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PERFORMANCE ANALYSIS
OF KALMAN FILTERS

by

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Abstract

Methods for the analysis of the performance of Kalman filters are considered in the paper. The methods are all based on the innovation sequence which has well defined statistical properties if the filter is optimal. Local and global teststatistics are presented and discussed. A global slippage teststatistic is introduced. This test, which has batch type properties, is given in a recursive form.

1. Introduction

With the advent of powerful microcomputers sophisticated algorithms for kinematic positioning are used more frequently. It is common practice to process data from different sensors simultaneously in a so-called integrated navigation system to obtain a best estimate of position. The algorithm most often used in these integrated navigation systems is the well known Kalman filter. Some typical examples of the application of Kalman filters in navigation systems is given in [1],[2],[3],[4], and [5].

In this paper we restrict ourselves to the performance analysis of Kalman filters. To obtain useful (positioning) results using an integrated navigation system it is important that the performance of the underlying filter is at an optimum. Methods for the detection of departures from optimality are all based on the so-called innovation sequence. The innovation sequence of an optimal filter has precisely defined characteristics which can be compared with the output of an actually implemented Kalman filter. Under normal conditions the innovation sequence is a zero mean Gaussian white noise sequence with known covariance. Independence can be tested with the so-called run test or reverse arrangements test, or with tests based on the autocorrelation function. These tests are all non parametric or distribution free. Tests for the covariance are based on the Wishart distribution. See e.g. [6], [7], and [8].

In this paper we focus on slippage tests, which basically test the zero mean of the innovation sequence. We present the local overall model test and the one-dimensional local slippage test. These tests are very easily implemented and we believe that every software package should at least have these two types of tests included. We also present the global overall model test statistic which is a simple weighted mean of the local overall model teststatistic. Finally we introduce a new powerful teststatistic, which is called the one-dimensional global slippage teststatistic. It is a teststatistic given in recursive form which has batch type properties.

The contents of the paper is as follows. The Kalman filter and the assumptions underlying its model are briefly described in the next section. In section 3 the innovation sequence is introduced and its characteristics are outlined. The local and global teststatistics are derived in sections 4 and 5 respectively. Section 6 contains some conclusions. Two appendices are attached to the paper. Appendix A contains some

necessary theory on adjustment and hypothesis testing. Appendix B gives a simple derivation of the Kalman filter using the principles of least squares.

2. The Linear Kalman Filter

In this section we present and briefly discuss the mathematical model and recursive relations of the linear discrete time Kalman filter. For a more extensive discussion the reader is referred to the literature, see e.g. [9] , [10]. The mathematical model which forms the basis of the Kalman filter is

$$(1a) \quad \underline{x}_k = \Phi_{k,k-1} \underline{x}_{k-1} + \underline{q}_k$$

$$(1b) \quad \underline{y}_k = A_k \underline{x}_k + \underline{e}_k$$

Equation (1a) represents the dynamic model and equation (1b) represents the measurement model. The dynamic model is a linear vector difference equation. The independent variable t , which is often time, can assume the values $t_0 \leq t_1 \leq \dots \leq t_N$, where the t_i are not necessarily equidistant. The state of the system at t_k is given by the n -dimensional vector \underline{x}_k . The underscore indicates that the state vector \underline{x}_k is a vector random variable.

In many applications (1a) is derived from the linearization or linear perturbation equations relating to a dynamic system, so that the $n \times n$ matrix $\Phi_{k,k-1}$ may be assumed to be a known state transition matrix with the following properties:

$$\Phi_{k,k} = I \quad \text{for all } k$$

$$\Phi_{k,l} \Phi_{l,m} = \Phi_{k,m}$$

The initial state \underline{x}_0 is considered to be a vector random variable with a Gaussian distribution and the known statistics

$$E\{\underline{x}_0\} = \underline{x}_0$$

$$E\{(\underline{x}_0 - \underline{x}_0)(\underline{x}_0 - \underline{x}_0)^*\} = P_0 | 0$$

The operator $E\{\cdot\}$ denotes the mathematical expectation and $(\cdot)^*$ denotes transpose. The n -dimensional vector random variable \underline{q}_k represents the dynamic system noise. It is assumed to have a Gaussian distribution with the known statistics

$$E\{\underline{q}_k\} = 0 \quad \text{for all } k$$

$$E\{\underline{q}_k \underline{q}_l^*\} = Q_k \delta_{kl},$$

where δ_{kl} is the Kronecker delta, i.e. $\delta_{kl} = 1$ for $k=l$ and $\delta_{kl} = 0$ otherwise.

The measurement model (1b) states that at each time t_k there are m_k measurements collected in the m_k -dimensional observation vector \underline{y}_k available. The vector of observations is linearly related to the state through the known $m_k \times n$ designmatrix A_k and is corrupted by additive measurement noise \underline{e}_k .

The vector random measurement noise \underline{e}_k is assumed to have a Gaussian distribution with the known statistics

$$E(\underline{e}_k) = 0 \text{ for all } k$$

$$E(\underline{e}_k \underline{e}_l^*) = R_k \delta_{kl}$$

The above defined covariance matrices $P_{0|0}$, Q_k , and R_k are all assumed to be positive definite and thus invertible.

Finally it is assumed that the random sequences \underline{q}_k and \underline{e}_k are mutually uncorrelated and also uncorrelated with the initial state:

$$E\{\underline{q}_k \underline{e}_l^*\} = 0 \text{ for all } k, l$$

$$E\{\underline{q}_k \underline{x}_0^*\} = 0 \text{ for all } k$$

$$E\{\underline{e}_k \underline{x}_0^*\} = 0 \text{ for all } k$$

The mathematical model described above provides the basis for all succeeding discussion.

Depending on the application one has in mind, one might wish to obtain an estimate of the state at a certain time k which depends on all observations taken up to and including time $k+l$. If $l < 0$ the process is called **prediction**. The state estimate depends then only on the observations taken prior to the desired time of estimation. If $l=0$ the process is called **filtering**. In this case the state estimate depends on all the observations taken prior to and at time k . Finally if $l > 0$ the process is called **smoothing**. The state estimate depends then on observations taken prior to, on, and after time k . (see also fig. 1). It will be clear that under normal conditions smoothed estimates are more precise and reliable than filtered estimates. Similarly filtered estimates are more precise and reliable than predicted estimates.

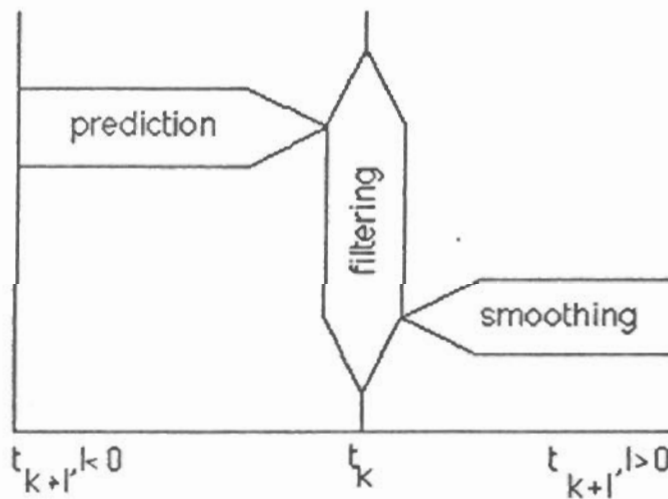


Fig. 1. Relation between prediction, filtering, and smoothing of the state at time k

Since we have real time applications of the estimation problem in mind, we shall restrict ourselves in the following to recursive prediction and filtering. The problem we stand for is to determine an estimate of the state at time k that is a linear combination of an estimate of the state at time $k-1$ and the observations at time k . Furthermore the estimate must be "best" in a certain sense. Kalman [11] was the first to solve this problem for the continuous time model using the minimum mean square error criterion. When Kalman's

method of derivation is applied to the discrete time model the so-called linear discrete time Kalman filter is obtained. It basically consists of two parts: the **time update** which gives the predicted state, and the **measurement update** which gives the filtered state. The time update of the state and its covariance matrix are given as

$$(2a) \quad \hat{x}_{k|k-1} = \Phi_{k,k-1} \hat{x}_{k-1|k-1}$$

$$(2b) \quad P_{k|k-1} = \Phi_{k,k-1} P_{k-1|k-1} \Phi_{k,k-1}^* + Q_k,$$

and the measurement updates of the state and its covariance matrix are given as

$$(3a) \quad \hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k (y_k - A_k \hat{x}_{k|k-1})$$

$$(3b) \quad P_{k|k} = (I - K_k A_k) P_{k|k-1}$$

where

$$(4) \quad K_k = P_{k|k-1} A_k (A_k P_{k|k-1} A_k + R_k)^{-1}$$

is the so-called Kalman gain matrix.

Equation (2a) gives the best estimate of the state at time k in the minimum mean square error sense using all observations prior to time k , whereas equation (3a) gives the best estimate of the state using both $\hat{x}_{k|k-1}$ and y_k .

Although Kalman used the minimum mean square error principle to derive his equations, it can be shown that if the model is linear and all vector random variables are Gaussian the methods of maximum likelihood, maximum a posteriori and least squares lead to identical results. Since surveyors and hydrographers are probably the most familiar with the principle of least squares, a simple derivation of the above Kalman filter equations based on this principle is given in appendix B.

3. The Innovation Sequence

As was pointed out in the previous section the recursive Kalman filter produces optimal estimates of the state vector with well defined statistical properties. These estimates are only optimal, however, as long as the given assumptions underlying the mathematical model hold. Misspecifications in the dynamic model and/or the measurement model will invalidate the results of estimation and thus also any conclusion based on them. It is therefore of crucial importance to have ways to verify the validity of the assumed mathematical model.

An important role in the process of model testing is played by the so-called **innovation sequence**. The innovation sequence is defined as the difference between the actual system output and the predicted output based on the predicted state. Thus the innovation sequence is given by

$$(5) \quad v_k = y_k - A_k \hat{x}_{k|k-1}, \quad k=1,2, \dots$$

It is called the innovation sequence since it represents the new information brought in by the latest observation vector. This can be seen from the measurement update equation

(3a), which shows that the filtered state is a linear combination of the predicted state and the innovation.

Under normal conditions the innovation is "small" and corresponds to random fluctuations in the output since all the systematic trends are eliminated by the model. If, however, the model is misspecified the innovation is "large" and contains systematic trends because the model no longer represents the physical system adequately.

Under normal conditions the innovation sequence has well defined statistical properties. It can be shown, see e.g. [12], that if the model is valid, the innovation sequence is a zero mean Gaussian white noise sequence with known covariance:

$$(6a) E\{\underline{v}_k\} = 0 \quad \text{for all } k$$

$$(6b) E\{\underline{v}_k \underline{v}_l^*\} = Q_{vk} \delta_{kl},$$

where

$$(7) Q_{vk} = R_k + A_k P_{k|k-1} A_k^*$$

is the covariance matrix of the innovation \underline{v}_k .

These properties can be used to test the innovation sequence for zero mean, whiteness and a given covariance. A sequence is called white if it consists of a sequence of uncorrelated random variables.

In the following we will restrict ourselves to misspecifications in the mean of the innovation sequence. That is, we will only consider slippage tests. The necessary teststatistics are all functions of the innovations and are optimized to detect a particular misspecification in the assumed mathematical model.

4. Local Slippage Tests

In this section we present some methods of hypothesis testing as applied to the linear discrete time Kalman filter. For a brief review of the theory of hypothesis testing the reader is referred to appendix A.

We consider the local overall model (LOM) test and the one-dimensional local slippage (LS) test. By local we mean that the tests when performed at time k only depend on the predicted state at time k and the observations at time k . Observations taken after time k have no effect on these tests. The influence of observations taken prior to time k is only felt indirectly via the predicted state.

The purpose of the LOM test is to detect misspecifications in the mathematical model occurring at time k . Since misspecifications in the model which have occurred prior to time k do affect the predicted state, the LOM test could detect them. The LOM test is however not optimal for these past misspecifications. This case will be considered in the next section.

The teststatistic of the LOM test is given as

$$(8) T_k = \frac{\underline{v}_k^* Q_v^{-1} \underline{v}_k}{m_k},$$

where m_k is the number of observations taken at time k .

The decision that a misspecification in the model has occurred at time k is made once

$$T_k \geq \chi^2_{\alpha}(m_k, 0),$$