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THE 1 AND 2D SYMMETRIC HELMERT TRANSFORMATION: AN EXACT NON-LINEAR LEAST-SQUARES SOLUTION

by

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Contents.

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Abstract

In this paper a particular class of non-linear least-squares problems for which it is possible to take advantage of the special structure of the non-linear model, is discussed. The non-linear models are of the ruled-type (Teunissen, 1985a). The proposed solution strategy for this class of problems is applied to the 1 and 2D non-linear Symmetric Helmert transformation. Exact non-linear least-squares solutions are derived.

1. Introduction

The aim of the present paper is to derive an exact non-linear least-squares solution for the 1 and 2D Symmetric Helmert transformation.
In section two we discuss a particular class of non-linear least-squares problems for which it is possible to take advantage of the special structure of the non-linear model. The non-linear models are manifolds of the ruled-type (see Teunissen, 1985a). We show that for this class of non-linear least-squares problems a two-step procedure can be devised. The first step consists of a linear least-squares problem, while the second step consists of a non-linear least-squares problem of a reduced dimension. In general the second step has to be solved through the use of linearization and iteration techniques, such as Gauss' method or variations thereof. A theorem is given which justifies the proposed two-step procedure.
In section three we first consider the linear 1D Helmert transformation. We show that the product of the scale estimator $\hat{\lambda}_H$ and the scale estimator $\hat{\lambda}'_H$ of the inverse 1D Helmert transformation does not satisfy $\hat{\lambda}_H.\hat{\lambda}'_H = 1$. This is unsatisfactory and a consequence of the fact that the linear Helmert trans-

formation does not treat the two coordinate sets on an equal basis. We therefore introduce our 1D Symmetric Helmert transformation. The corresponding model is non-linear, but a member of the class of models considered in section 1. The proposed two-step procedure is therefore applied and it is shown that the second step reduces to an eigenvalue problem which can be solved in an analytical way.

In section four we generalize the stochastic model of the classical linear 2D Helmert transformation to rotational-invariant covariance matrices. The linear least-squares solution is given.

In section five we introduce our new non-linear 2D Symmetric Helmert transformation. A rotational-invariant covariance structure is assumed. The non-linear least-squares solution is derived with the proposed two-step procedure. We show that the product of the scale estimators $\hat{\lambda}_{SH}$ and $\hat{\lambda}'_{SH}$ of the Symmetric Helmert transformation and its inverse satisfies $\hat{\lambda}_{SH} \cdot \hat{\lambda}'_{SH} = 1$. We also show that in general one systematically underestimates the scale when using the classical Helmert transformation.

The appendix contains a proof of an expression for the derivative of an orthogonal projector. This result is useful in itsself for perturbation analysis and is needed when one wants to apply Gauss' iteration method to the second step of the proposed two-step procedure.

2. A particular class of Non-Linear Least-Squares Problems

We will study a method that takes advantage of a special structure of an optimization problem, which is expressed so that the optimization with respect to some of the variables is easier than with respect to the others. Consider the unconstrained minimization problem

$$(2.1) \qquad \min_{w \in R^n} . \; f(w)$$

Suppose we can partition w into

$$(2.2) \qquad w = \begin{pmatrix} u \\ v \end{pmatrix}, \; u \in R^{n_1}, \; v \in R^{n_2}, \; n_1 + n_2 = n,$$

in such a way that the subproblem

$$(2.3) \qquad \min_{u \in R^{n_1}} . \quad f(u,v)$$

is easy to solve for every v in the domain of consideration. Let u(v) denote one solution of (2.3) and formulate the problem

$$(2.4) \qquad \min_{v \in R^{n_2}} . \quad f(u(v),v) \;.$$

We can now replace the original n dimensional minimization problem (2.1) by a n_2 dimensional one (2.4), where each evaluation of the object function needs the solution of a n_1 dimensional minimization problem (2.3).
In principle every partition of the variables is possible. But to be of advantage practically, (2.3) must be simpler to solve than the corresponding problem with respect to v.
If we restrict ourselves to <u>least-squares</u> problems, the above two-step procedure becomes particularly advantageous when some of the variables occur linearly. In this case (2.3) is simply a <u>linear</u> least-squares problem.

<u>Example</u>: Orthogonal projection onto a ruled surface (see Teunissen, 1985a)

A <u>ruled surface</u> is a surface which has the property that through every point of the surface there passes a straight line which lies entirely in the surface. Thus the surface is covered by straight lines, called <u>rulings</u> which form a family depending on one parameter.
In order to find a parametrization of a ruled surface choose on the surface a curve transversal to the rulings. Let this curve be given by c(v), $v \in R$.
At any point of this curve take a vector t of the ruling which passes through this point. This vector obviously depends on v. Thus we have t(v). Now we can write the equation of the surface as

$$(2.5) \qquad a(u,v) = c(u) + ut(v), \quad u,v \in R, \quad a,c,t \in R^3.$$

The parameter v indicates the ruling on the surface and the parameter u shows the position on the ruling.

Now let us assume that we have to solve for the following non-linear least-squares problem:

$$(2.6)\qquad \min_{u,v}.\ \|y-a(u,v)\|^2 ,$$

$$\text{with the norm } \|.\|^2 = (.)^* Q_y^{-1}(.) .$$

Since the ruled surface is flat in the directions of the rulings, whilst curved in the directions transversal to it, it becomes advantageous to perform the adjustment in two steps. In the first step one would then solve for a linear least-squares adjustment problem, and in the second step for a non-linear adjustment problem of a reduced dimension. That is, one first solves for

$$(2.7)\qquad \min_{u}.\ \|(y-c(v)) - t(v)u\|^2 ,$$

which gives

$$(2.8)\qquad u(v) = [t^*(v)Q_y^{-1}t(v)]^{-1}t^*(v)Q_y^{-1}(y-c(v)) .$$

Then in the second step one solves for the non-linear problem

$$(2.9)\qquad \min_{v}.\ \|y-(c(v) + t(v)u(v))\|^2 .$$

As a generalization of the foregoing example, we are interested in solving the non-linear model

$$(2.10)\qquad E\{y\} = A(z)x \quad , \ Cov.\{y\} = Q_y ,$$

in a least-squares sense; where y is the m dimensional vector of observational variates, $E\{.\}$ stands for the mathematical expectation, $A(z)$ is a $m\times n_1$ matrix, Q_y is the $m\times m$ positive definite covariance matrix of y, and x and z are respectively the n_1- and n_2 dimensional vectors of unknown parameters. We will assume that matrix $A(z)$ has constant full rank for all z of interest.

We can write (2.10) in index notation as

$$(2.10') \qquad E\{y^i\} = A^i_\alpha(z)x^\alpha, \quad Cov.\{y^i\} = g^{ij} \ .$$

We will assume that the mn_1 functions $A^i_\alpha(z)$ are continuously differentiable. We define

$$(2.11) \qquad \begin{cases} a) & f(x,z) \triangleq ||y-A(z)x||^2 \\ b) & f_1(z) \triangleq ||P^\perp_{A(z)}y||^2 \\ c) & x(z) \triangleq A^-(z)y \end{cases} ,$$

where $||.||^2 = (.)^* Q_y^{-1}(.)$, $P_{A(z)}$ is the orthogonal projector projecting onto the rangespace of $A(z)$, $P^\perp_{A(z)} = I-P_{A(z)}$ is the orthogonal projector projecting onto the orthogonal complement of the rangespace of $A(z)$ and $A^-(z)$ is the least-squares inverse of $A(z)$.

Since $x(z)$ is the solution of $\min_x . f(x,z)$ we have that

$$(2.12) \qquad \begin{cases} a) & f_1(z) = f(x(z),z) = \min_x . f(x,z) \quad \forall z \\ b) & f_1(z) \leqq f(x,z) \quad \forall x,z \ . \end{cases}$$

From (2.11) also follows that

$$(2.13) \qquad \begin{cases} a) & \partial_x f(x,z) = -2(y-A(z)x)^* Q_y^{-1} A(z) \\ b) & \partial_z f(x,z) = -2(y-A(z)x)^* Q_y^{-1} \partial_z A(z)x \\ c) & \partial_z f_1(z) = -2(P^\perp_{A(z)}y)^* Q_y^{-1} \partial_z P_{A(z)}y \ . \end{cases}$$

In the appendix it is proved that

$$(2.14) \qquad \partial_z P_{A(z)} = (I-P_{A(z)})\partial_z A(z)A^-(z) + [Q_y^{-1}(I-P_{A(z)})\partial_z A(z)A^-(z)Q_y]^* \ .$$

Since

$$(2.15) \qquad \begin{cases} a) & P^{\perp *}_{A(z)} Q_y^{-1} P^\perp_{A(z)} = Q_y^{-1} P^\perp_{A(z)} \\ b) & A^-(z)P^\perp_{A(z)} = 0 \ , \end{cases}$$

substitution of (2.14) into (2.13c) gives

$$(2.16) \qquad \partial_z f_1(z) = -2y^* Q_y^{-1} P_{A(z)}^{\perp} \partial_z A(z) A^-(z) y \ .$$

We are now ready to proof the following theorem, which gives a justification for the discussed two-step procedure.

Theorem

(i). If $\hat{x}$ and $\hat{z}$ are such that

$$(2.17) \qquad \text{a) } \partial_z f_1(\hat{z}) = 0, \quad \text{b) } \hat{x} = A^-(\hat{z})y \ ,$$

then

$$(2.18) \qquad \text{a) } f_1(\hat{z}) = f(\hat{x},\hat{z}), \quad \text{b) } \partial_x f(\hat{x},\hat{z}) = 0, \quad \text{c) } \partial_z f(\hat{x},\hat{z}) = 0 \,.$$

(ii). If $\hat{x}$ and $\hat{z}$ are such that

$$(2.19) \qquad \text{a) } f_1(\hat{z}) \leqq f_1(z) \ \forall \ z, \quad \text{b) } \hat{x} = A^-(\hat{z})y \ ,$$

then

$$(2.20) \qquad f(\hat{x},\hat{z}) \leqq f(x,z) \ \forall \ x,z.$$

(iii). If $\hat{x}$ and $\hat{z}$ are such that

$$(2.21) \qquad f(\hat{x},\hat{z}) \leqq f(x,z) \ \forall \ x,z \ ,$$

then

$$(2.22) \qquad \text{a) } f_1(\hat{z}) = f(\hat{x},\hat{z}), \quad \text{b) } f_1(\hat{z}) \leqq f_1(z) \ \forall \ z.$$

proof of (i):
(2.18a) follows from (2.11c), (2.17b) and (2.12a).
(2.18b) follows from (2.17b), (2.13a), (2.15a) and the fact that $P_{A(z)}^{\perp}A(z) = 0$.
(2.18c) follows from (2.17a), (2.17b), (2.13b), (2.15a) and (2.16).
Thus if $\hat{z}$ is a stationary point of $f_1(z)$ and $\hat{x}$ is defined by (2.17b) then $(\hat{x},\hat{z})$ forms a stationary point of $f(x,z)$.

proof of (ii):
We will give the proof by contradiction. Assume that a $\bar{x}$ and $\bar{z}$ exist such that $f(\bar{x},\bar{z}) < f(\hat{x},\hat{z})$. With (2.12b) this gives: $f_1(\bar{z}) \leqq f(\bar{x},\bar{z}) < f(\hat{x},\hat{z})$. With (2.19b), (2.11c) and (2.12a) this gives: $f_1(\bar{z}) \leqq f(\bar{x},\bar{z}) < f(\hat{x},\hat{z}) = f(x(\hat{z}),\hat{z}) = f_1(\hat{z})$. But this contradicts our assumption that $f_1(\hat{z}) \leqq f(z)\ \forall\ z$. Hence no $\bar{x}$ and $\bar{z}$ exist such that $f(\bar{x},\bar{z}) < f(\hat{x},\hat{z})$.
Thus if $\hat{z}$ is a global minimum of $f_1(z)$ and $\hat{x}$ is defined by (2.19b) then $(\hat{x},\hat{z})$ is a global minimum of $f(x,z)$.

proof of (iii):
First we will proof (2.22a).
From (2.12b) follows that $f_1(\hat{z}) \leqq f(\hat{x},\hat{z})$. Now let $\bar{x} = A^-(\hat{z})y$. With (2.11c) and (2.12a) follows then that $f_1(\hat{z}) = f(\bar{x},\hat{z}) \leqq f(\hat{x},\hat{z})$. Since $(\hat{x},\hat{z})$ is a global minimum of $f(x,z)$ we must have equality, i.e. $f_1(\hat{z}) = f(\bar{x},\hat{z}) = f(\hat{x},\hat{z})$.
We will proof (2.22b) by contradiction.
Assume that a $\bar{z}$ exists such that $f_1(\bar{z}) < f_1(\hat{z})$. Now let $\bar{x} = A^-(\bar{z})y$. With (2.11c) and (2.12a) this gives: $f_1(\bar{z}) = f(x(\bar{z}),\bar{z}) = f(\bar{x},\bar{z}) < f_1(\hat{z})$. According to (2.22a) we have $f_1(\hat{z}) = f(\hat{x},\hat{z})$ and thus $f_1(\bar{z}) = f(x(\bar{z}),\bar{z}) = f(\bar{x},\bar{z}) < f_1(\hat{z}) = f(\hat{x},\hat{z})$. But this contradicts our assumption that $f(\hat{x},\hat{z}) \leqq f(x,z)\ \forall\ x,z$.
Hence no $\bar{z}$ exists such that $f_1(\bar{z}) < f_1(\hat{z})$.
Thus if $(\hat{x},\hat{z})$ is a global minimum of $f(x,z)$ then $\hat{z}$ is a global minimum of $f_1(z)$.
This concludes the proof of the theorem.

From (ii) and (iii) of the Theorem follows that if the global minimum $(\hat{x},\hat{z})$ of $f(x,z)$ is unique, then also the global minimum of $f_1(z)$ is unique and is given by $\hat{z}$. Conversely, if the global minimum $\hat{z}$ of $f_1(z)$ unique then the global minimum of $f(x,z)$ is unique and is given by $(\hat{x},\hat{z})$. The uniqueness of the x-component follows from the uniqueness of the least-squares inverse $A^-(z)$, since $A(z)$ is assumed to be of constant full rank.

When one applies the above described two-step procedure one still has to solve for the non-linear problem $\min_z \|P^{\perp}_{A(z)}y\|^2$. This can be done by Gauss' iteration method or variations thereof. In this paper we will not discuss the application of the Gauss'iteration method to the above problem, but see e.g. (Teunissen, 1984, 1985a and b) for more details. Instead we will use the described two-step procedure to solve for the 1 and 2 dimensional non-linear Symmetric Helmert transformation in an analytical way.
First we consider the 1D Helmert transformation and its non-linear symmetrical generalization.

3. The 1D Helmert transformation

We define the model of the 1D Helmert transformation as

$$(3.1) \qquad E\{x_i\} = \lambda u_i + t \ , \quad i=1, \ .... \ ,n \quad ,$$

where x_i are the observed coordinates, u_i are the fixed given coordinates, and λ and t are respectively the unknown scale and translation parameters. With

$$(3.2) \qquad x = (\ldots x_i \ldots)^*, \quad u = (\ldots u_i \ldots)^* \quad \text{and } e = (1 \ldots . 1)^*,$$

we can write (3.1) in vector form as

$$(3.3) \qquad E\{x\} = (u \ e)\begin{pmatrix}\lambda \\ t\end{pmatrix}.$$

We assume the covariance matrix of x to be the unit matrix I. We also assume of course that u is not parallel to e. Similar assumptions are made for the other models to be discussed. Model (3.3) is linear and very easy to solve. The least-squares estimators of scale and translation read:

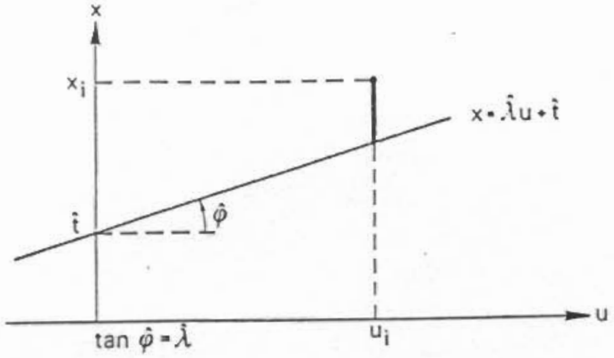
$$(3.4) \qquad \begin{cases} a) \quad \hat{\lambda} = \dfrac{\bar{u}^*\bar{x}}{\bar{u}^*\bar{u}} \\ \\ b) \quad \hat{t} = x_c - \hat{\lambda} u_c \ , \end{cases}$$

where

$$(3.5) \qquad \begin{cases} x_c = \dfrac{\sum_{i=1}^{n} x_i}{n} \ , \quad u_c = \dfrac{\sum_{i=1}^{n} u_i}{n} \\ \\ \bar{x} \ = x - x_c e \ , \quad \bar{u} \ = u - u_c e \end{cases}$$

Geometrically the least-squares adjustment of model (3.3) amounts to adjusting the parameters λ and t such that the sum of the squared <u>vertical</u> distances of the sample points (x_i, u_i) to the line $x = \lambda u + t$ is minimized (see fig.1). When we change the role of x and u in model (3.3) we get the model

$$(3.6) \qquad E\{u\} = (x \ e)\begin{pmatrix}\lambda' \\ t'\end{pmatrix},$$

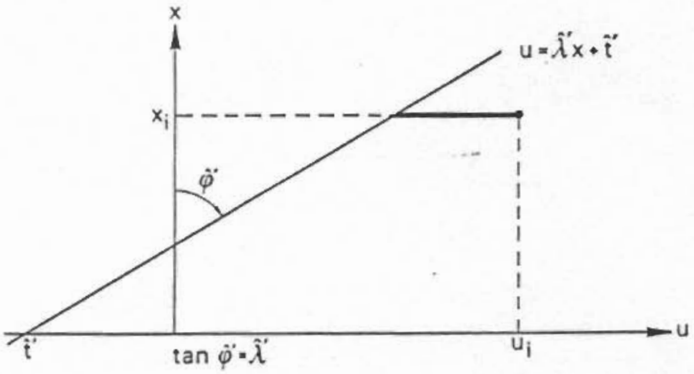


- fig. 1 -

with the solution

$$(3.7)\quad \begin{cases} a)\ \hat{\lambda}' = \dfrac{\bar{u}*\bar{x}}{\bar{x}*\bar{x}} \\ b)\ \hat{t}' = u_c - \hat{\lambda}' x_c \end{cases}$$

Geometrically this amounts to a minimization of the sum of the squared <u>horizontal</u> distances (see fig.2).



- fig. 2 -

Since in both models (3.3) and (3.6) x and u are treated in an a-symmetric way (that is x stochastic and u fixed, or vise versa), it follows that (see (3.4a) and (3.7a)):

$$(3.8)\qquad \hat{\lambda}.\hat{\lambda}' \neq 1.$$

This is unsatisfactory, especially if one considers that in general both x and u are observational variates with inherent observational errors. A more satisfactory model would therefore be

$$(3.9) \qquad E\{x\} = \lambda E\{u\}+t \quad ,$$

where both x and u are considered to be stochastic, each with the unit matrix as covariance matrix.
Introducing the coordinate unknowns u', we can write (3.9) as:

$$(3.10) \qquad E\{\begin{bmatrix} x \\ u \end{bmatrix}\} = \begin{bmatrix} \lambda I & e \\ I & 0 \end{bmatrix}\begin{bmatrix} u' \\ t \end{bmatrix} \; ; \; \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

This model will be called the <u>1D Symmetric Helmert transformation</u>. Note that this model is <u>non-linear</u> because of the product $\lambda u'$. Our problem is now to solve for model (3.10) in a least-squares sense. We define

$$(3.11) \qquad f(u',t,\lambda) \triangleq ||\begin{bmatrix} x \\ u \end{bmatrix}-\begin{bmatrix} \lambda I & e \\ I & 0 \end{bmatrix}\begin{bmatrix} u' \\ t \end{bmatrix}||^2,$$

where $||.||^2 = (.)^*(.)$.
In order to solve

$$(3.12) \qquad \min_{u',t,\lambda} . \; f(u',t,\lambda),$$

we proceed in two steps. First we fix λ and solve for

$$(3.13) \qquad \min_{u',t} . \; f(u',t,\lambda)$$

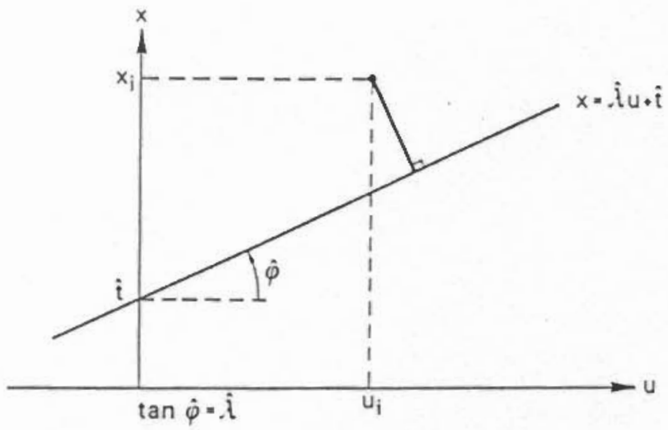
This is a <u>linear</u> least-squares problem. Its solution is denoted by $u'(\lambda),t(\lambda)$.
In the second step we solve for

$$(3.14) \qquad \min_{\lambda} . \; f_1(\lambda) = \min_{\lambda} . f(u'(\lambda),t(\lambda),\lambda)$$

Once we have found the solution $\hat{\lambda}$ of this <u>non-linear</u> least-squares problem, the complete solution of (3.12) is given by

$$(3.15) \qquad \begin{cases} \hat{u}' = u'(\hat{\lambda}) \\ \hat{t} = t(\hat{\lambda}) \\ \hat{\lambda} = \hat{\lambda} \end{cases}$$

Geometrically the least-squares problem (3.11) amounts to adjusting the parameters such that the sum of the squared smallest distances of the sample points (x_i,u_i) to the line $x = \lambda u+t$ is minimized (see fig.3).



- fig. 3 -

Step 1: (λ fixed)

For a fixed λ we find from (3.11) and (3.13) that

$$(3.16)\qquad \begin{bmatrix} u'(\lambda) \\ \\ t(\lambda) \end{bmatrix} = \begin{bmatrix} (1+\lambda^2)I & \lambda e \\ \\ \lambda e^* & e^*e \end{bmatrix}^{-1} \begin{bmatrix} \lambda x+u \\ \\ e^*x \end{bmatrix}$$

$$= \begin{bmatrix} (1+\lambda^2)^{-1}[I+\lambda^2 n^{-1}ee^*] & -n^{-1}\lambda e \\ \\ -n^{-1}\lambda e^* & n^{-1}(1+\lambda^2) \end{bmatrix} \begin{bmatrix} \lambda x+u \\ \\ e^*x \end{bmatrix}$$

or

$$(3.17)\qquad \begin{cases} u'(\lambda) = (1+\lambda^2)^{-1}[\lambda\bar{x}+\bar{u}]+u_c e \\ t(\lambda) \;= x_c-\lambda u_c \end{cases}$$

Step 2:

Substitution of (3.17) into (3.11) gives

$$(3.18)\qquad f_1(\lambda) = \left|\left|\begin{bmatrix} x \\ \\ u \end{bmatrix} - \begin{bmatrix} \lambda(1+\lambda^2)^{-1}[\lambda\bar{x}+\bar{u}]+x_c e \\ \\ (1+\lambda^2)^{-1}[\lambda\bar{x}+\bar{u}]+u_c e \end{bmatrix}\right|\right|^2$$

$$= \left|\left|\begin{bmatrix} (1+\lambda^2)^{-1}[\bar{x}-\lambda\bar{u}] \\ \\ -\ (1+\lambda^2)^{-1}[\bar{x}-\lambda\bar{u}] \end{bmatrix}\right|\right|^2$$

or

$$(3.19)\qquad f_1(\lambda) = (1+\lambda^2)^{-1}||\bar{x}-\lambda\bar{u}||^2$$

In order to find $\hat{\lambda}$ we need to minimize (3.19). Using the reparametrization

$$(3.20)\qquad \lambda = \tan\phi\ ,$$

we can write (3.19) as

$$(3.21)\qquad f_1(\tan\phi) = \begin{bmatrix} \cos\phi \\ \\ \sin\phi \end{bmatrix}^* \begin{bmatrix} \bar{x}^*\bar{x} & -\bar{x}^*\bar{u} \\ \\ -\bar{u}^*\bar{x} & \bar{u}^*\bar{u} \end{bmatrix} \begin{bmatrix} \cos\phi \\ \\ \sin\phi \end{bmatrix}$$

The minimization problem

$$(3.22)\qquad \min_{\phi}.\ f_1(\tan\phi),$$

now reduces to an <u>eigenvalue problem:</u>

$$(3.23)\qquad \left|\begin{bmatrix} \bar{x}^*\bar{x} & -\bar{x}^*\bar{u} \\ \\ -\bar{u}^*\bar{x} & \bar{u}^*\bar{u} \end{bmatrix} - \mu\begin{bmatrix} 1 & 0 \\ \\ 0 & 1 \end{bmatrix}\right| = 0$$

The two eigenvalues of (3.23) read

$$(3.24)\qquad \mu_{1,2} = \frac{(\bar{x}^*\bar{x}+\bar{u}^*\bar{u})\pm[(\bar{x}^*\bar{x}+\bar{u}^*\bar{u})^2-4(\bar{x}^*\bar{x}\bar{u}^*\bar{u}-(\bar{u}^*\bar{x})^2)]^{\frac{1}{2}}}{2}$$

Hence the smallest eigenvalue reads

$$\mu_{min} = \frac{(\bar{x}^*\bar{x}+\bar{u}^*\bar{u})-[(\bar{x}^*\bar{x}+\bar{u}^*\bar{u})^2-4(\bar{x}^*\bar{x}\bar{u}^*\bar{u}-(\bar{u}^*\bar{x})^2)]^{\frac{1}{2}}}{2} \tag{3.25}$$

From the two equations

$$\begin{bmatrix} (\bar{x}^*\bar{x}-\mu_{min}) & -\bar{x}^*\bar{u} \\ -\bar{u}^*\bar{x} & (\bar{u}^*\bar{u}-\mu_{min}) \end{bmatrix} \begin{bmatrix} \cos\hat{\phi} \\ \sin\hat{\phi} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.26}$$

we find that

$$\hat{\lambda} = \tan\hat{\phi} = \frac{\bar{x}^*\bar{u}}{\bar{u}^*\bar{u}-\mu_{min}} = \frac{\bar{x}^*\bar{x}-\mu_{min}}{\bar{x}^*\bar{u}} \tag{3.27}$$

From (3.17), (3.25) and (3.27) follows therefore that the final least-squares solution of the <u>non-linear 1D Symmetric Helmert transformation</u> is given by:

$$\boxed{\begin{aligned} \hat{u}' &= u_c e+(1+\hat{\lambda}^2)^{-1}[\hat{\lambda}\bar{x}+\bar{u}] \\ \hat{t} &= x_c-\hat{\lambda}u_c \\ \hat{\lambda} &= \frac{(\bar{x}^*\bar{x}-\bar{u}^*\bar{u})+[(\bar{x}^*\bar{x}-\bar{u}^*\bar{u})^2+4(\bar{u}^*\bar{x})^2]^{\frac{1}{2}}}{2\bar{x}^*\bar{u}} \end{aligned}} \tag{3.28}$$

Note that the smallest eigenvalue μ_{min} of (3.25) is not unique if

$$\bar{x}^*\bar{x} = \bar{u}^*\bar{u} \quad \text{and} \quad \bar{x}^*\bar{u} = 0.$$

If this is the case the matrix of (3.21) reduces to a scaled unit matrix and the solution $\hat{\lambda}$ becomes indeterminable. We shall disregard this exceptional case. Also note that if

$$\bar{x}^*\bar{u} = 0,$$

then $\mu_{min} = \bar{u}^*\bar{u}$ if $\bar{x}^*\bar{x} > \bar{u}^*\bar{u}$ and $\mu_{min} = \bar{x}^*\bar{x}$ if $\bar{u}^*\bar{u} > \bar{x}^*\bar{x}$. From (3.26) follows then that $\cos\hat{\phi} = 0$ or $\sin\hat{\phi} = 0$. We shall also disregard this exceptional case. Let us denote the scale estimator of the Helmert transformation (3.3) by $\hat{\lambda}_H$, of the Helmert transformation (3.6) by $\hat{\lambda}'_H$, of the Symmetric Helmert transformation (3.9) by $\hat{\lambda}_{SH}$ and of the Symmetric Helmert transformation when interchanging the role of x and u by $\hat{\lambda}'_{SH}$. From (3.4a),(3.7a),(3.25) and (3.27) follows then that

$$(3.29)\qquad \begin{cases} \hat{\lambda}_H = \dfrac{\bar{u}^*\bar{x}}{\bar{u}^*\bar{u}} & , \quad \hat{\lambda}'_H = \dfrac{\bar{u}^*\bar{x}}{\bar{x}^*\bar{x}} \\ \hat{\lambda}_{SH} = \dfrac{\bar{u}^*\bar{x}}{\bar{u}^*\bar{u}-\mu_{min}} = \dfrac{\bar{x}^*\bar{x}-\mu_{min}}{\bar{u}^*\bar{x}} & , \quad \hat{\lambda}'_{SH} = \dfrac{\bar{u}^*\bar{x}}{\bar{x}^*\bar{x}-\mu_{min}} = \dfrac{\bar{u}^*\bar{u}-\mu_{min}}{\bar{u}^*\bar{x}} \end{cases}$$

This shows that

$$(3.30)\qquad \hat{\lambda}_H.\hat{\lambda}'_H \neq 1, \text{ but } \hat{\lambda}_{SH}.\hat{\lambda}'_{SH} = 1.$$

From (3.29) also follows that

$$(3.31)\qquad \hat{\lambda}_{SH} = \hat{\lambda}_H.\frac{\bar{u}^*\bar{u}}{\bar{u}^*\bar{u}-\mu_{min}}$$

This shows that one, when using the Helmert transformation, in general systematically <u>underestimates</u> the scale parameter λ. The two scale estimates are identical if and only if $\mu_{min} = 0$. With (3.25) this gives the condition:

$$(3.32)\qquad \bar{x}^*\bar{x}\bar{u}^*\bar{u}-(\bar{u}^*\bar{x})^2 = 0,$$

or if α is defined to be the angle between the two vectors $\bar{x}$ and $\bar{u}$, the condition:

$$(3.33)\qquad \sin\alpha = 0, \quad \alpha = \measuredangle(\bar{x},\bar{u})$$

4. <u>The 2D Helmert transformation with a rotational invariant covariance structure</u>

The linear model of the 2D Helmert transformation reads

$$(4.1)\qquad E\{\begin{bmatrix} x_i \\ y_i \end{bmatrix}\} = \lambda\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} u_i \\ v_i \end{bmatrix}+\begin{bmatrix} t_x \\ t_y \end{bmatrix} \quad , \quad i=1,\ldots\ldots,n,$$

where (x_i,y_i) are the observed cartesian coordinates, (u_i,v_i) are the fixed given coordinates and λ,θ,t_x and t_y are respectively the four unknown scale, orientation and translation parameters.

We can write (4.1) in a more convenient form by making use of the Kronecker product $\otimes$, for which the following four properties hold (see e.g. Rao,1973):

$$(4.2)\qquad \begin{cases} (A\otimes B)^{*} = A^{*}\otimes B^{*} \ ; \ (A\otimes B)^{-} = A^{-}\otimes B^{-} \text{ using any inverse} \\ A_1A_2\otimes B_1B_2 = (A_1\otimes B_1)(A_2\otimes B_2) \ ; \\ (A+B)\otimes C = A\otimes C+B\otimes C \end{cases}$$

Take therefore the definitions:

$$(4.3)\qquad \begin{cases} x \triangleq (\ldots x_i \ldots)^{*},\ y \triangleq (\ldots y_i \ldots)^{*},\ z \triangleq (x^{*}\ y^{*})^{*}, \\ u \triangleq (\ldots u_i \ldots)^{*},\ v \triangleq (\ldots v_i \ldots)^{*},\ w \triangleq (u^{*}\ v^{*})^{*}, \\ S \triangleq \lambda\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix},\ e \triangleq (1\ \ldots.1)^{*},\ t \triangleq (t_x\ t_y)^{*}, \end{cases}$$

and write (4.1) as

$$(4.4)\qquad E\{z\} = (S\otimes I_n \quad I_2\otimes e)\begin{pmatrix} w \\ t \end{pmatrix}$$

We assume the covariance matrix of z to be <u>rotational invariant</u>, i.e.

$$(4.5)\qquad \text{Cov.}\{z\} = I_2\otimes Q_z \ ,$$

where Q_z is an arbitrary nxn positive definite matrix. The least-squares solution of the linear model (4.4)-(4.5) of the 2D Helmert transformation with a rotational invariant covariance structure was given in (Teunissen, 1986) and reads:

$$(4.6)\qquad \begin{cases} \hat{\lambda} = \dfrac{[[\bar{u}^{*}Q_z^{-1}\bar{x}+\bar{v}^{*}Q_z^{-1}\bar{y}]^2+[\bar{v}^{*}Q_z^{-1}\bar{x}-\bar{u}^{*}Q_z^{-1}\bar{y}]^2]^{\frac{1}{2}}}{[\bar{u}^{*}Q_z^{-1}\bar{u}+\bar{v}^{*}Q_z^{-1}\bar{v}]} \\ \hat{\theta} = \tan^{-1}\dfrac{\bar{v}^{*}Q_z^{-1}\bar{x}-\bar{u}^{*}Q_z^{-1}\bar{y}}{\bar{u}^{*}Q_z^{-1}\bar{x}+\bar{v}^{*}Q_z^{-1}\bar{y}} \\ \hat{t}_x = x_c-\hat{\lambda}\cos\hat{\theta}u_c-\hat{\lambda}\sin\hat{\theta}v_c \\ \hat{t}_y = y_c-\hat{\lambda}\cos\hat{\theta}v_c+\hat{\lambda}\sin\hat{\theta}u_c \end{cases} ,$$

where the weighted centred coordinates are defined by

$$
(4.7) \quad \begin{cases} \bar{x} \triangleq P_e^{\perp}x \ , \quad \bar{y} \triangleq P_e^{\perp}y \ , \quad \bar{u} \triangleq P_e^{\perp}u \ , \quad \bar{v} \triangleq P_e^{\perp}v \\ x_c \triangleq \dfrac{e^*Q_z^{-1}x}{e^*Q_z^{-1}e} \ , \quad y_c \triangleq \dfrac{e^*Q_z^{-1}y}{e^*Q_z^{-1}e} \ , \quad u_c \triangleq \dfrac{e^*Q_z^{-1}u}{e^*Q_z^{-1}e} \ , \quad v_c \triangleq \dfrac{e^*Q_z^{-1}v}{e^*Q_z^{-1}e} \\ P_e^{\perp} \triangleq I_n - e(e^*Q_z^{-1}e)^{-1}e^*Q_z^{-1} \end{cases}
$$

Note that if $Q_z = I_n$, solution (4.6) reduces to that of the well-known classical Helmert transformation (Helmert, 1893).

5. The 2D Symmetric Helmert transformation

We define the non-linear model of the 2D Symmetric Helmert transformation with a rotational invariant covariance structure as:

$$
(5.1) \qquad E\{\begin{bmatrix} z \\ w \end{bmatrix}\} = \begin{bmatrix} S \otimes I_n & I_2 \otimes e \\ I_{2n} & 0 \end{bmatrix}\begin{bmatrix} w' \\ t \end{bmatrix} \ , \ \begin{bmatrix} I_2 \otimes Q_z & 0 \\ 0 & I_2 \otimes Q_w \end{bmatrix} .
$$

We will assume that

$$
(5.2) \qquad Q_z = \sigma^2 Q_w \ , \ \sigma \in R^+ .
$$

Our problem is to solve for model (5.1) in a least-squares sense.
We define

$$
(5.3) \qquad f(w',t,\lambda,\theta) \triangleq \left|\left| \begin{bmatrix} z \\ w \end{bmatrix} - \begin{bmatrix} S \otimes I_n & I_2 \otimes e \\ I_{2n} & 0 \end{bmatrix}\begin{bmatrix} w' \\ t \end{bmatrix} \right|\right|^2 ,
$$

where

$$
(5.4) \qquad ||.||^2 = (.)^* \begin{bmatrix} I_2 \otimes Q_z^{-1} & 0 \\ 0 & I_2 \otimes Q_w^{-1} \end{bmatrix} (.) \quad .
$$

In order to solve

$$
(5.5) \qquad \min_{w',t,\lambda,\theta} . \quad f(w',t,\lambda,\theta) ,
$$

we again proceed in two steps. First we fix λ and θ, and solve for

$$(5.6) \qquad \min_{w',t}.\ f(w',t,\lambda,\theta)\ .$$

This is a <u>linear</u> least-squares problem. Its solution is denoted by $w'(\lambda,\theta)$, $t(\lambda,\theta)$. In the second step we solve for

$$(5.7) \qquad \min_{\lambda,\theta}.\ f_1(\lambda,\theta) = \min_{\lambda,\theta}.\ f(w'(\lambda,\theta),t(\lambda,\theta),\lambda,\theta)\ .$$

Once we have found the solution $\hat{\lambda}$ and $\hat{\theta}$ of this <u>non-linear</u> least-squares problem, the complete solution of (5.5) is given by

$$(5.8) \qquad \begin{cases} \hat{w}' = w(\hat{\lambda},\hat{\theta}) \\ \hat{t} = t(\hat{\lambda},\hat{\theta}) \\ \hat{\lambda} = \hat{\lambda} \\ \hat{\theta} = \hat{\theta} \end{cases}$$

<u>Step 1</u>: (λ and θ fixed)

For λ and θ fixed we find from (5.2),(5.3),(5.4) and (5.6) that

$$(5.9) \qquad \begin{bmatrix} w'(\lambda,\theta) \\ t(\lambda,\theta) \end{bmatrix} = \begin{bmatrix} (\lambda^2+\sigma^2)I_2\otimes Q_Z^{-1} & S^*\otimes Q_Z^{-1}e \\ S\otimes e^*Q_Z^{-1} & e^*Q_Z^{-1}eI_2 \end{bmatrix}^{-1} \begin{bmatrix} S^*\otimes Q_Z^{-1} & \sigma^2 I_2\otimes Q_Z^{-1} \\ I_2\otimes e^*Q_Z^{-1} & 0 \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix}$$

$$= \begin{bmatrix} (\lambda^2+\sigma^2)^{-1}[I_2\otimes Q_Z+(\lambda/\sigma)^2(e^*Q_Z^{-1}e)^{-1}I_2\otimes ee^*] & -\bar{\sigma}^2(e^*Q_Z^{-1}e)^{-1}S^*\otimes e \\ -\bar{\sigma}^2(e^*Q_Z^{-1}e)^{-1}S\otimes e^* & (\lambda^2+\sigma^2)\bar{\sigma}^2(e^*Q_Z^{-1}e)^{-1}I_2 \end{bmatrix} \begin{bmatrix} S^*\otimes Q_Z^{-1} & \sigma^2 I_2\otimes Q_Z^{-1} \\ I_2\otimes e^*Q_Z^{-1} & 0 \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix}$$

or

$$(5.10) \qquad \begin{cases} w'(\lambda,\theta) = (\lambda^2+\sigma^2)^{-1}[S^*\otimes I_n)\bar{z}+\sigma^2\bar{w}]+w_c\otimes e \\ t(\lambda,\theta) = z_c - Sw_c \quad , \end{cases}$$

where the weighted centred coordinates are defined as

$$(5.11) \qquad \begin{cases} z_c\otimes e \triangleq (I_2\otimes P_e)z \ , \quad w_c\otimes e \triangleq (I_2\otimes P_e)w \\ \bar{z} \triangleq z - z_c\otimes e \ , \quad \bar{w} \triangleq w - w_c\otimes e \ , \quad P_e \triangleq e(e^*Q_Z^{-1}e)^{-1}e^*Q_Z^{-1} \end{cases}$$

Step 2:

Substitution of (5.10) into (5.3) gives

$$(5.12) \qquad f\,(\lambda,\theta) = \left|\left|\begin{bmatrix} z \\ w \end{bmatrix} - \begin{bmatrix} S\otimes I_n & I_2\otimes e \\ I_{2n} & 0 \end{bmatrix}\begin{bmatrix} w'(\lambda,\theta) \\ t(\lambda,\theta) \end{bmatrix}\right|\right|^2$$

$$= \left|\left|\begin{bmatrix} \sigma^2(\lambda^2+\sigma^2)^{-1}[\bar{z}-(S\otimes I_n)\bar{w}] \\ -(\lambda^2+\sigma^2)^{-1}S^*\otimes I_n[\bar{z}-(S\otimes I_n)\bar{w}] \end{bmatrix}\right|\right|^2$$

or

$$(5.13) \qquad f_1(\lambda,\theta) = \sigma^2(\lambda^2+\sigma^2)^{-1}||\bar{z}-(S\otimes I_n)\bar{w}||^2 \ ,$$

where

$$(5.14) \qquad ||.||^2 = (.)^* I_2\otimes Q_z^{-1}(.) \ .$$

In order to find $\hat{\lambda}$ and $\hat{\theta}$ we need to minimize (5.13). Using the reparametrization

$$(5.15) \qquad \lambda = \sigma\tan\phi \ ,$$

we can write (5.13) as

$$(5.16) \qquad f_1(\sigma\tan\phi,\theta) = \begin{bmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{bmatrix}^* \begin{bmatrix} a & 0 & b \\ 0 & a & c \\ b & c & d \end{bmatrix} \begin{bmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{bmatrix} \ ,$$

where

$$(5.17) \qquad \begin{cases} a = \sigma^2(\bar{u}^*Q_z^{-1}\bar{u}+\bar{v}^*Q_z^{-1}\bar{v}) \ , & d = \bar{x}^*Q_z^{-1}\bar{x}+\bar{y}^*Q_z^{-1}\bar{y} \\ b = -\sigma(\bar{u}^*Q_z^{-1}\bar{x}+\bar{v}^*Q_z^{-1}\bar{y}) \ , & c = -\sigma(\bar{v}^*Q_z^{-1}\bar{x}-\bar{u}^*Q_z^{-1}\bar{y}) \end{cases}$$

The minimization problem

$$(5.18) \qquad \min_{\phi,\theta}.\ f_1(\sigma\tan\phi,\theta) \ ,$$

again reduces to an eigenvalue problem:

$$\text{(5.19)} \qquad \begin{vmatrix} a-\mu & 0 & b \\ 0 & a-\mu & c \\ b & c & d-\mu \end{vmatrix} = 0$$

The three eigenvalues of (5.19) read:

$$\text{(5.20)} \qquad \begin{cases} \mu = a \\ \mu_{1,2} = \dfrac{(a+d)\pm[(a+d)^2+4(b^2+c^2-ad)]^{\frac{1}{2}}}{2} \end{cases}$$

Hence, the smallest eigenvalue reads:

$$\text{(5.21)} \qquad \mu_{min} = \frac{(a+d)-[(a+d)^2+4(b^2+c^2)]^{\frac{1}{2}}}{2}$$

From the three equations

$$\text{(5.22)} \qquad \begin{bmatrix} a-\mu_{min} & 0 & b \\ 0 & a-\mu_{min} & c \\ b & c & d-\mu_{min} \end{bmatrix} \begin{bmatrix} \sin\hat{\phi}\cos\hat{\theta} \\ \sin\hat{\phi}\cos\hat{\theta} \\ \cos\hat{\phi} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

we find that

$$\text{(5.23)} \qquad \begin{cases} \hat{\lambda}\cos\hat{\theta} = \sigma\tan\hat{\phi}\cos\hat{\theta} = \dfrac{-\sigma b}{a-\mu_{min}} \\ \hat{\lambda}\sin\hat{\theta} = \sigma\tan\hat{\phi}\sin\hat{\theta} = \dfrac{-\sigma c}{a-\mu_{min}} \end{cases}$$

From (5.10), (5.17), (5.21) and (5.23) follows therefore that the final least-squares solution of the <u>non-linear 2D Symmetric Helmert transformation</u> (5.1)-(5.2) is given by:

$$\hat{w}' = w_c \otimes e + (\sigma^2+\hat{\lambda}^2)^{-1}[(\hat{S}^* \otimes I_n)\bar{z}+\sigma^2\bar{w}] \ ,$$

$$\hat{t} = z_c - \hat{S}w_c \ , \quad \hat{S} = \hat{\lambda}\begin{bmatrix} \cos\hat{\theta} & \sin\hat{\theta} \\ -\sin\hat{\theta} & \cos\hat{\theta} \end{bmatrix} \ ,$$

$$\hat{\lambda}\cos\hat{\theta} = \frac{2\sigma^2<\bar{z},\bar{w}>}{\sigma^2||\bar{w}||^2-||\bar{z}||^2+[(\sigma^2||\bar{w}||^2-||\bar{z}||^2)^2+4\sigma^2(<\bar{z},\bar{w}>^2+<\bar{z},\bar{w}''>^2)]^{\frac{1}{2}}} \ ,$$

(5.24)

$$\hat{\lambda}\sin\hat{\theta} = \frac{2\sigma^2<\bar{z},\bar{w}''>}{\sigma^2||\bar{w}||^2-||\bar{z}||^2+[(\sigma^2||\bar{w}||^2-||\bar{z}||^2)^2+4\sigma^2(<\bar{z},\bar{w}>^2+<\bar{z},\bar{w}''>^2)]^{\frac{1}{2}}} \ ,$$

with the inner product

$$<.,.> = (.)^* I_2 \otimes Q_z^{-1}(.) \ ,$$

and $\bar{w}'' = \begin{pmatrix} \bar{v} \\ -\bar{u} \end{pmatrix}$.

Note that this solution reduces to that of (4.6) if $\sigma^2 \to \infty$. Also note that the smallest eigenvalue μ_{min} of (5.21) is not unique if

$$||\bar{z}||^2 = ||\bar{w}||^2 \ , \quad <\bar{z},\bar{w}> = 0 \quad \text{and} \quad <\bar{z},\bar{w}''> = 0$$

If this is the case the matrix of (5.16) reduces to a scaled unit matrix and the solution for $\hat{\lambda}$ and $\hat{\theta}$ becomes indeterminable. We shall disregard this exceptional case.

Furthermore if

$$<\bar{z},\bar{w}> = 0 \quad \text{and} \quad <\bar{z},\bar{w}''> = 0 \ ,$$

then $\mu_{min} = \sigma^2||\bar{w}||^2$ if $\sigma^2||\bar{w}||^2 < ||\bar{z}||^2$ and $\mu_{min} = ||\bar{z}||^2$ if $||\bar{z}||^2 < \sigma^2||\bar{w}||^2$. From (5.22) follows then that $\hat{\theta}$ is indeterminable and $\cos\hat{\phi} = 0$ or $\sin\hat{\phi} = 0$. We shall also disregard this exceptional case.

Let λ be a scale parameter, i.e. positive. From (5.22) follows then that

$$\hat{\lambda}/\sigma = \tan\phi = \frac{(b^2+c^2)^{\frac{1}{2}}}{a-\mu_{min}} = \frac{d-\mu_{min}}{(b^2+c^2)^{\frac{1}{2}}} \tag{5.25}$$

Let us denote the scale estimators of the Helmert transformation (4.4) by $\hat{\lambda}_H$, of the corresponding transformation when interchanging the role of z and w by $\hat{\lambda}'_H$, of the Symmetric Helmert transformation (5.1) by $\hat{\lambda}_{SH}$ and of the corresponding transformation when interchanging the role of z and w by $\hat{\lambda}'_{SH}$. From (4.6), (5.17) and (5.25) follows then when $\sigma = 1$ that:

$$(5.26) \quad \begin{cases} \hat{\lambda}_H = \dfrac{(b^2+c^2)^{\frac{1}{2}}}{a} & , \quad \hat{\lambda}'_H = \dfrac{(b^2+c^2)^{\frac{1}{2}}}{d} \\ \hat{\lambda}_{SH} = \dfrac{(b^2+c^2)^{\frac{1}{2}}}{a-\mu_{min}} = \dfrac{d-\mu_{min}}{(b^2+c^2)^{\frac{1}{2}}} & , \quad \hat{\lambda}'_{SH} = \dfrac{(b^2+c^2)^{\frac{1}{2}}}{d-\mu_{min}} = \dfrac{a-\mu_{min}}{(b^2+c^2)^{\frac{1}{2}}} \end{cases}$$

This shows that also for the 2D transformation:

$$(5.27) \quad \hat{\lambda}_H . \hat{\lambda}'_H \neq 1, \text{ but } \hat{\lambda}_{SH} . \hat{\lambda}'_{SH} = 1$$

From (5.26) also follows that:

$$(5.28) \quad \hat{\lambda}_{SH} = \hat{\lambda}_H . \frac{a}{a-\mu_{min}}$$

Hence, we see again that in general the classical Helmert transformation systematically underestimates the scale.
The two scale estimates are identical if

$$(5.29) \quad b^2+c^2-ad = 0$$

That is, when $\bar{z}$ is parallel to $\bar{w}$ or to $\bar{w}''$.

6. Concluding remarks

In this paper we discussed a paricular class of non-linear least-squares problems for which a useful two-step procedure can be devised. Exact least-squares solutions are given for the 1 and 2D Helmert transformation and their non-linear symmetrical generalizations. For the two dimensional case a rotational-invariant covariance structure was assumed. Solutions of the linearized versions and teststatistics were already given in (Teunissen, 1984). Our exact non-line-

ar least-squares solutions make the computation of approximate values, linearization and iteration superfluous.
Although we had to make some simplifying assumptions in the covariance structure of the observational variates, it is felt that these assumptions are sufficiently general for many practical applications. When digitizing maps, the covariance matrix of the digitized coordinates can often even be simplified to a scaled unit matrix. The assumption of the rotational invariant covariance structure is also in many cases sufficient for geodetic networks. For instance, the Baarda-Alberda substitute matrix (see e.g. Brouwer et al., 1982 or Teunissen, 1984a):

$$
\begin{array}{c|cccc}
 & x_i & x_j & y_i & y_j \\
\hline
x_i & d^2 & d^2-d_{ij}^2 & & \\
x_j & d^2-d_{ij}^2 & d^2 & \multicolumn{2}{c}{0} \\
y_i & & & d^2 & d^2-d_{ij}^2 \\
y_j & \multicolumn{2}{c}{0} & d^2-d_{ij}^2 & d^2
\end{array}
$$

d^2 : a parameter

d_{ij}^2: a covariance function

is an example of a rotational-invariant covariance matrix. It describes the precision of many geodetic networks to a sufficient degree and can therefore be used in our formulae.
In a forthcoming contribūtion we will derive some local and global distributional properties of our non-linear least-squares estimators. The approach will be based on the geometric theory of non-linear adjustment (Teunissen, 1984, 1985a,b).
For a discussion of the 3D Helmert transformation we refer to (Sansõ, 1973), (Köchle, 1982) and (Krarup, 1985) and for the 3D Helmert transformation with its symmetrical generalization to (Teunissen, 1985)

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APPENDIX

Proof of:

$$\text{(A1)}\qquad \partial_z P_{A(z)} = (I-P_{A(z)})\partial_z A(z)A^-(z) + [Q_y^{-1}(I-P_{A(z)})\partial_z A(z)A^-(z)Q_y]^*$$

The orthogonal projector $P_{A(z)}$ and least-squares inverse $A^-(z)$ are given by

$$\text{(A2)}\qquad \begin{cases} \text{a)} \quad P_{A(z)} = A(z)N^{-1}(z)A^*(z)Q_y^{-1} \\ \text{b)} \quad A^-(z) = N^{-1}(z)A^*(z)Q_y^{-1}\ , \end{cases}$$

where

$$\text{(A3)}\qquad N(z) = A^*(z)Q_y^{-1}A(z)$$

From

$$\partial_z(N^{-1}(z)N(z)) = \partial_z I = 0 = \partial_z N^{-1}(z)N(z) + N^{-1}(z)\partial_z N(z)$$

follows that

$$\text{(A4)}\qquad \partial_z N^{-1}(z) = -N^{-1}(z)\partial_z N(z)N^{-1}(z)$$

From

$$\partial_z N(z) = \partial_z(A^*(z)Q_y^{-1}A(z))$$

follows that

$$\text{(A5)}\qquad \partial_z N(z) = \partial_z A^*(z)Q_y^{-1}A(z) + A^*(z)Q_y^{-1}\partial_z A(z)$$

From (A2) follows that

$$\text{(A6)}\qquad \partial_z P_{A(z)} = \partial_z A(z)N^{-1}(z)A^*(z)Q_y^{-1} + A(z)\partial_z N^{-1}(z)A^*(z)Q_y^{-1} + A(z)N^{-1}(z)\partial_z A^*(z)Q_y^{-1}$$

With (A4) and (A5) this gives:

$$\partial_z P_{A(z)} = \partial_z A(z) N^{-1}(z) A^*(z) Q_y^{-1} + A(z)[-N^{-1}(z)(\partial_z A^*(z) Q_y^{-1} A(z)$$

$$+ A^*(z) Q_y^{-1} \partial_z A(z)) N^{-1}(z)] A^*(z) Q_y^{-1} + A(z) N^{-1}(z) \partial_z A^*(z) Q_y^{-1}$$

$$= [I - A(z) N^{-1}(z) A^*(z) Q_y^{-1}] \partial_z A(z) N^{-1}(z) A^*(z) Q_y^{-1}$$

$$+ A(z) N^{-1}(z) \partial_z A^*(z) [I - Q_y^{-1} A(z) N^{-1}(z) A^*(z)] Q_y^{-1}$$

or with (A2):

(A7) $$\partial_z P_{A(z)} = (I - P_{A(z)}) \partial_z A(z) A^-(z) + Q_y A^{-*}(z) \partial_z A^*(z) (I - P_{A(z)})^* Q_y^{-1} ,$$

which is identical to (A1).