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A recursive slippage test for use in state-space filtering

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Abstract

New test statistics for (recursive) state estimation in dynamic systems are introduced. Particular attention is paid to slippage-type test statistics. A distinction is made between local and global testing. Local testing only takes one epoch of the dynamic system into account,.whereas global testing is based on more than one epoch. Recursive forms for the global slippage test statistics are derived, thus rendering real-time global testing procedures feasible.

1 Introduction

Real-time estimation of parameters in dynamic systems becomes increasingly important in the field of high precision navigation. The real-time estimation procedure inevitably requires real-time testing of the models underlying the dynamic system. We will discuss testing procedures that can be used in conjunction with the well-known Kalman filter algorithm.

In this paper we focus on slippage tests. First we consider local testing. Local tests only take one epoch of the dynamic system into account. With local tests it may be difficult to detect unmodelled trends. It is therefore expedient to include global testing as well. The detection power of the global tests is larger than of the local tests. However, in order to render the global tests useful for real-time applications, recursive forms are needed, that parallel the Kalman filter algorithm. In this paper we will derive such recursive forms for our global slippage test statistics.

The contents of the paper is as follows. First we briefly discuss the model underlying (linear) dynamic systems and introduce the Kalman filter algorithm. Then local model testing is discussed, of which various forms are presented. Some of the pitfalls of the local tests are pointed out. Then global model testing is introduced and recursive forms for the global slippage test statistic are derived. Finally some practical considerations are given.

2 Recursive Filtering

In this section we present the mathematical model of discrete time linear(ized) dynamic systems, and briefly discuss the recursive filtering equations that define the optimal estimators of the system state. The mathematical model is defined as (see, e.g. [Koch, 1982; Teunissen and Salzmann, 1988]):

$$\begin{array}{lcl} \underline{x}_0 & \sim & N(x_0, P_0) \\ \underline{d}_k & \sim & N(0, Q_k) \\ \underline{y}_k & \sim & N(y_k, R_k) \\ & & \text{for } k = 1, 2, \ldots \ , \end{array} \tag{1}$$

with

$$\begin{array}{lcl} x_k & = & \Phi_{k,k-1} x_{k-1} + d_k \\ y_k & = & A_k x_k \ . \end{array}$$

An underscore indicates the random character of a variable. The n-vector x_k denotes the system state, a set of parameters that completely describes the system at time k; $\Phi_{k,k-1}$ is the state transition matrix that describes the non-random transition of the state from time $k-1$ to k; $\underline{d}_k$ is an unobservable random disturbance vector that models the uncertainty in the state transition at time k; $\underline{y}_k$ is the m_k-vector of observables at time k; and A_k is the $m_k \times n$ designmatrix that relates the mean of $\underline{y}_k$ to the state vector x_k.

Furthermore it is assumed that $\underline{x}_0$ is uncorrelated with $\underline{d}_k$ and $\underline{y}_k$ for $k = 1, 2, \ldots$; $\underline{d}_k$ is uncorrelated with $\underline{d}_l$ for $k \neq l$; $\underline{y}_k$ is uncorrelated with $\underline{y}_l$ for $k \neq l$; and $\underline{d}_k$ is uncorrelated with $\underline{y}_l$ for all k, l.

The matrices $P_0, Q_k, R_k, \Phi_{k,k-1}$, and A_k are assumed to be known, with P_0 and R_k positive definite, and Q_k

semi-positive definite. The sample for the disturbance $\underline{d}_k$ is taken equal to the zero-vector.

Depending on the application one has in mind, one might wish to obtain an estimate of the state at a certain time k, which depends on all observations taken up to and including time $k+l$. If $l < 0$ the process is called *prediction*. The state estimates then only depends on the observations taken prior to the desired time of estimation. If $l = 0$ the process is called *filtering*. In this case the state estimate depends on all the observations prior to and at time k. Finally, if $l > 0$ the process is called *smoothing*. The state estimate then depends on observations taken prior to, on, and after time k.

Since we have real time applications of the estimation problem in mind, we shall restrict ourselves in the following to recursive prediction and filtering. The problem we stand for is to determine an estimate of the state at time k that is a linear combination of an estimate of the state at time $k-1$ and the observations at time k. Furthermore the estimate must be "best" in a certain sense. Kalman [1960] was the first to solve this problem using the minimum mean square error criterion. It can be shown, however, (see, e.g., Koch [1982], Teunissen and Salzmann [1988]) that the methods of maximum likelihood, maximum a posteriori, and least squares lead to identical results.

The Kalman filter basically consists of two parts: the *time update* and the *measurement update*. The time update of the state estimator and its covariance matrix are given as:

$$\hat{\underline{x}}_{k|k-1} = \Phi_{k,k-1}\hat{\underline{x}}_{k-1|k-1} + \underline{d}_k \quad (2)$$

$$P_{k|k-1} = \Phi_{k,k-1}P_{k-1}\Phi^*_{k,k-1} + Q_k \quad (3)$$

Equation (2) gives the best estimator of the state at time k using all observables prior to time k. The time update equation is also known as the one-step prediction equation. Note that since the sample of $\underline{d}_k$ is taken to be equal to the zero-vector, the time update of the state *estimate* reads $\hat{x}_{k|k-1} = \Phi_{k,k-1}\hat{x}_{k-1|k-1}$. The measurement update of the state and its covariance matrix are given as:

$$\hat{\underline{x}}_{k|k} = \hat{\underline{x}}_{k|k-1} + K_k(\underline{y}_k - A_k\hat{\underline{x}}_{k|k-1}) \quad (4)$$

$$P_k = (I - K_kA_k)P_{k|k-1} \quad , \quad (5)$$

where

$$K_k = P_{k|k-1}A^*_k(R_k + A_kP_{k|k-1}A^*_k)^{-1} \quad (6)$$

is the so-called Kalman gain matrix.

Equation (4) gives the best estimator of the state at time k using both $\hat{\underline{x}}_{k|k-1}$ and $\underline{y}_k$. The measurement update equation is also known as the filter equation.

The (recursive) Kalman filter produces optimal estimators of the state vector with well defined statistical properties. The state estimators are unbiased, have minimum variance, and are normally distributed. Optimality is, however, only guaranteed as long as the assumptions underlying the model hold. Misspecifications in the model will invalidate the results of estimation and thus also any conclusion based on them. It is therefore of crucial importance to have ways to verify the validity of the assumed mathematical model. An important role in the process of model testing is played by the so-called *predicted residual*. The predicted residual is defined as the difference between the actual system output and the predicted output based on the predicted state

$$\underline{v}_k = \underline{y}_k - A_k\hat{\underline{x}}_{k|k-1} \; . \quad (7)$$

The predicted residual represents the new information brought in by the latest observable $\underline{y}_k$. This can be seen from the measurement update equation (4), which shows that the filtered state is a linear combination of the predicted state and the predicted residual. Under the working hypothesis that the mathematical model is specified correctly, the predicted residual has well defined statistical properties:

$$\underline{v}_k \sim N(0, Q_{v_k}) \quad , \quad (8)$$

where

$$Q_{v_k} = (R_k + A_kP_{k|k-1}A^*_k) \; . \quad (9)$$

This result follows directly from the defining equation (7). Note that the predicted residual and its covariance matrix are available during each measurement update.

Knowledge of the distribution of the predicted residual can be used for testing the validity of the assumed mathematical model, consisting of the functional and stochastic model. In this paper we will restrict ourselves to the functional model and consider only misspecifications in the mean of the predicted residual. That is, we will only consider slippage tests.

3 Local Model Testing

In the class of slippage tests we make a distinction between *local* model testing and *global* model testing. We speak of local model testing when the tests performed at time k only depend on the predicted state at time k and the observations at time k. If the test takes more than one epoch into account we speak of global model testing. From this definition it follows that in contrast with the global tests, the local tests can be executed in real-time and thus corrective action can be taken immediately. Since we have real-time applications in mind, we will, for the moment, restrict ourselves to local model testing.

We will base our local model testing on the predicted residual $\underline{v}_k$. The following two hypotheses are considered:

$$\begin{array}{llll} H_{0\_k}: & \underline{v}_k & \sim & N(0, Q_{v\_k}) \\ H_{A\_k}: & \underline{v}_k & \sim & N(\nabla v_k, Q_{v\_k}) \end{array} \tag{10}$$

We will assume that the m_k-vector ∇v_k can be parametrized as

$$\underset{m_k \times 1}{\nabla v_k} = \underset{m_k \times b_k}{C_{v\_k}} \; \underset{b_k \times 1}{\nabla_k} , \tag{11}$$

with the full rank matrix C_{v_k} known, the b_k-vector ∇_k unknown, and $b_k \leq m_k$.

It is well-known (see, e.g., Graybill [1976]; Teunissen [1985,1986]; Koch [1988]) that the appropriate test statistic for testing H_{0_k} against H_{A_k} is given by:

$$\underline{T}^k_{b\_k} = \underline{v}_k^* Q_{v\_k}^{-1} C_{v\_k} [C_{v\_k}^* Q_{v\_k}^{-1} C_{v\_k}]^{-1} C_{v\_k}^* Q_{v\_k}^{-1} \underline{v}_k \; . \tag{12}$$

Geometrically $\underline{T}^k_{b\_k}$ can be interpreted [Teunissen, 1986] as the square of the length of the vector that follows from projecting $\underline{v}_k$ orthogonally on the range space of C_{v_k} (see Figure 1). The test statistic $\underline{T}^k_{b\_k}$ has the following

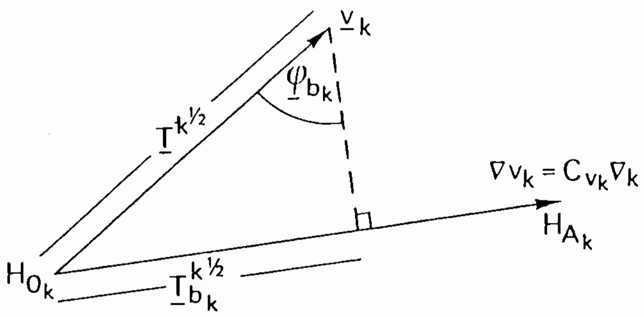

Figure 1: Predicted residual space with metric $Q_{v\_k}^{-1}$ and the two hypotheses H_{0_k} and H_{A_k}.

distributions under H_{0_k} and H_{A_k}:

$$\begin{array}{llll} H_{0\_k}: & \underline{T}^k_{b\_k} & \sim & \chi^2(b_k, 0) \\ H_{A\_k}: & \underline{T}^k_{b\_k} & \sim & \chi^2(b_k, \lambda_k) \end{array} \tag{13}$$

where

$$\lambda_k = \nabla_k^* C_{v\_k}^* Q_{v\_k}^{-1} C_{v\_k} \nabla_k \; .$$

The test of size α is now as follows: Reject H_{0_k} if and only if $T^k_{b\_k}$ satisfies $T^k_{b\_k} \geq \chi^2_{\alpha:b_k}$ where $\chi^2_{\alpha:b_k}$ is the upper α probability point of the central χ^2-distribution with b_k degrees of freedom. If the hypothesis H_{0_k} is rejected in favour of H_{A_k}, the best estimator of ∇_k under H_{A_k} is given by

$$\hat{\underline{\nabla}}_k = [C_{v\_k}^* Q_{v\_k}^{-1} C_{v\_k}]^{-1} C_{v\_k}^* Q_{v\_k}^{-1} \underline{v}_k \; . \tag{14}$$

For most practical applications there are two particular forms of $\underline{T}^k_{b\_k}$ that are of special importance. They correspond to the case that $b_k = m_k$, and to the case that $b_k = 1$. If b_k is chosen to be equal to m_k, the matrix C_{v_k} of (11) becomes a square and non singular matrix, implying that the vector ∇v_k of (10) remains completely unspecified. In this case the invertible matrix C_{v_k} can be eliminated from (12) and the test statistic can be written as:

$$\boxed{\underline{T}^k = \underline{v}_k^* Q_{v\_k}^{-1} \underline{v}_k} \; . \tag{15}$$

This test statistic can be used to perform an overall model test for detecting possible unspecified model errors in H_{0_k}, and is called *local overall model* (LOM) test statistic.

If b_k is chosen equal to 1, the matrix C_{v_k} of (11) reduces to a vector, which will be denoted by c_{v_k}, and the vector ∇_k reduces to a scalar. In this case the test statistic can be written as:

$$\boxed{\underline{t}^k = \frac{(c_{v\_k}^* Q_{v\_k}^{-1} \underline{v}_k)^2}{c_{v\_k}^* Q_{v\_k}^{-1} c_{v\_k}}} \; . \tag{16}$$

The lower case kernel letter t will be used for our one-dimensional slippage test statistics. This test statistic can be used to identify particular one-dimensional misspecifications in H_{0_k}, such as a slippage in the mean of the predicted state, a slippage in the mean of the observables, or a slippage in the mean of a combination of observables and the predicted state. Hence we call (16) a *local slippage* (LS) test statistic. For instance, if one suspects sensor failures or outlying observations one can follow the *datasnooping* approach [Baarda, 1968; Teunissen, 1985] by choosing m_k number of vectors c_{v_k} of the form

$$\underset{m_k \times 1}{c_i} = (\underset{1}{0} \; \cdots \; \underset{i}{1} \; \cdots \; \underset{m_k}{0})^* , \tag{17}$$

for $i = 1, \ldots, m_k$.

So far the covariance matrix Q_{v_k} of the predicted residual was assumed to be known. However, if the covariance matrix Q_{v_k} is known up to a scale factor, alternative test statistics have to be introduced. The appropriate test statistic [Teunissen, 1986] in this case is (see Figure 1):

$$\sin^2 \underline{\phi}_{b\_k} = \frac{\underline{v}_k^* Q_{v\_k}^{-1} C_{v\_k} [C_{v\_k}^* Q_{v\_k}^{-1} C_{v\_k}]^{-1} C_{v\_k}^* Q_{v\_k}^{-1} \underline{v}_k}{\underline{v}_k^* Q_{v\_k}^{-1} \underline{v}_k} \tag{18}$$

The test statistic $\sin^2 \underline{\phi}_{b\_k}$ has the following distributions under H_{0_k} and H_{A_k}:

$$\begin{aligned} H_{0\_k}: \quad \sin^2 \underline{\phi}_{b\_k} &\sim B(b_k, m_k, 0) \\ H_{A\_k}: \quad \sin^2 \underline{\phi}_{b\_k} &\sim B(b_k, m_k, \lambda_k) \end{aligned} \tag{19}$$

where $B(f_1, f_2, \lambda)$ is the Beta-distribution with f_1, f_2 degrees of freedom and non-centrality parameter λ. Instead of the test statistic $\sin^2 \underline{\phi}_{b\_k}$ one may also take $\cos^2 \underline{\phi}_{b\_k}$ or $\tan^2 \underline{\phi}_{b\_k}$ as test statistic. Because of their functional dependency, they will give identical outcomes for the testing. It should be noted that since $\sin^2 \underline{\phi}_{b\_k} \equiv 1$ for $b_k = m_k$, no overal model test exists for the case that Q_{v_k} is known up to a scale factor. Thus the test statistic $\sin^2 \underline{\phi}_{b\_k}$ is only applicable for $1 \leq b_k < m_k$.

In our derivations so far the vector ∇_k of (11) was assumed to be unknown under H_{A_k}. A very special case occurs, however, if besides the matrix C_{v_k} also the vector ∇_k is assumed known under H_{A_k}. In this case the testing problem reduces to one of *discriminant analysis.* From figure 1 follows that H_{0_k} is in this case the most likely hypothesis if $T_{m\_k}^{k\ 1/2} \sin\phi < \frac{1}{2}||\nabla v_k||$. The decision rule for discriminating between H_{0_k} and H_{A_k} becomes in this case: accept H_{0_k} if $(\nabla v_k)^* Q_{v\_k}^{-1}(v_k - \frac{1}{2}\nabla v_k) < 0$; accept H_{A_k} otherwise. Although this test is conceptually very simple, the possibility of two fully specified hypotheses H_{0_k} and H_{A_k} very rarely occurs in practical applications. One of the few exceptions is perhaps the laneslip identification problem that occurs in radiopositioning systems.

The test statistics given above are easily executed in a Kalman filter environment. This follows since the predicted residual $\underline{v}_k$ and its covariance matrix Q_{v_k} are readily available during each measurement update. A disadvantage of the above given test statistics is, however, that they are local. It will be clear that observations taken after time k have no effect on these local tests at time k. Thus any misspecification in the mathematical model that may occur after time k has no effect on these tests. A somewhat similar situation exists for observations taken prior to time k. That is, although local model testing is dependent on the observations taken prior to time k, this dependency is rather weak, since misspecifications that occur prior to time k are only felt indirectly via the predicted state. This is illustrated by the following example.

Example I

Assume that $\Phi_{k,k-1} = 1$, $A_k = 1$, $P_0 = p$, $R_k = r$, and $Q_k = 0$. Furthermore assume that a sensor failure of size ∇ starts at time l. The influence of this sensor failure on the predicted state at time k is then

$$\nabla x_{k|k-1} = \frac{k-l}{r/p+k-1}\nabla .$$

Hence, the influence on the predicted residual at time k becomes

$$\nabla v_k = \frac{r/p+l-1}{r/p+k-1}\nabla ,$$

with

$$Q_{v\_k}^{-1} = \frac{r/p+k-1}{r[r/p+k]} .$$

The influence of the sensor failure on the one-dimensional local slippage test statistic $\underline{t}^k$ of (16) is then

$$\nabla t^k = \frac{[r/p+l-1]^2}{r[r/p+k][r/p+k-1]}\nabla^2 . \tag{20}$$

This result shows that ∇t^k has its maximum at $k = l$ and that ∇t^k decreases rapidly the larger $k - l$ gets for $k > l$ and l fixed. Thus, since ∇t^k is the noncentrality parameter of the distribution of $\underline{t}^k$ under H_{A_k}: $\underline{v}_k \sim N(\nabla v_k, Q_{v\_k})$, and the probability $P\{\underline{t}^k > \chi^2_{\alpha;1}|H_{A\_k}\}$ is an increasing function of ∇t^k, it follows that the probability of rejecting H_{0_k} correctly (with the one-dimensional local slippage test) is a decreasing function of $k \geq l$. □

The above example indicates that the local tests may be unable to detect global unmodelled trends. Therefore it seems expedient to include global testing as well. One way to perform global tests is to store all the collected data in computer memory and then to apply a batch type solution. Better results than for local tests can be expected as smoothing is involved. The disadvantage of batch solutions is however, that the test statistics are only available with a delay, and more important, the recursiveness, which makes the Kalman filter algorithm so attractive is lost. We feel that a small delay is acceptable, because after all it may be more important in practice to detect a possible misspecification with a delay, than not to detect it at all. In the following section we will therefore develop our global test statistics with batch type properties in *recursive form.*

4 Global Model Testing

We will base our global model testing on the predicted residuals $\underline{v}_i$, $i = l, l{+}1, \ldots$. The vector of predicted residuals will be denoted by $\underline{v}$. Thus $\underline{v} = (\underline{v}_l^*, \underline{v}_{l+1}^*, \ldots, \underline{v}_k^*)^*$. The following two global hypotheses are considered:

$$\begin{aligned} H_0: \quad \underline{v} &\sim N(0, Q_v) \\ H_A: \quad \underline{v} &\sim N(\nabla v, Q_v) \end{aligned} \tag{21}$$

We will assume that the $(\sum_{i=l}^k m_i)$-vector ∇v can be parametrized as

$$\nabla v = C_v \nabla \tag{22}$$

with

$$C_v = (C_{v\_l}^*, C_{v\_l+1}^*, \ldots, C_{v\_k}^*)^* ,$$

where C_v is a $(\sum_{i=l}^{k} m_i \times b)$ matrix and ∇ is a vector of dimension b. The matrix C_v is assumed to be known and of full rank b, and the b-vector ∇ is assumed to be unknown.

In analogy with our results of the previous section, the appropriate global slippage test statistic for testing the above two hypotheses is

$$\underline{T}_b^{l,k} = \underline{v}^* Q_v^{-1} C_v [C_v^* Q_v^{-1} C_v]^{-1} C_v^* Q_v^{-1} \underline{v} \,. \tag{23}$$

This test statistic, however, is not yet in a form which is suitable for real-time applications. We will develop our global recursive slippage test statistic from (23) in two steps. First we will show that (23) can be written as

$$\underline{T}_b^{l,k} = \\ [\sum_{i=l}^{k} C_{v\_i}^* Q_{v\_i}^{-1} \underline{v}_i]^* [\sum_{i=l}^{k} C_{v\_i}^* Q_{v\_i}^{-1} C_{v\_i}]^{-1} [\sum_{i=l}^{k} C_{v\_i}^* Q_{v\_i}^{-1} \underline{v}_i] \,. \tag{24}$$

This simplification is essential and is based on the observation that the predicted residuals of different time epochs are uncorrelated. As a second step we will show how the matrices C_{v_i}, $i = l, \ldots, k$, can be computed recursively in a manner that parallels the Kalman filter algorithm.

In order to proof (24), we have to proof that

$$E\{\underline{v}_k \underline{v}_l^*\} = 0, \text{ for } k \neq l \,. \tag{25}$$

The proof goes as follows. First note that

$$\begin{aligned} E\{\underline{v}_k \underline{v}_l^*\} &= Q_{y_k y_l} - Q_{y_k \hat{x}_{l|l-1}} A_l^* - \\ & A_k Q_{\hat{x}_{k|k-1} y_l} + \\ & A_k Q_{\hat{x}_{k|k-1} \hat{x}_{l|l-1}} A_l^* \,. \end{aligned} \tag{26}$$

We will restrict ourselves to the case $l < k$. The case $l > k$ follows on the basis of symmetry. By definition the first two terms on the right hand side of (26) vanish for the case $l < k$. Thus

$$\begin{aligned} E\{\underline{v}_k \underline{v}_l^*\} &= -A_k Q_{\hat{x}_{k|k-1} y_l} + \\ & A_k Q_{\hat{x}_{k|k-1} \hat{x}_{l|l-1}} A_l^* \,. \end{aligned} \tag{27}$$

The two covariance matrices on the right hand side of (27) can be computed once the relation between $\hat{x}_{k|k-1}$ and the original input data of the Kalman filter is established. From a combination of the time and measurement update equations of the Kalman filter it follows

$$\begin{aligned} \underline{\hat{x}}_{i|i-1} &= \Phi_{i,i-1}(I - K_{i-1}A_{i-1})\underline{\hat{x}}_{i-1|i-2} + \\ & \underline{d}_i + \Phi_{i,i-1} K_{i-1} \underline{y}_{i-1} \,. \end{aligned} \tag{28}$$

If we let i run from $l+1$ to k we get by combining the equations

$$\begin{aligned} \underline{\hat{x}}_{k|k-1} &= \prod_{i=l}^{k-1} [\Phi_{i+1,i}(I - K_i A_i)] \underline{\hat{x}}_{l|l-1} + \\ & \sum_{j=l}^{k-1} \prod_{i=j+1}^{k-1} [\Phi_{i+1,i}(I - K_i A_i)] \underline{d}_{j+1} + \\ & \sum_{j=l}^{k-1} \prod_{i=j+1}^{k-1} [\Phi_{i+1,i}(I - K_i A_i)] \Phi_{j+1,j} K_j \underline{y}_j \,. \end{aligned} \tag{29}$$

Note that for $j = k-1$ the product in the last two terms of (29) reduces to the identity matrix I. From (29) follows that

$$Q_{\hat{x}_{k|k-1} y_l} = \\ \{\prod_{i=l+1}^{k-1} [\Phi_{i+1,i}(I - K_i A_i)]\} \, \Phi_{l+1,l} K_l R_l \tag{30}$$

and

$$Q_{\hat{x}_{k|k-1} \hat{x}_{l|l-1}} A_l^* = \\ \{\prod_{i=l+1}^{k-1} [\Phi_{i+1,i}(I - K_i A_i)]\} \Phi_{l+1,l}(I - K_l A_l) P_{l|l-1} A_l^* \,. \tag{31}$$

What remains to be shown is that the difference of (30) and (31) vanishes, or that

$$K_l R_l - (I - K_l A_l) P_{l|l-1} A_l^* = 0 \,, \tag{32}$$

which is easily verified upon substitution of (6). This concludes the proof of (25) and thus of (24).

Just as in the previous section, there are two particular forms of $\underline{T}_b^{l,k}$ that are of special importance for practical applications. The first one corresponds to the case that b is chosen equal to $\sum_{i=l}^{k} m_i$. Then the matrices C_{v_i}, $i = l, \ldots, k$ can be eliminated from (24) and the test statistic can be written as

$$\boxed{\underline{T}^{l,k} = \sum_{i=l}^{k} \underline{v}_i^* Q_{v_i}^{-1} \underline{v}_i} \,, \tag{33}$$

which under H_0 is distributed as $\chi^2(\sum_{i=l}^{k} m_i, 0)$. Note that this test statistic reduces to the LOM test statistic (15) for $l = k$. The global recursive test statistic $\underline{T}^{l,k}$ can be used to perform an overall model test for detecting possible unspecified global model errors, and is consequently called *global overall model* (GOM) test statistic. The test statistic covers the complete mathematical model if l is chosen equal to one.

The second form of $\underline{T}_b^{l,k}$ that is of special importance follows if b is chosen equal to one. Then the matrices C_{v_i}, $i = l, \ldots, k$ reduce to vectors. These vectors will be denoted by c_{v_i}, $i = l, \ldots, k$. The corresponding one-dimensional *global slippage* (GS) test statistic reads then

$$\boxed{\underline{t}^{l,k} = \frac{[\sum_{i=l}^{k} c_{v\_i}^* Q_{v\_i}^{-1} \underline{v}_i]^2}{\sum_{i=l}^{k} c_{v\_i}^* Q_{v\_i}^{-1} c_{v\_i}}}, \tag{34}$$

which under H_0 is distributed as $\chi^2(1,0)$. Note that this test statistic reduces to the one-dimensional LS test statistic (16) for $l = k$. The one-dimensional global slippage test statistic $\underline{t}^{l,k}$ can be used to identify particular one-dimensional global misspecifications in H_0. However, in order to be able to use the test statistic $\underline{t}^{l,k}$ in real-time we still need a recursive scheme for updating the vectors c_{v_i}, $i = l, \ldots, k$. Various cases, depending on the choice of alternative hypothesis, can be considered. We will consider the following three cases:

a) A permanent slip c_x in the state vector that starts at time l.

b) A single slip c_y in the vector of observables that starts at time l.

c) A sensor failure that starts at time l.

The corresponding recursive schemes for the vectors c_{v_i}, $i = l, \ldots, k$ can be derived from combining the time and measurement update equations of the Kalman filter.

The *recursive* scheme for a permanent slip in the state vector (which after time l manifests itself as a systematic disturbance $d_i \neq 0$) reads

$$\boxed{\begin{array}{rcl} c_{v\_i} & = & -A_i x_{i,l}, \quad i = l, \ldots, k \\ x_{i+1,l} & = & c_x + \Phi_{i+1,i}(I - K_i A_i) x_{i,l}; \; x_{l,l} = c_x \end{array}} \tag{35}$$

The *recursive* scheme for a single slip in the vector of observables at time l reads

$$\boxed{\begin{array}{rcl} c_{v\_i} & = & \left\{ \begin{array}{ll} c_y & \text{for } i = l \\ -A_i x_{i,l} & \text{for } i = l+1, \ldots, k \end{array} \right. \\ x_{i+1,l} & = & \Phi_{i+1,i}(x_{i,l} + K_i c_{v\_i}); \; x_{l,l} = 0 \end{array}} \tag{36}$$

The vector c_y is not specified explicitely here, but can, e.g., be chosen as in (17). Finally the *recursive* scheme for a failure in the j^{th} sensor reads

$$\boxed{\begin{array}{rcl} c_{v\_i} & = & c_j - A_i x_{i,l}, \quad i = l, \ldots, k \\ x_{i+1,l} & = & \Phi_{i+1,i}(x_{i,l} + K_i c_{v\_i}); \; x_{l,l} = 0 \end{array}} \tag{37}$$

where

$$c_j = (0 \; \cdots \; \underset{j}{1} \; \cdots \; 0)^* \; .$$

With (35), (36), or (37) we are now able to compute the one-dimensional global test statistic (34) recursively and perform the testing in a manner that parallels the Kalman filter algorithm.

It will be clear that our global recursive test statistics are more sensitive to global model errors than the local test statistics. The difference in detection power between the local and global one-dimensional slippage tests follows when one compares the noncentrality parameters ∇t^k and $\nabla t^{l,k}$ of the two test statistics. For the local test statistic we have

$$\nabla t^k = c_{v\_k}^* Q_{v\_k}^{-1} c_{v\_k} \nabla^2 \; , \tag{38}$$

and for our global test statistic we have

$$\nabla t^{l,k} = [\sum_{i=l}^{k} c_{v\_i}^* Q_{v\_i}^{-1} c_{v\_i}] \nabla^2 \; . \tag{39}$$

Since the matrices Q_{v_i}, $i = l, \ldots, k$ are positive definite, this result shows that $\nabla t^{l,k}$ is an increasing function of k and that $\nabla t^{l,k} > \nabla t^k$ for $k > l$. Hence, with increasing k the detection power of the global test increases and is never less than the detection power of the local test. That the difference in detection power between the local and global tests can be considerable is shown in the following example.

Example I (continued)

The influence of a sensor failure of size ∇ starting at time l on the one-dimensional global slippage test statistic is

$$\nabla t^{l,k} = \frac{[r/p + l - 1][k - l + 1]}{r[r/p + k]} \nabla^2 \; . \tag{40}$$

The ratio of (40) and (20) gives

$$\frac{\nabla t^{l,k}}{\nabla t^k} = \frac{r/p + k - 1}{r/p + l - 1} [k - l + 1] \; . \tag{41}$$

This result shows that for this particular example the relative sensitivity of the global test increases more than linearly. □

5 Practical Considerations

Although our global recursive test statistics can be computed in real-time through a scheme that parallels the Kalman filter equations, there may still be the practical problem of a delay in time of detection. Furthermore, although the necessary computations are less than when done in a batch mode, they can still be somewhat involved.

We will first discuss some computational aspects. Up to this point we tacitly assumed that one has to compute the test statistic $\underline{t}^{l,k}$ for each alternative hypothesis considered, and for each $k \geq l$. Moreover, since l is unknown one usually will start with $l = 1$. This implies that one has to compute k number of test statistics per

$$l \downarrow \begin{bmatrix} \overset{k}{\rightarrow} & & & k=4 \\ t^{1,1} & t^{1,2} & t^{1,3} & t^{1,4} \\ & t^{2,2} & t^{2,3} & t^{2,4} \\ & & t^{3,3} & t^{3,4} \\ & & & t^{4,4} \end{bmatrix}$$

(a)

$$\begin{bmatrix} t^{1,1} & t^{1,2} & t^{1,3} & \bullet \\ & t^{2,2} & t^{2,3} & t^{2,4} \\ & & t^{3,3} & t^{3,4} \\ & & & t^{4,4} \end{bmatrix}$$

(b)

$$\begin{bmatrix} \bullet & t^{1,2} & t^{1,3} & \bullet \\ & \bullet & t^{2,3} & t^{2,4} \\ & & \bullet & t^{3,4} \\ & & & \bullet \end{bmatrix}$$

(c)

Figure 2: The one dimensional recursive test statistic $\underline{t}^{l,k}$ with (a) no window, (b) a moving window with $N = 3, M = 0$, and (c) a moving window with $N = 3, M = 1$.

alternative hypothesis at the time of testing k. This is shown in Figure 2a. In order to reduce the number of computations and the delay time of detection, it is worthwhile to consider introducing a moving *window* of length N by constraining l to $k - N + 1 \leq l \leq k$. This is shown in Figure 2b. With this window the delay time of detection is at the most equal to $N - 1$. When choosing N one of course has to make sure that the detection power of the test statistic $\underline{t}^{k-N+1,k}$ is still sufficient. This is typical a problem one should take into consideration when designing the filter. Instead of constraining l to $k - N + 1 \leq l \leq k$ one may achieve further computational savings by constraining l to $k-N+1 \leq l \leq k-M$. This is shown in Figure 2c. The rationale behind this constraint is that the test statistic $\underline{t}^{l,k}$ may be too insensitive for detecting global model errors if $l > k - M$.

Example II

To illustrate the performance of the global slippage test statistic we give a small numerical example. The (time invariant) model used to simulate and process the data is given as follows (all quantities are dimensionless)

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \Phi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$R = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \quad Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The influence of the initial state is assumed to have died out. At two distinct epochs permanent slips of +7.0 and −6.0 are introduced in the first measurement (this corresponds to 4.0 and 3.5 times the standard deviation of the measurement). For a level of significance of $\alpha = 0.001$ we find a critical value of 10.82 for the one-dimensional slippage test statistic. The results of the simulations are given in Figure 3, where the notation of Figure 2(a) has been maintained. The test statistics larger than the critical value in Fig. 3 are denoted by an asteriks. Note that the occurence of the slip is only detected with a delay in both cases. □

$$\begin{bmatrix} 8.41 & 9.71 & 10.12 & 11.00 \;(*) \\ & 1.44 & 1.89 & 2.66 \\ & & 0.45 & 1.32 \\ & & & 1.31 \end{bmatrix}$$

(a)

$$\begin{bmatrix} 9.63 & 10.80 & 13.90 \;(*) & 13.60 \;(*) \\ & 1.42 & 4.27 & 3.97 \\ & & 4.39 & 3.33 \\ & & & 0.00 \end{bmatrix}$$

(b)

Figure 3: The one dimensional recursive test statistic $\underline{t}^{l,k}$ for the first measurement; simulation results for (a) a slip of 7.0, (b) a slip of −6.0.

A delay in time of detection renders real-time corrective action impossible. A strategy for handling the delayed detection of slips depends to a large extent on the application one has in mind.

It will be clear that optimal results are obtained if one is able to design a filter that is capable of following the correct alternative hypothesis at the correct time of occurence. Such an approach could be followed if one designs a whole bank of filters, each one taylored for one particular hypothesis. If the main objective is to obtain a clean dataset, a similar result can be obtained with one parallel filter, operating with a delay $N + 1$, using information provided by the real-time filter and the original data. For this method the data in the time interval $[k - N, k]$ have to be stored. If, however, the number of filters is restricted to one, the following simple approach might be followed. One computes recursively the best estimates of the slip $\nabla^{l,k}$ for each alternative hypothesis considered. This estimator for $\nabla^{l,k}$ reads

$$\hat{\underline{\nabla}}^{l,k} = \frac{\sum_{i=l}^{k} c_{v\_i}^{*} Q_{v\_i}^{-1} \underline{v}_i}{\sum_{i=l}^{k} c_{v\_i}^{*} Q_{v\_i}^{-1} c_{v\_i}} \quad . \tag{42}$$

At time k when the correct hypothesis has been identified, one then *resets* the real-time filter by correcting the filtered state as

$$\hat{\underline{x}}_{k|k}^{\mathrm{corr}} = \hat{\underline{x}}_{k|k} - x_{k,l} \hat{\underline{\nabla}}^{l,k} \quad , \tag{43}$$

where $x_{k,l}$ can be computed recursively in a way anagolous to the second formula of (35). The state $\hat{\underline{x}}_{k|k}^{\mathrm{corr}}$, with appropriate covariance matrix which follows from an error propagation of (43), is then used as the new initial state for time k. In this way the state estimate can immediately be repaired at time k of detection. Note that this simple approach is not optimal in the sense that the filter has not followed the correct hypothesis from time l onward.

6 Concluding Remarks

In this paper new test statistics for use in state-space filtering were introduced. The test statistics derived are all functions of the predicted residuals. The concepts of local and global testing were introduced. It was demonstrated that local tests can fail to detect global unmodelled trends. Therefore we introduced our global test statistics and it was shown how global overall model tests and global slippage tests can be computed in *recursive* form. We stressed that for the design of filters the detection power of the various tests should be taken into account. Besides it has to be mentioned that in filter design one also has to consider how well one can discriminate between different alternative hypotheses.

We presented global slippage test statistics for one-dimensional alternative hypotheses, where the variance factor of the covariance matrix of the predicted residual is known, and ∇ is unknown. One can, however, following the approach given in Section 3 also derive recursive global slippage test statistics for 1) the case of b-dimensional alternative hypotheses, 2) for the case that Q_{v_i}, $i = 1, 2, \ldots$ is known up to a scale factor, and 3) for the discriminant analysis case that ∇ is known. The solutions have a form similar to the ones derived in Section 4.

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