1990, pp. 35-41.

can be computed that are detectable with a certain prefixed reference probability. These biases are termed minimal detectable biases (MDB). An outline is given of how the MDB's can be used for the design of a navigation filter that allows for a sufficient control on the presence of bias.

In the last section some extensions of the presented quality control theory are discussed.

RECURSIVE FILTERING UNDER H_o

In this section we present the mathematical model of the discrete time linear(ized) dynamic system under the nullhypothesis H_o and the corresponding recursive filtering equations that define the optimal estimators of the system state. The *dynamics* of the system are modelled by the equation

$$x_{k+1} = \Phi_{k+1,k} x_k + d_k \ , \quad k = 0, 1, \ldots \tag{1}$$

where x_k is the n-dimensional statevector at time k, $\Phi_{k+1,k}$ is the known n-by-n transition matrix and d_k is the process noise, assumed to be Gaussian distributed with mean zero and known covariance

$$E\{d_k d_l^*\} = Q_k \delta_{kl}. \tag{2}$$

The initial state is also Gaussian distributed with known mean $x_{o|o}$ and known covariance $P_{o|o}$, independent of d_k. The *measurements* of the system are modelled by the equation

$$y_k = A_k x_k + e_k \ , \quad k = 1, 2, \ldots \tag{3}$$

where A_k is a known m_k-by-n design matrix and the measurement noise e_k, independent of d_l and x_o, is Gaussian distributed with mean zero and known covariance

$$E\{e_k d_l^*\} = R_k \delta_{kl}. \tag{4}$$

Based on the above model, the optimal recursive prediction and filtering equations read

$$\begin{aligned} \hat{x}_{k|k-1} &= \Phi_{k,k-1} \hat{x}_{k-1|k-1} \\ \hat{x}_{k|k} &= \hat{x}_{k|k-1} + K_k (y_k - A_k \hat{x}_{k|k-1}) \end{aligned} \tag{5}$$

with corresponding covariance matrices

$$\begin{aligned} P_{k|k-1} &= \Phi_{k,k-1} P_{k-1|k-1} \Phi_{k,k-1}^* + Q_k \\ P_{k|k} &= [I - K_k A_k] P_{k|k-1} \end{aligned} \tag{6}$$

where

$$K_k = P_{k|k-1} A_k^* [R_k + A_k P_{k|k-1} A_k^*]^{-1}$$

is the so-called Kalman gain matrix.

The above filter produces optimal estimators of the statevector with well defined statistical properties. The state estimators are unbiased, are Gaussian distributed and have minimum variance within the class of linear unbiased estimators. It is important to realize however, that optimality is only guaranteed as long as the assumptions underlying the mathematical model hold. Misspecifications in the model will invalidate the results of estimation and thus also any conclusion based on them. It is therefore of importance to have ways to verify the validity of the working hypothesis H_o.

An important role in the process of model testing is played by the *predicted residual*. The predicted residual is defined as the difference between the actual system output and the predicted output based on the predicted state

$$v_k = y_k - A_k \hat{x}_{k|k-1}. \tag{7}$$

Under the working hypothesis H_o, the predicted residual is Gaussian distributed with mean zero and covariance

$$E\{v_k v_l^*\} = Q_{v_k} \delta_{kl} \tag{8}$$

where

$$Q_{v_k} = [R_k + A_k P_{k|k-1} A_k^*].$$

This knowledge of the distribution of the predicted residual under H_o enables us to test the validity of the assumed mathematical model. Although it is possible to develop teststatistics for the second and higher moments of v_k, we will restrict ourselves in this paper to the first moments of the predicted residuals. That is, we will only consider misspecifications in the mean of the predicted residual, caused by *underparametrizations* in the dynamic model and/or the measurement model.

RECURSIVE TESTING OF H_o AGAINST H_a

In this section we present the uniformly-most-powerful-invariant teststatistic for testing the nullhypothesis H_o against an alternative hypothesis H_a. The teststatistic is a quadratic form in the predicted residuals. We define the vector of predicted residuals as

$$v = (v_1^*, v_2^*, \ldots, v_k^*)^*.$$

The following two hypotheses are considered

$$H_0: \ v \sim N(0, Q_v) \quad and \quad H_a: \ v \sim N(\nabla v, Q_v). \tag{9}$$

We will assume that the $(\Sigma_{i=1}^{k} m_i)$ vector ∇v can be parametrized as

$$\nabla v = C_v \nabla \tag{10}$$

with

$$C_v = (C_{v_1}^*, C_{v_2}^*, \ldots, C_{v_k}^*)^*,$$

where C_v is a $(\Sigma_{i=1}^{k} m_i)$ matrix and ∇ is a vector of dimension b. The matrix C_v is assumed to be known and of full rank b, and the vector ∇ is assumed to be unknown.

The specification of appropriate alternative hypotheses for a particular application is non-trivial and probably the most difficult task in the process of quality control. It depends to a great extent on experience and ones knowledge of the dynamic system. The following two types of alternative hypotheses do contain however the most frequently occurring model errors:

1. A slip $C_{y,k}\nabla$ in the vector of measurements that starts at time l:
$$y_k = A_k x_k + C_{y,k} \nabla + e_k$$
with
$$C_{y,k} = 0 \quad for \quad k < l.$$
This alternative hypothesis can accommodate sensor failures and outliers in the data. The matrix $C_{y,k}$ propagates into the matrices C_{v_i} of (10) as

$$\begin{cases} C_{v_i} = C_{y,i} - A_i X_{i,l} \ , \ i = l, \ldots, k \\ X_{i+1,l} = \Phi_{i+1,i}[X_{i,l} + K_i C_{v_i}] \ , \ X_{l,l} = 0 \end{cases} \quad (11)$$

In case of a failure in the jth sensor, the matrix $C_{y,i}$ becomes a vector that reads $(0,\ldots, 1,0,\ldots)^*$, with a one at the jth position.

2. A slip $C_{x,k}\nabla$ in the statevector that starts at time l:

$$x_k = \Phi_{k,k-1}x_{k-1} + C_{x,k}\nabla + d_k$$

with

$$C_{x,k} = 0 \quad for \quad k < l.$$

This alternative hypothesis can accommodate under-parameterizations in the statevector. For instance, assume that the dynamics of a moving vehicle is based on a constant velocity model under H_o. Then, if at time l the vehicle starts accelerating linearly the constant velocity model becomes inadequate and an additional parametrization in the form of $C_{x,k}\nabla$ is needed. The matrix $C_{x,k}$ propagates into the matrices C_{v_i} of (10) as:

$$\begin{cases} C_{v_i} = -A_i X_{i,l} \ ; \ i = l, \ldots, k \\ X_{i+1,l} = C_{x,i} + \Phi_{i+1,i}[I - K_i A_i] X_{i,l} \ ; \ X_{l,l} = C_{x,l} \end{cases} \quad (12)$$

With (11) and (12) it becomes possible to construct the matrix C_v of (10) *recursively*.

Once the alternative hypothesis has been specified, the appropriate teststatistic can be derived. The appropriate teststatistic for testing H_o against H_a reads (see e.g., (1),(2)):

$$T = v^* Q_v^{-1} C_v [C_v^* Q_v^{-1} C_v]^{-1} C_v^* Q_v^{-1} v \ . \quad (13)$$

Geometrically T can be interpreted as the square of the length of the vector that follows from projecting v orthogonally on the rangespace of C_v. Since the variance matrix Q_v is block-diagonal (see (8)) the teststatistic T may also be written as

$$\boxed{T^{l,k} = [\sum_{i=l}^{k} C_{v_i}^* Q_{v_i}^{-1} v_i]^* [\sum_{i=l}^{k} C_{v_i}^* Q_{v_i}^{-1} C_{v_i}]^{-1} [\sum_{i=l}^{k} C_{v_i}^* Q_{v_i}^{-1} v_i]} \quad (14)$$

in which l is the time that the slips start to occur.

Hence, the teststatistic can be computed *recursively* in a manner that parallels the recursive filter algorithm of the previous section. An alternative expression for $T^{l,k}$ is given by

$$T^{l,k} = \hat{\nabla}^{l,k*} Q_{\hat{\nabla}^{l,k}}^{-1} \hat{\nabla}^{l,k},$$

where

$$\hat{\nabla}^{l,k} = Q_{\hat{\nabla}^{l,k}} \sum_{i=l}^{k} C_{v_i}^* Q_{v_i}^{-1} v_i$$

is the *best linear unbiased estimator* of the b-vector ∇ under H_a, with variance matrix $Q_{\hat{\nabla}^{l,k}} = [\sum_{i=l}^{k} C_{v_i}^* Q_{v_i}^{-1} C_{v_i}]^{-1}$.

The teststatistic $T^{l,k}$ is distributed under H_o and H_a as

$$H_0: \ T^{l,k} \sim \chi^2(b,0) \quad and \quad H_a: \ T^{l,k} \sim \chi^2(b,\lambda) \quad (15)$$

with noncentrality parameter

$$\lambda = \nabla^* Q_{\hat{\nabla}^{l,k}}^{-1} \nabla.$$

The *uniformly-most-powerful-invariant* test of size α is now as follows: Reject H_o in favor of H_a if and only if $T^{l,k} \geq X_\alpha^2(b,O)$, where $X_\alpha^2(b,0)$ is the upper α probability point of the central X^2-distribution with b degrees of freedom.

A TESTING PROCEDURE

In this section we will develop a testing procedure for use in integrated navigation systems. The testing procedure is based on the teststatistic of the previous section. Our testing procedure consists of the following three steps:

1. *Detection:* An overall model test is performed to diagnose whether an unspecified model error has occurred.
2. *Identification:* After detection of a model error, identification of the potential source of the model error is needed. A search for the most likely alternative hypothesis and for the most likely starting time l is performed. Also the likelihood of their occurrence is tested.
3. *Adaptation:* After identification of an alternative hypothesis, adaptation of the recursive filter is needed to eliminate statevector biases.

These three steps will now be discussed in more detail. In a particular application one will never be sure whether the class of alternative hypotheses specified indeed contains the true hypothesis. It is therefore expedient to have ways to detect unspecified model errors as well. This is possible if the degrees of freedom b of the teststatistic $T^{l,k}$ is chosen to be equal to $\sum_{i=l}^{k} m_i$. If b equals the maximum possible degrees of freedom, the matrix C_v of (13) becomes a square and regular matrix, implying that the vector ∇v of (10) remains completely unspecified. In this case the invertible matrix C_v can be eliminated from (13) and the teststatistic (14) can be written as

$$T^{l,k} = \sum_{i=l}^{k} v_i^* Q_{v_i}^{-1} v_i.$$

Hence, $T^{l,k}$ can be computed *recursively* as

$$\boxed{T^{l,k} = T^{l,k-1} + T^{k,k}} \quad (16$$

This teststatistic can be used to perform an overall model test for *detecting* unspecified model errors in the nullhypothesis H_o. The *overall model test* is now as follows: An unspecified model error in H_o is considered to be present if and only if $T^{l,k} \geq X_\alpha^2(\sum_{i=l}^{k} m_i, 0)$.

The next step after detection is the identification of the most likely alternative hypothesis. As with detection, identification is based on the teststatistic (13). The dimension of the b-vector ∇ depends on the alternative hypothesis considered and can range for identification purposes from 1 to $\sum_{i=l}^{k} m_i$. Experience has shown however that it suffices for most practical applications to restrict oneselves to the case $b = 1$. We will consider therefore in this paper only the one-dimensional case. If $b = 1$, the matrix C_v of (13) reduces to a vector, which will be denoted by the lower case kernel letter c_v, and the vector ∇ reduces to a scalar. In this case the teststatistic (14) can be written as

$$T^{l,k} = (t^{l,k})^2$$

with

$$t^{l,k} = \frac{\sum_{i=l}^{k} c_{v_i}^* Q_{v_i}^{-1} v_i}{(\sum_{i=l}^{k} c_{v_i}^* Q_{v_i}^{-1} c_{v_i})^{1/2}}$$

Hence, with $\sigma_{\hat{\nabla}^{l,k}}^{-2} = \sigma_{\hat{\nabla}^{l,k-1}}^{-2} + \sigma_{\hat{\nabla}^{k,k}}^{-2}$, $t_{l,k}$ can be computed *recursively* as

$$t^{l,k} = \frac{\sigma_{\hat{\nabla}^{l,k}}}{\sigma_{\hat{\nabla}^{l,k-1}}} t^{l,k-1} + \frac{\sigma_{\hat{\nabla}^{l,k}}}{\sigma_{\hat{\nabla}^{k,k}}} t^{k,k} \quad (17)$$

Strictly speaking, the teststatistic $t^{l,k}$ has to be computed for each alternative hypothesis considered and for each $k \geq l$. Moreover, since l, the time that the model error starts to occur, is unknown a priori, one has to start in principle with $l = 1$. This implies that one has to compute k number of teststatistics $t^{l,k}$ per alternative hypothesis H_a at the time of testing k. This is shown in Figure 1a for an alternative hypothesis H_a.

Our identification procedure can now be described as follows: At the time of testing k one first determines per alternative hypothesis the value of l for which $|t^{l,k}|, 1 \leq l \leq k$, is at a maximum. This value of l would then be the most likely time of occurrence of the model error if the corresponding alternative hypothesis would be true. In order to find both the most likely alternative hypothesis and most likely value of l the values $\max_{l} |t^{l,k}|$ for the different alternative hypotheses are compared. The maximum of this set identifies then both the most likely time of occurrence l and the most likely alternative hypothesis H_a.

Since the one-dimensional teststatistic $t^{l,k}$ is distributed under H_o and H_a as

$$H_0: \; t^{l,k} \sim N(0,1) \quad , \quad H_a: t^{l,k} \sim N(\nabla/\sigma_{\hat{\nabla}^{l,k}}, 1), \quad (18)$$

the likelihood of the corresponding alternative hypothesis can be tested by comparing $|t^{l,k}|$ with the critical value $N_{1/2\alpha_0}(0,1)$. If $|t^{l,k}| \leq N_{1/2\alpha_0}(0,1)$, the corresponding most likely alternative hypothesis can be considered likely enough to have occurred. However, if for the most likely alternative hypothesis $|t^{l,k}| \leq N_{1/2\alpha_0}(0,1)$ holds, and the overall model test diagnoses an unspecified model error, one should reconsider the aptitude of the specified class of alternative hypotheses.

It will be clear that the necessary computations for the above described recursive identification procedure are less than when done in a batch mode. The computations, however, can still be somewhat involved. Furthermore, there is also still the practical problem of a *delay* in time of detection and identification. In order to reduce the number of computations and the time of delay, it is worthwhile to consider introducing a moving *window* of length N by constraining l to $k - N + 1 \leq l \leq k$. This is shown in Figure 1b. With this window the time of delay is at most equal to $N - 1$. When choosing N one of course has to make sure that the detection power of the teststatistics is still sufficient. This is typical a problem one should take into consideration when designing the filter. Instead of constraining l to $k - N + 1 \leq l \leq k$ one may achieve some further computational savings by constraining l to $k - N + 1 \leq l \leq k - M$. This is shown in Figure 1c. The rationale behind this constraint is that the teststatistic may be too insensitive for identifying global model errors if $l > k - M$.

After identification of the most likely alternative hypothesis, adaptation of the recursive filter is needed to eliminate the presence of biases in the filtered statevectors. In principal one has to correct the filtered states from time l to the present time k. This however may be a too heavy computational burden. Instead the following simple approach is suggested. At time k, when the most likely alternative hypothesis has been identified, one resets the real-time filter by correcting the filtered state as

$$l \downarrow \begin{bmatrix} \overset{k}{\rightarrow} & & & k=4 \\ t^{1,1} & t^{1,2} & t^{1,3} & t^{1,4} \\ & t^{2,2} & t^{2,3} & t^{2,4} \\ & & t^{3,3} & t^{3,4} \\ & & & t^{4,4} \end{bmatrix}_{H_a}$$

Fig. 1A. No Window

$$\begin{bmatrix} t^{1,1} & t^{1,2} & t^{1,3} & \bullet \\ & t^{2,2} & t^{2,3} & t^{2,4} \\ & & t^{3,3} & t^{3,4} \\ & & & t^{4,4} \end{bmatrix}_{H_a}$$

Fig. 1B. A Moving Window With $N = 3, M = 0$.

$$\begin{bmatrix} \bullet & t^{1,2} & t^{1,3} & \bullet \\ & \bullet & t^{2,3} & t^{2,4} \\ & & \bullet & t^{3,4} \\ & & & \bullet \end{bmatrix}_{H_a}$$

Fig. 1C. A Moving Window With $N = 3, M = 1$.

Fig. 1. The One-Dimensional Recursive Teststatistic $t^{l,k}$

$$\hat{x}^a_{k|k} = \hat{x}^0_{k|k} - \Phi_{k,k+1} X_{k+1,l} \hat{\nabla}^{l,k} \quad (19)$$

where $x_{k|k}^o$ and $x_{k|k}^a$ are the filtered statevectors corresponding with H_o and H_a respectively. The statevector $x_{k|k}^a$, with appropriate variance matrix which follows from an error propagation of (19) as

$$P^a_{k|k} = P^0_{k|k} + \Phi_{k,k+1} X_{k+1,l} Q_{\hat{\nabla}^{l,k}} X^*_{k+1,l} \Phi^*_{k,k+1}, \quad (20)$$

is then used as the new initial state for time k.

Note that the estimate $\hat{\nabla}^{l,k}$ and its variance matrix $Q_{\hat{\nabla}^{l,k}}$ can be computed *recursively* as

$$\begin{cases} \hat{\nabla}^{l,k} & = \hat{\nabla}^{l,k-1} + G_k[v_k - C_{v_k} \hat{\nabla}^{l,k-1}] \\ Q_{\hat{\nabla}^{l,k}} & = [I - G_k C_{v_k}] Q_{\hat{\nabla}^{l,k-1}}, \end{cases} \quad (21)$$

with gain matrix

$$G_k = Q_{\hat{\nabla}^{l,k-1}} C^*_{v_k} [Q_{v_k} + C_{v_k} Q_{\hat{\nabla}^{l,k-1}} C^*_{v_k}]^{-1} \quad (22)$$

Also the n-by-b matrix $X_{k+1,l}$ of (19) can be computed recursively. For instance, in case of a slip $C_{y,k}\nabla$ in the measurement vector, $X_{k+1,l}$ reads:

$$X_{k+1,l} = \sum_{i=l}^{k} \left[\prod_{j=i+1}^{k} \Phi_{j+1,j}[I - K_j A_j] \right] \Phi_{i+1,i} K_i C_{y,i},$$

which can be computed recursively as shown in (11).

ON MINIMAL DETECTABLE BIASES (MDB)

In this section we introduce

1. A *bias-to-noise ratio* for diagnosing the impact of model errors in H_o on the statevector.

2. A measure for the *separability* of alternative hypotheses
3. The concept of *minimal detectable biases* (MDB)

Undetected model errors in H_o generally influence the mathematical expectation of the statevector estimator $\hat{x}_{k/k}$. It is therefore of importance to know how particular misspecifications in H_o manifest themselves as biases in the statevector. Knowledge of the impact of model errors in H_o on $\hat{x}_{k/k}$ can then be used to set acceptance criteria for the sizes of these model errors. This is of importance for the design of an appropriate filter and for the design of a powerful enough testing procedure.

The bias $\nabla \hat{x}_{k,k}$ in $\hat{x}_{k/k}$ due to a model error in H_o can be computed rather straightfowardly from the equations that define the filter algorithm. For instance, an outlier $C_{y,l}\nabla$ in the data at time l results in a bias in $\hat{x}_{k,k}$ of

$$\nabla\hat{x}_{k|k} = \left[\prod_{j=l+1}^{k} [I - K_j A_j]\Phi_{j,j-1}\right] K_l C_{y,l}\nabla \ .$$

This bias can be computed recursively in essentially the same manner as shown in section 3. The significance of the biases in the statevector can be tested with the following scalar bias-to-noise ratio

$$\boxed{\lambda_{\hat{x}_{k|k}} = \nabla\hat{x}^*_{k|k} P^{-1}_{k|k} \nabla\hat{x}_{k|k}} \tag{23}$$

Using Cauchy-Schwarz' inequality, the scalar $\lambda_{\hat{x}k/k}$ can be shown to give an upperbound on the bias-to-noise ratio of an arbitrary linear function of $\hat{x}_{k/k}$:

$$\boxed{\frac{(a^*\nabla\hat{x}_{k|k})^2}{a^* P_{k|k} a} \leq \lambda_{\hat{x}_{k|k}} \ \forall\, a \in \mathcal{R}^n} \tag{24}$$

Hence, $\lambda_{\hat{x}k/k}$ also gives an upperbound for the bias-to-noise ratios of the individual elements of the statevector $\hat{x}_{k/k}$.

Assuming that for a particular application a quantification of the demands is given in terms of criteria for $\lambda_{\hat{x}k/k}$, (23) can be used to determine the sizes of the model errors that should be detectable by the statistical tests at a certain level of probability. This brings us then to the important concept of the power of a statistical test. The power γ of a statistical test, being the probability of rejecting H_o when an alternative hypothesis H_a is true, depends on the chosen level of significance α (probability of false alarm), the number of degrees of freedom b, and the non-centrality parameter λ of the corresponding teststatistic:

$$\gamma = \gamma(\alpha, b, \lambda) \ . \tag{25}$$

The power γ is a monotonic increasing function in α and λ, and a monotonic decreasing function in b. Since λ depends on the assumed model errors in H_o, the power function (25) can be used to determine how well particular model errors can be detected with the associated test. A low probability corresponds with poor detectability and a high probability with good detectability. Instead of using the power function (25), we propose to use the *inverse power function*

$$\lambda = \lambda(\alpha, b, \gamma) \ . \tag{26}$$

The rationale for using the "inverse power function" is that in practical applications one is usually much more interested in the size of the model error that can be detected with a certain probability γ, than in the power γ itself. If we assume that under the true hypothesis H_{true}

$$E\{v \mid H_{true}\} = \bar{C}_v \nabla \ , \tag{27}$$

the non-centrality parameter of the teststatistic T becomes

$$\lambda = \nabla^* \bar{C}_v^* Q_v^{-1} C_v [C_v^* Q_v^{-1} C_v]^{-1} C_v^* Q_v^{-1} \bar{C}_v \nabla \ . \tag{28}$$

This may be written in geometric terms as

$$\lambda = \| P_{C_v} \bar{C}_v \nabla \|^2 = \| \bar{C}_v \nabla \|^2 \cos^2\phi, \tag{29}$$

where P_{CV} is the orthogonal projector that projects orthogonally onto the range space of C_v, $\|.\|$ is the norm defined by the metric of Q_v^{-1}, and ϕ is the angle between $\bar{C}_v\nabla$ and the range space $R(C_v)$.

With (26) and (29) we are now in the position to compute the hyperellipsoidal boundary region of biases that can be detected with a chosen reference probability γ_o. For the one-dimensional case ($b = 1$) we get

$$| \nabla | = [\lambda_o / \| \bar{c}_v \|^2 \cos^2\phi]^{1/2} \tag{30}$$

with

$$\lambda_o = \lambda(\alpha = \alpha_o, b = 1, \gamma = \gamma_o).$$

The angle ϕ in (30) is a measure of the *separability* between H_a and H_{true}. It becomes more difficult to distinguish between the hypotheses H_a and H_{true} if ϕ decreases. If we assume that H_a and H_{true} only differ in their time of occurrences l and l_o respectively, then

$$\boxed{\cos^2\phi_{l_o}^{l,k} = \sigma^2_{t^{l,k}, t^{l_o,k}}} \tag{31}$$

where $\sigma_{t^{l,k}, t^{l_o,k}}$ is the covariance (or correlation coefficient) between the teststatistics $t^{l,k}$ and $t^{l_o,k}$. The covariance $\sigma_{t^{l,k}, t^{l_o,k}}$ reaches its maximum of 1 for $l = l_o$. The result (31) can be used as a *diagnostic tool* for inferring how well the true starting time l_o can be determined. The following simple example should make this clear.

Example

Assume that $\Phi_{k,k-1} = 1, A_k = 1, P_{o/o} = p, R_k = r$ and $Q_k = 0$. The following two cases are considered:

1. An outlier in the data at time l_o. Then

$$\cos^2\phi_{l_o}^{l,k} = \begin{cases} [r/p + k - 1]^{-2} & for \quad l \neq l_o \\ 1 & for \quad l = l_o \end{cases}$$

2. A sensor failure that starts at time l_o. Then

$$\cos^2\phi_{l_o}^{l,k} = \begin{cases} \dfrac{(k - l_o + 1)(r/p + l - 1)}{(k - l + 1)(r/p + l_o - 1)} & for \quad l \leq l_o \\[2ex] \dfrac{(k - l + 1)(r/p + l_o - 1)}{(k - l_o + 1)(r/p + l - 1)} & for \quad l \geq l_o \end{cases}$$

A plot of the above two correlation functions is given in Figure 2. Note the distinct difference in behavior of the two correlation functions. Figure 2a shows that the maximum of $\cos^2\phi_{l_o}^{l,k}$ gets more pronounced for increasing k. Hence it becomes easier to determine the correct time of occurrence, l_o, of an outlier when k increases. Figure 2b, on the other hand, shows the reversed situation. In this case the maximum of $\cos^2\phi_{l_o}^{l,k}$ gets less pronounced for increasing k. In this case it becomes

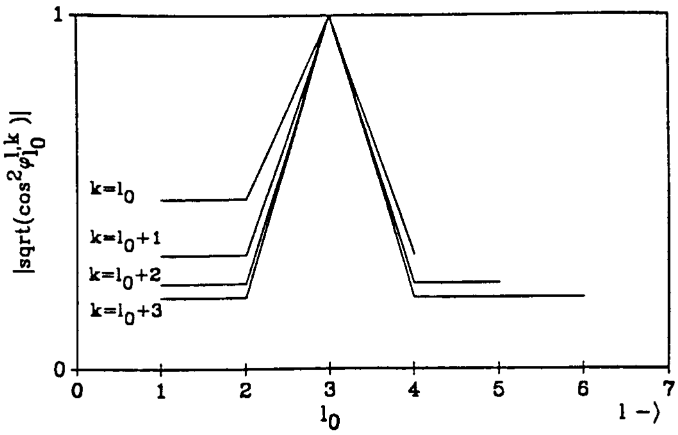


Fig. 2A. Correlation of Teststatistics for the Case of an Outlier in the Data at Time l_o

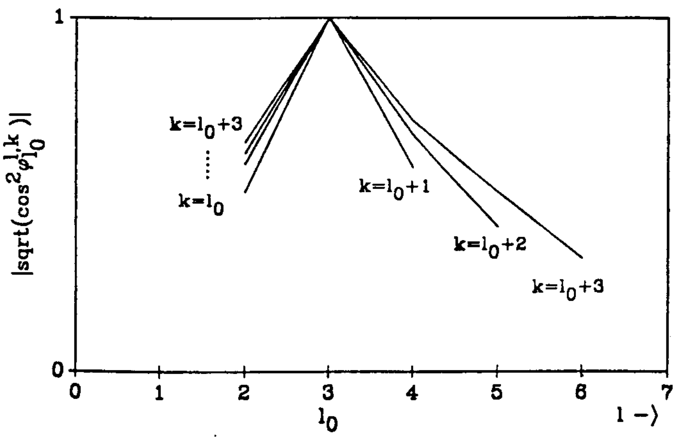
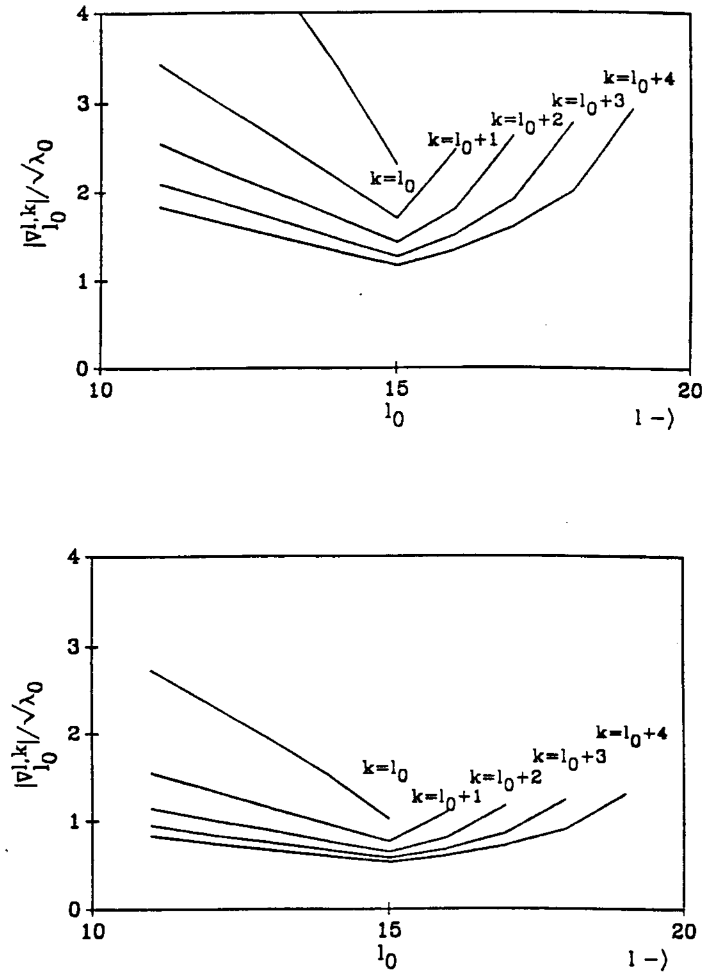


Fig. 2B. Correlation of Teststatistics for the Case of a Sensor Failure that Starts at Time l_o



therefore more difficult to locate the true starting time l_o of the sensor failure when k increases. This implies that one should make sure when designing the filter that a not too large value of k relative to l_o is needed for detecting a bias of a prefixed size.

We now introduce the concept of a *minimal detectable bias* (MDB). A MDB is defined as the size of the model error that can just be detected with a probability γ_o with the one-dimensional teststatistic $t_{l,k}$. The MDB follows with $|\bar{c}_\nabla|^2 = \sigma^{-2}_{\hat{\nabla}_{l_o,k}}$ from (30) and (31) as

$$| \nabla_{l_o}^{l,k} | = \sigma_{\hat{\nabla}_{l_o,k}} \left[\frac{\lambda_o}{\sigma^2_{t^{l,k},t^{l_o,k}}} \right]^{1/2} \qquad (32)$$

The MDB's provide an important *diagnostic tool* for inferring how well particular model errors can be detected. Since $\sigma_{\hat{\nabla}_{l_o,k}}$ is independent of l, $|\nabla_{l_o}^{l,k}|$ has its minimum at $l = l_o$. This is in agreement with the fact that the teststatistic $t^{l,k}$ is most powerful for $l = l_o$. The minimum of $|\Delta_{l_o}^{l,k}|$ decreases for increasing k. That is, when $l = l_o$, larger values of k correspond with biases of a smaller size that can be detected with a probability γ_o. This result can be used to make an appropriate choice for the windowlength N. For instance, if in a particular application a criterion is set at a level of probability γ_o, for the size of the MDB, k and therefore N can be chosen such that $|\Delta_{l_o}^{l,k}|$ meets the criterion. An appropriate level of the MDB can of course also be obtained through the design of the filter (e.g., sampling rate, number of sensors, measurement precision and geometry).

Example *(continued)*

The MDB that corresponds with a sensor failure starting at time l_o reads:

$$| \nabla_{l_o}^{l,k} | = \begin{cases} \left[\lambda_o \dfrac{r(r/p+k)}{(k-l+1)(r/p+l-1)} \right]^{\frac{1}{2}} \left[\dfrac{k-l+1}{k-l_o+1} \right] \\ for\ l \leq l_o \\ \left[\lambda_o \dfrac{r(r/p+k)}{(k-l+1)(r/p+l-1)} \right]^{\frac{1}{2}} \left[\dfrac{r/p+l-1}{r/p+l_o-1} \right] \\ for\ l \geq l_o \end{cases}$$

In Figure 3 the ratio $|\Delta_{l_o}^{l,k}| / \lambda_o^{1/2}$ is plotted as function of l for different values of $k \leq l_o$ and two different values of r. The figure shows that $|\Delta_{l_o}^{l,k}| / \lambda_o^{1/2}$ decreases for $l < l_o$, that it obtains its minimum at $l = l_o$, and that it increases at a somewhat faster rate $l > l_o$. The figure also clearly shows the decrease in the minimum for increasing k. But note that since

$$\lim_{k\to\infty} | \nabla_{l_o}^{l,k} | = \begin{cases} \left[\lambda_o \frac{r}{r/p+l-1}\right]^{1/2} & for \quad l \leq l_o \\ \left[\lambda_o \frac{r}{r/p+l-1}\right]^{1/2} \left[\frac{r/p+l-1}{r/p+l_o-1}\right] & for \quad l \geq l_o \end{cases}$$

the improvement in the MDB is not without bounds. Hence, it does not pay to test with $t^{l_o,k}$ after a certain time delay.

Comparison of the two figures a and b shows the influence of an increase in measurement precision on the size of the MDB. The gradient with respect to k of $|\Delta_{l_o}^{l,k}| / \lambda_o^{1/2}$ at $l = l_o$ increases, and the minimum of $|\Delta_{l_o}^{l,k}| / \lambda_o^{1/2}$ decreases for decreasing r.

CONCLUSION

In this paper a recursive testing procedure for use in integrated navigation systems was introduced. Our procedure can accommodate slippages in the mean of the predicted residuals caused by for instance: outliers in the data, sensor failures or switches in the dynamic model. The method is therefore also applicable to the important problem of GPS failure detection and integrity checking as discussed in e.g., (6-11).

One of the assumptions on which our method is based is that the complete variance matrix of the predicted residuals is known. Different teststatistics are needed however in case Q_v is unknown or only partially known. It can be shown that it is also possible to develop for this more general case recursive algorithms. The procedures can become quite complex however (4). The simplest extension occurs when Q_v is known up to an unknown scale factor. In this case the teststatistic T of (13) has to be replaced by the teststatistic

$$sin^2\varphi = \frac{v^* Q_v^{-1} C_v^* [C_v^* Q_v^{-1} C_v]^{-1} C_v^* Q_v^{-1} v}{v^* Q_v^{-1} v}$$

The teststatistic $sin^2\varphi$ is distributed under H_o and H_a as

$$H_o : sin^2\varphi \sim B(b, \sum_{i=l}^{k} m_i, 0) \; ; \; H_a : sin^2\varphi \sim B(b, \sum_{i=l}^{k} m_i, \lambda)$$

where $B(f_1, f_2 \lambda)$ *is the Beta-distribution with* f_1, f_2 degrees of freedom and non-centrality parameter λ. It should be noted that since $sin^2\varphi = 1$ for $b = \Sigma_{i=l}^{k} m_i$, *no overall model test exists for the case that* Q_v is known up to a scale factor.

REFERENCES

1. Teunissen, P.J.G. (1984): *"QUALITY CONTROL IN GEODETIC NETWORKS."* In: Grafarend, E. and F. Sansò (eds.), Optimization and Design of Geodetic Networks, Springer Verlag, Berlin, pp. 526-547.

2. Teunissen, P.J.G. (1986):, *"ADJUSTING AND TESTING WITH THE MODELS OF THE AFFINE AND SIMILARITY TRANSFORMATION."* Manuscripta Geodaetica, Vol. 11, pp. 214-225.

3. Teunissen, P.J.G. and M.A. Salzmann (1988):, *"PERFORMANCE ANALYSIS OF KALMAN FILTER."* Report 88.2. Geodetic Computing Centre, Delft University of Technology.

4. Salzmann, M.A. and P.J.G. Teunissen (1989):, *"QUALITY CONTROL IN KINEMATIC DATA PROCESSING."* In: Land Vehicle Navigation 1989. Deutsche Gesellschaft für Ortung und Navigation e.V. Verlag TÜV Rheinland, pp. 353-366.

5. Teunissen, P.J.G. and M.A. Salzmann (1989):, *"A RECURSIVE SLIPPAGE TEST FOR USE IN STATE-SPACE FILTERING."* Will be published in Manuscripta Geodaetica, Vol. 14, No. 6.

6. Brown, G. and P.Y.C. Hwang (1987), *"GPS FAILURE DETECTION BY AUTONOMOUS MEANS WITHIN THE COCKPIT."* Navigation, Journal of the Institute of Navigation, Vol. 33, No. 4, pp. 335-353.

7. McBurney, P.W. and R.G. Brown (1987):, *"SELF-CONTAINED GPS INTEGRITY CHECK USING MAXIMUM SOLUTION SEPARATION AS THE TESTSTATISTIC."* Proc. ION Satellite Division Technical Meeting, Colorado Springs, 1987.

8. Parkinson, B.W. and P. Axelrad (1987):, *"A BASIS FOR THE DEVELOPMENT OF OPERATIONAL ALGORITHMS FOR SIMPLIFIED GPS INTEGRITY CHECKING."* Proc. ION Satellite Division Technical Meeting, Colorado Springs, 1987.

9. Lee, Y.C. (1988):, *"NEW CONCEPT FOR INDEPENDENT GPS INTEGRITY MONITORING."* Navigation: Journal of the Institute of Navigation, Vol. 35, No. 2, pp. 239-254.

10. Sturza, M.A. (1988):, *"NAVIGATION SYSTEM INTEGRITY MONITORING USING REDUNDANT MEASUREMENTS."* Navigation: Journal of the Institute of Navigation, Vol. 35, No. 4, pp. 483-501.

11. Parkinson, B.W. and P. Axelrad (1988):, *"AUTONOMOUS GPS INTEGRITY MONITORING USING THE PSEUDORANGE RESIDUAL."* Navigation: Journal of the Institute of Navigation, Vol. 35, No. 2, pp. 255-274.

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