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Adjusting and testing with the models of the affine and similarity transformation

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Abstract

In this paper inversion-free formulae are derived for the adjustment and testing of the models of the Affine and Similarity transformation. For the Affine transformation we assume the covariance matrices of the plane coordinates in the two coordinate systems to be block diagonal with equal 2x2 blocks. The coordinate covariance structure for the Similarity transformation is assumed to be rotational invariant and thus allows the use of substitute matrices of the type as developed by e.g. Baarda and Alberda.
It is felt that our results are general enough for many applications, such as the transformation of digitized maps and the connection of geodetic networks using rotational invariant substitute covariance matrices. As a very special case our formulae include the well-known results which hold true for the classical Helmert transformation.

I. Introduction

The aim of the present paper is to derive inversion-free formulae which are needed for the adjustment and testing of the models of the Affine and Similarity transformation. By "inversion-free" we mean that the normal equations are explicitly solved for. A well-known example of a linear adjustment problem where inversion-free formulae are derivable, is given by the classical Helmert transformation (see Helmert, 1893). Our results are, however, more general since we do not start from such stringent requirements on the properties of the coordinate covariance matrices used, as is the case with the Helmert transformation. For the Affine transformation we assume the coordinate covariance structure to be such that the resulting covariance matrices of the plane coordinates in the two coordinate systems are block diagonal with equal 2 x 2 blocks. Our formulae for the least-squares adjustment of the Affine transformation and hypothesis testing seem to be particularly applicable to the problem of transforming digitized maps.
For the Similarity transformation we assume a more complicated coordinate covariance structure, namely one that is rotational invariant. Our results therefore allow the use of criterion or substitute matrices of the type as developed by e.g. Baarda and Alberda.
In order to make the present paper selfcontained we start in the next section with a brief review of the theory of adjustment, testing and reliability. In this we take geometric stand point, thus allowing to derive the various results in a compact and intuitively easily understandable way. In section III we discuss the model of the Affine transformation. With the application of transforming digitized maps in mind, we have included here some results assuming that the points are distributed over a grid of rectangles. Section IV deals with the model of the Similarity transformation. The final section V contains some concluding remarks in which we also briefly touch upon the nature of the non-linearity present in the non-linear functional models of the Affine and Similarity transformation.

II. Some Adjustment theory

II.1. Linear Least-Squares Adjustment (LLSA)

Let us start by introducing the linear mathematical model M_1:

(1) $\quad M_1: \underset{m\times 1}{E\{y\}} = \underset{m\times n}{A}\,\underset{n\times 1}{x}, \quad \text{Cov.}\{y\} = \sigma^2 \underset{m\times m}{Q_y},$

where y is the m dimensional vector of observational variates; $E\{.\}$ stands for the mathematical expectation operator; A is the design matrix of order mxn; x is the n-dimensional vector of unknown parameters; $\text{Cov.}\{y\}$ is the covariance matrix of y and σ^2 is the variance factor of unit weight, which we assume to be known.
The LLS-solution of model M_1 reads

$$(2) \quad \begin{cases} \hat{y}_1 = A(A^tQ_y^{-1}A)^{-1}A^tQ_y^{-1}y \overset{\text{call}}{=} P_A y \\ \hat{x}_1 = (A^tQ_y^{-1}A)^{-1}A^tQ_y^{-1}y = A^- P_A y\ , \end{cases}$$

where A^- may be any arbitrary inverse of A. We use the lower index "1" in (2) to indicate that the solution refers to model M_1.
Note that since $P_A.P_A = P_A$ (idem potence) holds, P_A is a projector. In fact, with the metric of the observation space given by Q_y^{-1}, one can interpret P_A as an orthogonal projector, projecting onto the range space of A, i.e. R(A), and along the orthogonal complement of R(A), i.e. $R(A)^\perp$ (see figure 1).

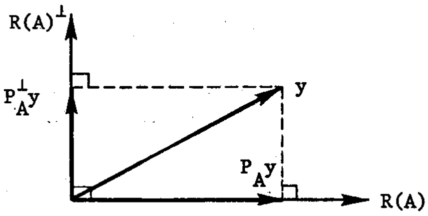

Figure 1

For the orthogonal projector projecting onto $R(A)^\perp$ we will use the notation $P_A^\perp$, i.e. $P_A^\perp = I - P_A$.
Let us now relax model M_1 by introducing more explanatory variables ∇_b. We will call this relaxed model, model M_2:

(3) $\quad M_2: \underset{m\times 1}{E\{y\}} = (\underset{m\times n}{A}\ \underset{m\times b}{C_b}) \underset{(n+b)\times 1}{\begin{pmatrix} x \\ \nabla_b \end{pmatrix}}, \quad \text{Cov.}\{y\} = \sigma^2 \underset{m\times m}{Q_y}.$

The LLS-solution for y of this model is

(4) $\quad \hat{y}_2 = P_{(A\vdots C_b)}y.$

Since the subspace $R(A\vdots C_b)$ is identical to the direct sum of the two mutually orthogonal subspaces $R(P_{C_b}^\perp A)$ and $R(C_b)$, we have $R(A\vdots C_b) = R(P_{C_b}^\perp A) \oplus R(C_b)$. That is, the two subspaces $R(P_{C_b}^\perp A)$ and $R(C_b)$ are each others orthogonal complement in $R(A\vdots C_b)$ (see figure 2).

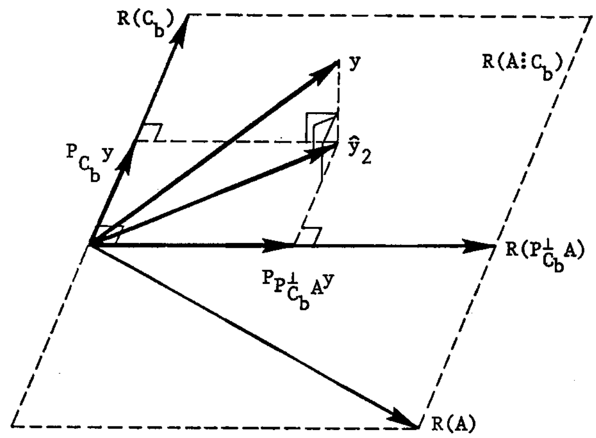

Figure 2

Instead of (4) we can therefore write

(5) $\quad \hat{y}_2 = (P_{P_{C_b}^\perp A} + P_{C_b})\, y.$

In order to find the LLS-solution for the parameters, note that we can write the consistent set of equations

$$\hat{y}_2 = Ax + C_b\nabla_b$$

as $\quad (P_{P_{C_b}^\perp A} + P_{C_b})y = P_{C_b}^\perp Ax + (P_{C_b}Ax + C_b\nabla_b),$

which due to the orthogonality can be split into

$$P_{P_{C_b}^\perp A}y = P_{C_b}^\perp Ax \quad \text{and} \quad P_{C_b}y = P_{C_b}Ax + C_b\nabla_b.$$

Hence, the LLS-solution of model M_2 can be written as

$$(6) \quad \begin{cases} \hat{y}_2 = (P_{P_{C_b}^\perp A} + P_{C_b})y\ , \\ \hat{x}_2 = (P_{C_b}^\perp A)^- P_{P_{C_b}^\perp A}y = (P_{C_b}^\perp A)^- y\ , \\ \hat{\nabla}_{b,2} = C_b^- P_{C_b}(y - A\hat{x}_2) = C_b^-(y - A\hat{x}_2)\ , \end{cases}$$

where $(.)^-$ denotes the *least-squares inverse* of the corresponding matrix. For a definition of the least-squares inverse and a further geometric elaboration of the linear inverse mapping problem, see (Teunissen, 1984a or 1985b). In the sequel all inverses will be least-squares inverses where the appropriate metric will be clear from the context.
Note that by switching the rôle of the two subspaces R(A) and $R(C_b)$ in the above orthogonalization procedure an alternative representation of the LLS-solution can be found.

II.2. Testing Hypotheses

The results of estimation depend on the assumptions underlying the applied mathematical model, e.g. M_1. Misspecification in, for instance, the covariance matrix $\sigma^2 Q_y$ or in the functional model $E\{y\} = Ax$ will invalidate the results of estimation and thus also any conclusion based on them. It is therefore of importance to have a way to verify the validity of the assumed working hypothesis M_1, which we call the null hypothesis.

In order to trace possible misspecifications in the mathematical model we can try to test the null hypothesis against one, or possibly more, alternative hypotheses. For this we need a decision rule that, based on the observation vector which we from now on shall assume to be a sample vector of a continuous m dimensional random vector having a multivariate normal distribution with mean $E\{y\}$ and covariance matrix $\sigma^2 Q_y$, decides whether to accept or reject the assumptions underlying the mathematical model. In principle it is possible to find a set-up which aims at finding misspecifications in all the assumed distributional properties of the random vector y. In the following we will restrict ourselves, however, to the case where only misspecifications in the mean $E\{y\}$ are considered.

We know that the null hypothesis M_1 is violated if $E\{y\} \notin R(A)$. It seems therefore reasonable to oppose the null hypothesis to a more relaxed alternative hypothesis, e.g. M_2. In order to have a way to discriminate between the possible validity of M_1 or M_2, we first recall the geometric interpretation of the method of least-squares (see figure 3).

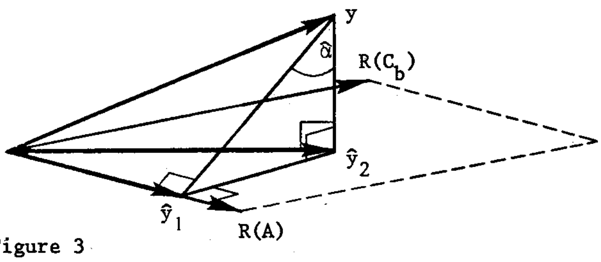

Figure 3

From this geometric interpretation follows that the properties of the triangle spanned by the two vectors $y - \hat{y}_1$ and $y - \hat{y}_2$ seem to be decisive for the acceptance or rejection of M_1 and M_2. For instance, it seems reasonable to consider M_1 valid if the observation point y is close enough to the subspace R(A). Hence, if $||.||^2 = (.)^t Q_y^{-1}(.)$ denotes the norm squared with respect to the metric Q_y^{-1}, one can consider M_1 valid if $||y-\hat{y}_1||^2$ is small enough. On the other hand a misspecification in M_1 as formulated by M_2 can considered to be present if $\hat{y}_2$ is too far apart from $\hat{y}_1$. Hence, one can consider M_1 invalidated by M_2 if $||\hat{y}_1-\hat{y}_2||^2$ is too large.

In order to obtain an objective and workable decision rule on the basis of which one can decide to accept or reject M_1, we should specify what is meant by "small enough" and "too large". This is possible in a statistical meaningful way once we know the distributional properties of the above mentioned two indicators.

It is well known (see e.g. Rao, 1973) that under M_1:

$$(7)\quad \begin{aligned} ||y||^2 &\sim \sigma^2\chi^2(m, \sigma^{-2}x^t A^t Q_y^{-1} Ax) \text{ and} \\ ||\hat{y}_1||^2 &\sim \sigma^2\chi^2(n, \sigma^{-2}x^t A^t Q_y^{-1} Ax). \end{aligned}$$

From this and the fact that $||y-\hat{y}_1||^2$ has a χ^2-distribution independent of $||\hat{y}_1||^2$ follows that under M_1:

$$(8)\quad ||y-\hat{y}_1||^2 = ||y||^2 - ||\hat{y}_1||^2 \sim \sigma^2\chi^2(m-n, o).$$

In a similar way we find that under M_1:

$$(9)\quad ||y-\hat{y}_2||^2 = ||y||^2 - ||\hat{y}_2||^2 \sim \sigma^2\chi^2(m-n-b, o).$$

And since $||\hat{y}_1-\hat{y}_2||^2$ has a χ^2-distribution independent of $||y-\hat{y}_2||^2$ it also follows with (8) and (9) that under M_1:

$$(10)\quad ||\hat{y}_1-\hat{y}_2||^2 = ||y-\hat{y}_1||^2 - ||y-\hat{y}_2||^2 \sim \sigma^2\chi^2(b, o).$$

Note that if $R(C_b)$ is chosen to be complementary to R(A), b equals m-n, $\hat{y}_2$ reduces to y and the distance of (10) reduces to that of (8). Hence, the decision rule which makes use of $||y-\hat{y}_1||^2$ can be considered an (m-n) dimensional *overall model test*. Also note that instead of (10), one can make use of $\tan^2\hat{\alpha}$, $\sin^2\hat{\alpha}$ or $\cos^2\hat{\alpha}$ (see figure 3). However, since we assume the variance factor of unit weight σ^2 to be known a priori, we will restrict ourselves to the simpler test statistics of (8) and (10).

With (8) the decision rule to reject the null hypothesis M_1 becomes:

$$(11)\quad \boxed{M_1 \text{ invalid if } T_{m-n} = \frac{||y-\hat{y}_1||^2}{\sigma^2(m-n)} = \frac{||P_A^{\perp}y||^2}{\sigma^2(m-n)} > \frac{\chi^2(m-n, o)_{\alpha_{m-n}}}{m-n},}$$

where α_{m-n} is a chosen level of significance of the (m-n) dimensional overall model test. If this test comes to reject M_1 one can try to trace the misspecification in M_1 by testing M_1 against an alternative hypothesis, say M_2. For this decision rule we use the distance squared of (10). The decision rule reads then:

$$(12)\quad \boxed{\begin{array}{l} M_1 \text{ invalidated by } M_2 \text{ if } T_b = \\ \dfrac{\|\hat{y}_1-\hat{y}_2\|^2}{\sigma^2 b} = \dfrac{\|P_{P_A^\perp C_b} y\|^2}{\sigma^2 b} > \dfrac{\chi^2(b,o)_{\alpha_b}}{b}. \end{array}}$$

In many practical applications one dimensional tests (b=1) are used, e.g. for single blunder detection. Since $R(C_b)$ is then a one dimensional subspace, i.e. $R(C_1)$, the test statistic T_b of (12) reduces to a simple form. To see this, note that $\hat{y}_1-\hat{y}_2$ lies in the subspace $R(A\vdots C_1)$ and is orthogonal to $R(A)$. Hence,

$$(13)\quad \hat{y}_1-\hat{y}_2 = \alpha P_A^\perp C_1 \text{ for some } \alpha \in R.$$

Furthermore, since $y-\hat{y}_2$ is orthogonal to the subspace $R(A\vdots C_1)$ in which $P_A^\perp C_1$ lies, and $P_A^\perp C_1$ is orthogonal to the subspace $R(A)$ in which $\hat{y}_1$ lies, we also have

$$(14)\quad y-\hat{y}_2 \perp P_A^\perp C_1 \quad \text{and} \quad P_A^\perp C_1 \perp \hat{y}_1.$$

Hence, it follows from (13), (14) and $\hat{y}_1 - \hat{y}_2 = (-y) + (\hat{y}_1) + (y-\hat{y}_2)$ that

$$(15)\quad \begin{array}{l} \|\hat{y}_1-\hat{y}_2\|^2 = \dfrac{\langle P_A^\perp C_1, \hat{y}_1-\hat{y}_2\rangle^2}{\|P_A^\perp C_1\|^2} = \\ \dfrac{\langle P_A^\perp C_1, y\rangle^2}{\|P_A^\perp C_1\|^2} = \dfrac{\langle C_1, P_A^\perp y\rangle^2}{\|P_A^\perp C_1\|^2}, \end{array}$$

where $\langle .,.\rangle = (.)^t Q_y^{-1}(.)$ denotes the innerproduct with respect to the metric Q_y^{-1}. From (15) and (12) follows then that in the one dimensional case

$$(16)\quad \boxed{\begin{array}{l} M_1 \text{ invalidated by } M_2 \text{ (b=1) if} \\ |t_1| = \left|\dfrac{\langle C_1, P_A^\perp y\rangle}{\|P_A^\perp C_1\|}\right| > \sqrt{\chi^2(1,o)_{\alpha_1}}. \end{array}}$$

II.3. Reliability

When one considers the problem of testing the null hypothesis against one or more alternative hypotheses, it is of interest to know how well possible misspecifications in M_1 as specified by M_2 are checked by the applied tests. This concerns the concept of reliability. Let us first consider the one dimensional tests. We know that under M_1 the quadratic form $\|P_{P_A^\perp C_1} y\|^2$ has a central χ^2-distribution with b=1 degrees of freedom, i.e. $\sigma^2\chi^2(1,o)$. Under M_2(b=1) however, this quadratic form has a non-central χ^2-distribution, namely $\sigma^2\chi^2(1,\lambda_1)$, with the non-centrality parameter

$$(17)\quad \lambda_1 = \sigma^{-2}\|P_{P_A^\perp C_1} C_1\nabla_1\|^2.$$

By fixing the level of significance α_1 and power β_1 of the one dimensional test, we can compute the non-centrality parameter λ_1. We write this symbolically as

$$(18)\quad \lambda_1 = \lambda(\alpha_1,\beta_1,1).$$

From (17) and (18) follows then that we can compute the absolute value of the scalar ∇_1 from

$$\lambda(\alpha_1,\beta_1,1) = \sigma^{-2}\|P_{P_A^\perp C_1} C_b\nabla_b\|^2$$

as

$$(19)\quad \boxed{|\nabla_1| = \sigma\sqrt{\frac{\lambda(\alpha_1,\beta_1,1)}{\|P_A^\perp C_1\|^2}}.}$$

Thus a misspecification of $|\nabla_1|$ in M_1 in the direction of vector C_1 will dislocate the quadratic form $\sigma^{-2}\|P_{P_A^\perp C_1} y\|^2$ by the amount of λ_1. And the power associated with this dislocation is β_1. Hence, small values of $|\nabla_1|$ are more favourable than large values. In order to make a connection between the one dimensional test (16) and the overall model test (11) possible, Baarda proposed (see e.g. Baarda, 1968; Brouwer et al. 1982 or Teunissen, 1984b) to choose the non-centrality parameter λ_{m-n} and power β_{m-n} of the overall model test equal to λ_1 and β_1 respectively, i.e.

$$(20)\quad \begin{array}{l} \lambda_{m-n} = \lambda_1,\ \beta_{m-n} = \beta_1 \text{ where} \\ \lambda_{m-n} = \lambda(\alpha_{m-n},\beta_{m-n},m-n),\ \lambda_1 = \lambda(\alpha_1,\beta_1,1). \end{array}$$

From $\lambda_1 = \lambda(\alpha_{m-n},\beta_1,m-n)$ one can then compute implicitly the appropriate level of significance α_{m-n} for the overall model test.

Baarda's procedure of comparing the statistical parameters α, β and λ has the favourable consequence that the same value (19) also follows for the overall model test if M_1 is misspecified in the direction C_1. This follows from the fact that $||P_A^{\perp}C_1\nabla_1||^2 = ||P_{P_A^{\perp}C_1}C_1\nabla_1||^2$. As to the b dimensional tests, we can now start from the known values λ_{m-n} and β_{m-n}. By setting

$$(21)\quad \begin{aligned} &\lambda_b = \lambda_{m-n},\ \beta_b = \beta_{m-n} \quad \text{where} \\ &\lambda_b = \lambda(\alpha_b,\beta_b,b),\ \lambda_{m-n} = \lambda(\alpha_{m-n},\ \beta_{m-n},m-n), \end{aligned}$$

we can compute the appropriate level of significance α_b for the b dimensional test. And analogously to (19) we can compute from

$$(22)\quad ||P_{P_A^{\perp}C_b}C_b\nabla_b||^2 = (P_A^{\perp}C_b\nabla_b)^t Q_y^{-1}(P_A^{\perp}C_b\nabla_b) = \sigma^2\lambda_b.$$

the ellipsoidal region on which misspecifications of M_1 must lie in order to dislocate the quadratic form $\sigma^{-2}||P_{P_A^{\perp}C_b}y||^2$ by the amount of λ_b. We can bring (22) into a form which more closely resembles (19), if we make use of the eigenvalues μ_k and normalized eigenvectors d_k, $k = 1,\ldots,b$ of the matrix $(P_A^{\perp}C_b)^t Q_y^{-1}(P_A^{\perp}C_b)$:

$$(22')\quad \boxed{(\nabla_b)_k = \sigma d_k\sqrt{\frac{\lambda(\alpha_b,\beta_b,b)}{\mu_k}},\quad k = 1,\ldots,b.}$$

As with the one dimensional test also here the choice of the statistical parameters has been such that the same ellipsoidal region as described by (22) or (22') follows for the overall model test if M_1 is misspecified by the subspace $R(C_b)$. This follows from the fact that $||P_A^{\perp}C_b\nabla_b||^2 = ||P_{P_A^{\perp}C_b}C_b\nabla_b||^2$. In fact the choice of the statistical parameters is such that even the same value (19) follows for the b dimensional test if M_1 is misspecified in the direction $C_1 \in R(C_b)$. This is a consequence of the fact that $||P_{P_A^{\perp}C_b}C_1\nabla_1||^2 = ||P_{P_A^{\perp}C_1}C_1\nabla_1||^2$ if $C_1 \in R(C_b)$.

Now that we have discussed most of the important aspects of adjustment and testing we can start to apply the above derived results to the models of the Affine and Similarity transformation.

III. The model of the Affine transformation

III.1. Adjustment

The functional model of the 2-dimensional Affine

$$(23)\quad E\{\begin{pmatrix}x_i\\y_i\end{pmatrix}\} = E\{u_i\}\begin{pmatrix}a\\c\end{pmatrix} + E\{v_i\}\begin{pmatrix}b\\d\end{pmatrix} + \begin{pmatrix}t_x\\t_y\end{pmatrix},\ i=1,\ldots,n,$$

where (x_i,y_i) and (u_i,v_i) are the observed cartesian coordinates of point i in respectively the first and second coordinate system; n = number of points involved and a, b, c, d, t_x and t_y are the six transformation parameters of the Affine transformation. We can write (23) in a more convenient form by making use of the Kronecker product $\otimes$, for which the following four properties hold (see e.g. Rao, 1973):

$$(24)\quad \begin{cases} (A\otimes B)^t = A^t\otimes B^t;\quad (A\otimes B)^- = A^-\otimes B^- \text{ using any inverse} \\ A_1A_2\otimes B_1B_2 = (A_1\otimes B_1)(A_2\otimes B_2); \\ (A+B)\otimes C = A\otimes C + B\otimes C. \end{cases}$$

Take therefore the definitions

$$(25)\quad \begin{cases} x=(..x_i..)^t;\ y=(..y_i..)^t;\ z=(x^ty^t)^t; \\ u=(..u_i..)^t;\ v=(..v_i..)^t;\ w=(u^tv^t)^t;\ e_n=(1,\ldots,1)^t \\ A=(uve_n);\ B=\begin{pmatrix}a & b\\ c & d\end{pmatrix};\ T=(abt_xcdt_y)^t;\ t=(t_xt_y)^t, \end{cases}$$

and write (23) as

$$(23')\quad E\{z\} = (I_2\otimes E\{A\})T = (B\otimes I_n)E\{w\} + (I_2\otimes e_n)t.$$

We assume that the covariance matrices of the observed coordinates are given by

$$(26)\quad \text{Cov.}\{z\} = Q_z\otimes I_n,\ \text{Cov.}\{w\} = Q_w\otimes I_n,$$

and that there is no correlation between coordinates in different coordinate systems. Since model (23') is non-linear we shall need to linearize it in order to apply linear estimation techniques. Linearization of (23') gives then:

$$(27)\quad M_1: \begin{cases} \text{a)}\quad E\{\Delta r\} \overset{\text{call}}{=} E\{\Delta z-(B^o\otimes I_n)\Delta w\} = (I_2\otimes A^o)\Delta T \\ \text{b)}\quad \text{Cov.}\{\Delta r\} = (Q_z+B^oQ_wB^{ot})\otimes I_n, \end{cases}$$

where we have made use of the properties in (24) of the Kronecker product. In what follows we shall make a frequent use of these four properties. The upper index "o" in (27) indicates that approximate values are taken for the entries of the corresponding matrices.

By partitioning the matrix A^o as $A^o = (u^ov^o:e_n)$ and using the orthogonalization technique of section II we find from (27) that

(28) $(\Delta\hat{a}\ \Delta\hat{b}\ \Delta\hat{c}\ \Delta\hat{d})^t = (P^{\perp}_{I_2\otimes e_n} I_2\otimes(u^o\ v^o))^- \Delta r.$

With $P^{\perp}_{I_2\otimes e_n} = I_2\otimes\tilde{P}^{\perp}_{e_n}$, where the tilde sign indicates that the corresponding projector is defined with respect to the ordinary cartesian metric, and the following definition of *centered coordinates*:

(29) $\bar{u}^o = \tilde{P}^{\perp}_{e_n} u^o,\ \bar{v}^o = \tilde{P}^{\perp}_{e_n} v^o,$

equation (28) becomes

$$(\Delta\hat{a}\ \Delta\hat{b}\ \Delta\hat{c}\ \Delta\hat{d})^t = (I_2\otimes(\bar{u}^o\ \bar{v}^o))^-\Delta r =$$

$$= (I_2\otimes \begin{pmatrix} \bar{u}^{ot}\bar{u}^o & \bar{u}^{ot}\bar{v}^o \\ \bar{u}^{ot}\bar{v}^o & \bar{v}^{ot}\bar{v}^o \end{pmatrix}^{-1} \begin{pmatrix} \bar{u}^{ot} \\ \bar{v}^{ot} \end{pmatrix})\Delta r.$$

Hence, with $\Delta r = (\Delta p^t\ \Delta q^t)^t$ we have

$$(30)\quad \boxed{\begin{aligned} \Delta\hat{a} &= \frac{\bar{v}^{ot}(\bar{v}^o\bar{u}^{ot}-\bar{u}^o\bar{v}^{ot})}{\bar{v}^{ot}(\bar{v}^o\bar{u}^{ot}-\bar{u}^o\bar{v}^{ot})\bar{u}^o}\,\Delta p \\ \Delta\hat{b} &= \frac{\bar{u}^{ot}(\bar{v}^o\bar{u}^{ot}-\bar{u}^o\bar{v}^{ot})}{\bar{u}^{ot}(\bar{v}^o\bar{u}^{ot}-\bar{u}^o\bar{v}^{ot})\bar{v}^o}\,\Delta p \end{aligned}}$$

The solution for $\Delta\hat{c}$ and $\Delta\hat{d}$ follows from (30) by replacing p by q, a by c and b by d. Note the remarkable result that these least-squares estimates are *independent* of the chosen coordinate covariance matrices in (26).

The LLS-solution for the translational increments follows from (27) as

$$\Delta\hat{t} = (I_2\otimes e_n)^-[\Delta r-(I_2\otimes(u^o v^o))(\Delta\hat{a}\ \Delta\hat{b}\ \Delta\hat{c}\ \Delta\hat{d})^t]$$

$$= (I_2\otimes(e_n^t e_n)^{-1}e_n^t)[\Delta r-(I_2\otimes(u^o v^o))(\Delta\hat{a}\ \Delta\hat{b}\ \Delta\hat{c}\ \Delta\hat{d})^t].$$

Hence,

$$(30')\quad \boxed{\Delta\hat{t}_x = \Delta p_c - u^o_c\Delta\hat{a} - v^o_c\Delta\hat{b}}$$

The solution for $\Delta\hat{t}_y$ follows from (30') by replacing p by q, a by c and b by d. The subindex "c" in (30') indicates the centre of the point set, i.e. $\Delta x_c = (e_n^t e_n)^{-1}e_n^t\Delta x$.

In order to derive the LLS-solution of the coordinate increments in the two coordinate systems, we apply the following well-known formula (see e.g. Teunissen, 1985b):

(31) $\hat{\Delta w} = \Delta w + \mathrm{Cov.}\{\Delta w,\Delta r\}\ \mathrm{Cov.}\{\Delta r\}^{-1}(\hat{\Delta r}-\Delta r).$

With

$$\mathrm{Cov.}\{\Delta w,\Delta r\} = -Q_w B^{ot}\otimes I_n,\quad \hat{\Delta r} = P_{I_2\otimes A^o}\Delta r,$$

and (27.b) it follows then that

(32) $\hat{\Delta w} = \Delta w + (Q_w B^{ot}\otimes I_n)[(Q_z+B^oQ_wB^{ot})\otimes I_n]^{-1}\ P^{\perp}_{I_2\otimes A^o}\Delta r.$

Since $P^{\perp}_{I_2\otimes A^o} = I_2\otimes\tilde{P}^{\perp}_{A^o}$ we can write (32) finally as

$$(33)\quad \boxed{\Delta\hat{w} = \Delta w+(I_2\otimes\tilde{P}^{\perp}_{A^o})[Q_wB^{ot}(Q_z+B^oQ_wB^{ot})^{-1}\otimes I_n]\Delta r}$$

In a similar way one can derive that

$$(33')\quad \boxed{\Delta\hat{z} = \Delta z-(I_2\otimes\tilde{P}^{\perp}_{A^o})[Q_z(Q_z+B^oQ_wB^{ot})^{-1}\otimes I_n]\Delta r\quad .}$$

The computation of the estimates in (33) and (33') becomes straightforward once we have an explicit expression for the projector $\tilde{P}^{\perp}_{A^o}$. The inversion of the 2x2 matrix $(Q_z+B^oQ_wB^{ot})$ is namely a matter of routine. To find an explicit expression for $\tilde{P}^{\perp}_{A^o}$, note that

$$\tilde{P}^{\perp}_{A^o} = I_n - \tilde{P}_{A^o} = I_n - \tilde{P}_{e_n} - \tilde{P}_{(\bar{u}^o\bar{v}^o)}.$$

Hence, with

$$\tilde{P}_{(\bar{u}^o\bar{v}^o)} = \frac{(\bar{v}^o\bar{u}^{ot}-\bar{u}^o\bar{v}^{ot})(\bar{v}^o\bar{u}^{ot}-\bar{u}^o\bar{v}^{ot})^t}{\bar{v}^{ot}(\bar{v}^o\bar{u}^{ot}-\bar{u}^o\bar{v}^{ot})\bar{u}^o},$$

we find that

$$(34)\quad \boxed{\tilde{P}^{\perp}_{A^o} = I_n - \frac{e_ne_n^t}{n} - \frac{(\bar{v}^o\bar{u}^{ot}-\bar{u}^o\bar{v}^{ot})(\bar{v}^o\bar{u}^{ot}-\bar{u}^o\bar{v}^{ot})^t}{\bar{v}^{ot}(\bar{v}^o\bar{u}^{ot}-\bar{u}^o\bar{v}^{ot})\bar{u}^o}}$$

III.2. Testing and Reliability

In this section we will derive the overall model, the b dimensional- and the one dimensional test with the corresponding realibility measures, in terms of the easily computable projector $\tilde{P}^{\perp}_{A^o}$.

For the overall model test (11) we need to compute in the notation of section II:

$$||P^{\perp}_A y||^2 = y^tQ_y^{-1}A(A^tQ_y^{-1}A)^{-1}A^tQ_y^{-1}y\quad .$$

With (27) the corresponding form for the Affine transformation reads then

$$||P^{\perp}_A y||^2 := (\Delta r-\Delta\hat{r})^t[(Q_z+B^oQ_wB^{ot})\otimes I_n]^{-1}(\Delta r-\Delta\hat{r})^t.$$

Hence, with $\Delta r-\Delta\hat{r} = (I_2\otimes\tilde{P}^{\perp}_{A^o})\Delta r$ the overall model test becomes

$$(35)\quad \boxed{\begin{array}{l} M_1 \text{ invalid if} \\ \dfrac{\Delta r^t((Q_z+B^oQ_wB^{ot})^{-1} \otimes \tilde{P}^{\perp}_{Ao})\Delta r}{2n-6} > \dfrac{\chi^2(2n-6,0)\alpha_{2n-6}}{2n-6} \end{array}}$$

For the b dimensional test (12) we need to compute in the notation of section II:

$$(36)\quad \begin{array}{l} \|P_{P^{\perp}_A C_b} y\|^2 = \\ ((P^{\perp}_A C_b)^t Q_y^{-1} y)^t [(P^{\perp}_A C_b)^t Q_y^{-1} (P^{\perp}_A C_b)]^{-1} (P^{\perp}_A C_b)^t Q_y^{-1} y \,. \end{array}$$

Now let us assume that one point, say point i, has been misspecified in the first coordinate system. To test this assumption we have to take

$$(37)\quad C_b^t := I_2 \otimes c_i^t, \quad \text{with } c_i^t = (0 \ldots 1\; 0 \ldots)^t \,. \\ \qquad\qquad\qquad\qquad\qquad 1 \ldots i^{th} \ldots n$$

With this choice we have

$$(38)\quad \begin{cases} (P^{\perp}_A C_b)^t Q_y^{-1} y := [(Q_z+B^oQ_wB^{ot})^{-1} \otimes c_i^t \tilde{P}^{\perp}_{Ao}]\Delta r \\ [(P^{\perp}_A C_b)^t Q_y^{-1} (P^{\perp}_A C_b)]^{-1} := (c_i^t \tilde{P}^{\perp}_{Ao} c_i)^{-1} (Q_z+B^oQ_wB^{ot}). \end{cases}$$

From (12), (36) and (38) follows then that

$$(39)\quad \boxed{\begin{array}{l} M_1 \text{ is invalidated by a misspecification in point} \\ i \text{ of the first coordinate system if} \\ \dfrac{\Delta r^t (I_2 \otimes c_i^t \tilde{P}^{\perp}_{Ao})^t (Q_z+B^oQ_wB^{ot})^{-1} (I_2 \otimes c_i^t \tilde{P}^{\perp}_{Ao})\Delta r}{2\; c_i^t \tilde{P}^{\perp}_{Ao}\; c_i} > \\ \dfrac{\chi^2(2,0)\alpha_2}{2} \end{array}}$$

In a similar way one can find the test which tests whether M_1 is invalidated by a misspecification in point i of the second coordinate system. In this case one should take $C_b^t := B^o \otimes c_i^t$. It will be clear, however, that then exactly the same result (39) follows. The with test (39) corresponding reliability ellipsoid follows readily form (22) and (38) as

$$(40)\quad \nabla_2^t\, (Q_z+B^oQ_wB^{ot})^{-1} \nabla_2 = \frac{\lambda_2}{c_i^t \tilde{P}^{\perp}_{Ao} c_i} \,.$$

An alternative way to represent this ellipsoid is

$$(40')\quad \boxed{\begin{array}{l} (\nabla_2)_k = d_k \sqrt{\dfrac{\lambda_2}{d_k^t (Q_z+B^oQ_wB^{ot})^{-1} d_k c_i^t \tilde{P}^{\perp}_{Ao} c_i}} \,, \\ \text{with } d_k^t = (\cos\alpha_k \;\; \sin\alpha_k) \end{array}}$$

For the one dimensional test (16) we need to compute in the notation of section II:

$$(41)\quad \frac{\langle c_1, P^{\perp}_A y\rangle}{\|P^{\perp}_A c_1\|} = \frac{(P^{\perp}_A c_1)^t Q_y^{-1} y}{(P^{\perp}_A c_1)^t Q_y^{-1} (P^{\perp}_A c_1)} \,.$$

Now let us assume that one point, say point i, has been misspecified in the first coordinate system in a *particular direction*. That is, we choose

$$(42)\quad c_1^t := d^t \otimes c_i^t, \quad \text{with } d^t = (\cos\alpha \;\; \sin\alpha)$$

We then find that

$$(43)\quad \begin{cases} (P^{\perp}_A c_1)^t Q_y^{-1} y := [d^t (Q_z+B^oQ_wB^{ot})^{-1} \otimes c_i^t \tilde{P}^{\perp}_{Ao}]\Delta r \\ [(P^{\perp}_A c_1)^t Q_y^{-1} (P^{\perp}_A c_1)] := d^t (Q_z+B^oQ_wB^{ot})^{-1} d \,.\, c_i^t \tilde{P}^{\perp}_{Ao} c_i \end{cases}$$

Hence, with (16) and (41) the one dimensional test becomes

$$(44)\quad \boxed{\begin{array}{l} M_1 \text{ is invalidated by a misspecification in} \\ \text{point } i \text{ of the first coordinate system in} \\ \text{the direction } d^t = (\cos\alpha \;\; \sin\alpha) \\ \text{if } \left| \dfrac{(d^t (Q_z+B^oQ_wB^{ot})^{-1} \otimes c_i^t \tilde{P}^{\perp}_{Ao})\Delta r}{\sqrt{d^t (Q_z+B^oQ_wB^{ot})^{-1} d\;\, c_i^t \tilde{P}^{\perp}_{Ao} c_i}} \right| > \sqrt{\chi^2(1,o)\alpha_1} \end{array}}$$

The with this test corresponding reliability measure reads

$$(45)\quad \boxed{|\nabla_1| = \sqrt{\dfrac{\lambda_1}{d^t (Q_z+B^oQ_wB^{ot})^{-1} d\; c_i^t \tilde{P}^{\perp}_{Ao} c_i}}}$$

Compare this with (40'). The scalar $c_i^t \tilde{P}^{\perp}_{Ao} c_i$ occuring in (40'), (44) and (45) is readily computed from (34) as

$$(46)\quad \boxed{c_i^t \tilde{P}^{\perp}_{Ao} c_i = 1 - \frac{1}{n} - \frac{(\bar{u}_i^o)^2 \bar{v}^{ot} \bar{v}^o - 2\bar{u}_i^o \bar{v}_i^o \bar{u}^{ot} \bar{v}^o + (\bar{v}_i^o)^2 \bar{u}^{ot} \bar{u}^o}{\bar{v}^{ot} (\bar{v}^o \bar{u}^{ot} - \bar{u}^o \bar{v}^{ot}) \bar{u}^o}}$$

III.3. A special case

In some applications the distribution of points is such that the results we obtained in the preceding sections can still be simplified a bit further. For instance, when digitizing maps the grid of squares or rectangles on the map is used to calibrate the digitizer and/or to check certain affine properties of the map involved (see e.g. Bisselink, 1975). Let us therefore assume that the points in the second coordinate system are distributed over a grid of rectangles with sides s_u and s_v as shown in figure 4.

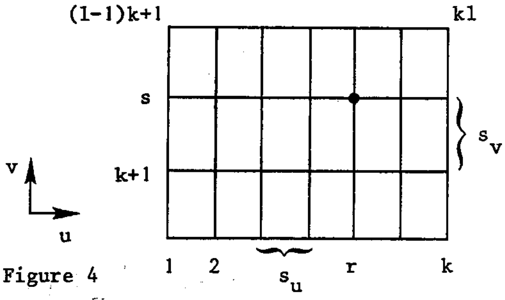

Figure 4

We assume to have $n=kl$ number of points and that the numbering is as shown in figure 4. Since the orthogonal projector $\tilde{P}^{\perp}_{A^o}$ of (34) occurs in all important formulae of the preceding sections, let us investigate how the expression for this projector simplifies when assuming the "grid of rectangles"-distribution.

From figure 4 it is easily verified that

$$(47)\quad \begin{cases} u=e_l\otimes u';\; u'_i=u_1+(i-1)s_u;\; i=1,\ldots k;\; e_l^t=(1,\ldots,1) \\ v=v'\otimes e_k;\; v'_i=v_1+(i-1)s_v;\; i=1,\ldots l;\; e_k^t=(1,\ldots,1) \end{cases}$$

Hence,

$$(48)\quad \begin{cases} \bar{u}=\tilde{P}^{\perp}_{e_n}u=e_l\otimes\tilde{P}^{\perp}_{e_k}u'=e_l\otimes\bar{u}';\; \bar{v}=\tilde{P}^{\perp}_{e_n}v=\tilde{P}^{\perp}_{e_l}\bar{v}'\otimes e_k=\bar{v}'\otimes e_k; \\ \bar{u}\bar{u}^t=e_le_l^t\otimes\bar{u}'\bar{u}'^t;\; \bar{v}\bar{v}^t=\bar{v}'\bar{v}'^t\otimes e_ke_k^t \end{cases}$$

From this follows then that

$$(49)\quad \begin{aligned} &\bar{u}^t\bar{u}=l\bar{u}'^t\bar{u}'=\frac{1}{12}s_u^2kl(k^2-1);\; \bar{v}^t\bar{v}=k\bar{v}'^t\bar{v}'=\frac{1}{12}s_v^2kl(l^2-1); \\ &\bar{u}^t\bar{v}=0, \end{aligned}$$

where we have made use of the fact that

$$(50)\quad \sum_{i=1}^{n} i=\frac{1}{2}n(n+1) \quad\text{and}\quad \sum_{i=1}^{n} i^2=\frac{1}{6}n(n+1)(2n+1).$$

With (48), (49) and (50) we can now simplify the expression for $\tilde{P}^{\perp}_{A^o}$ to

$$\boxed{\begin{aligned} &\tilde{P}^{\perp}_{A^o}=I_l\otimes I_k-\tilde{P}_{e_l}\otimes\tilde{P}_{e_k}-\tilde{P}_{e_l}\otimes\tilde{P}_{\bar{u}^{o\prime}}-\tilde{P}_{\bar{v}^{o\prime}}\otimes\tilde{P}_{e_k}, \\ &\text{where:} \\ &\tilde{P}_{\bar{u}^{o\prime}}=\frac{12\,\bar{u}^{o\prime}\bar{u}^{o\prime t}}{s_u^2k(k^2-1)},\quad \bar{u}^{o\prime}=\tilde{P}^{\perp}_{e_k}u^{o\prime}=(\ldots,[s_u(i-\tfrac{1}{2}(k+1))],\ldots)^t \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 1\ldots\ldots\ldots i\ldots\ldots\ldots k \\ &\tilde{P}_{\bar{v}^{o\prime}}=\frac{12\,\bar{v}^{o\prime}\bar{v}^{o\prime t}}{s_v^2l(l^2-1)},\quad \bar{v}^{o\prime}=\tilde{P}^{\perp}_{e_l}v^{o\prime}=(\ldots,[s_v(i-\tfrac{1}{2}(l+1))],\ldots)^t \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 1\ldots\ldots\ldots i\ldots\ldots\ldots l \end{aligned}}$$

(51)

Note that the projector $\tilde{P}^{\perp}_{A^o}$ is independent of the chosen side lengths s_u and s_v. In order to compute the scalar $c_i^t\tilde{P}^{\perp}_{A^o}c_i$ of (46), we assume that point i has been identified as the point (r,s) in figure 4. Then

$$(52)\quad c_i=c_s\otimes c_r,\quad \text{with } i=(s-1)p+r.$$

With (51) this gives

$$\begin{aligned} &c_i^t\tilde{P}^{\perp}_{A^o}c_i= \\ &c_s^t\otimes c_r^t(I_l\otimes I_k-\tilde{P}_{e_l}\otimes\tilde{P}_{e_k}-\tilde{P}_{e_l}\otimes\tilde{P}_{\bar{u}^{o\prime}}-\tilde{P}_{\bar{v}^{o\prime}}\otimes\tilde{P}_{e_k})c_s\otimes c_r= \\ &1-\frac{1}{kl}-\frac{1}{k}(c_s^t\tilde{P}_{\bar{v}^{o\prime}}c_s)-\frac{1}{l}(c_r^t\tilde{P}_{\bar{u}^{o\prime}}c_r), \end{aligned}$$

or

$$(53)\quad \boxed{c_i^t\tilde{P}^{\perp}_{A^o}c_i=1-\frac{1}{kl}\left(1+12\left[\frac{(s-\frac{1}{2}(l+1))^2}{(l^2-1)}+\frac{(r-\frac{1}{2}(k+1))^2}{(k^2-1)}\right]\right)}$$

IV. The model of the Similarity transformation

IV.1. Adjustment

The non-linear functional model of the 2-dimensional Similarity transformation reads

$$(54)\quad E\{z\}=E\{S\}T=(B\otimes I_n)E\{w\}+(I_2\otimes e_n)t.$$

$$\text{where } S=\begin{pmatrix} u & v & e_n & o \\ v & -u & o & e_n \end{pmatrix},\quad T=(a\;\, b\;\, t_x\;\, t_y)^t$$

$$\text{and}\quad B=\begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

We assume the coordinate covariance matrices to be *rotational invariant*, i.e.

$$(55)\quad \text{Cov.}\{z\}=I_2\otimes Q_z \quad\text{and}\quad \text{Cov.}\{w\}=I_2\otimes Q_w.$$

Note that the Q-matrices of the preceding section III are of order 2, whereas in this section they are of order n.

Linearization of (54) gives with (55)

$$(56)\quad \begin{cases} \qquad\text{a) } E\{\Delta r\}\overset{\text{call}}{=}E\{\Delta z-(B^o\otimes I_n)\Delta w\}=S^o\Delta T \\ M_1:\text{ b) } \text{Cov.}\{\Delta r\}=I_2\otimes Q_r=I_2\otimes(Q_z+\lambda^{o2}Q_w), \\ \qquad\text{with } \lambda^{o2}=a^{o2}+b^{o2}. \end{cases}$$

If we partition S as $S=(A\,\vdots\, I_2\otimes e_n)$, with $A=\begin{pmatrix} u & v \\ v & -u \end{pmatrix}$, we find with the orthogonalization technique of section II from (56) that

$$(57)\quad \begin{aligned} (\Delta\hat{a}\;\,\Delta\hat{b})^t&=(P^{\perp}_{I_2\otimes e_n}A^o)^{-}\Delta r \\ &=((I_2\otimes P^{\perp}_{e_n})A^o)^{-}\Delta r, \end{aligned}$$

With the following definition of centred coordinates:

$$\bar{u}^o = P^\perp_{e_n} u^o, \quad \bar{v}^o = P^\perp_{e_n} v^o,$$

equation (57) becomes

$$(\Delta\hat{a}\ \Delta\hat{b})^t = \begin{pmatrix}\bar{u}^o & \bar{v}^o\\ \bar{v}^o & -\bar{u}^o\end{pmatrix}^- \Delta r =$$

$$= (\bar{u}^{ot} Q_r^{-1}\bar{u}^o + \bar{v}^{ot}Q_r^{-1}\bar{v}^o)^{-1}\begin{pmatrix}\bar{u}^o & \bar{v}^o\\ \bar{v}^o & -\bar{u}^o\end{pmatrix}^t (I_2\otimes Q_r^{-1})\Delta r.$$

Hence,

$$(58)\quad \boxed{\begin{aligned}\Delta\hat{a} &= \frac{\bar{u}^{ot}Q_r^{-1}\Delta p + \bar{v}^{ot}Q_r^{-1}\Delta q}{\bar{u}^{ot}Q_r^{-1}\bar{u}^o + \bar{v}^{ot}Q_r^{-1}\bar{v}^o}\\ \Delta\hat{b} &= \frac{\bar{v}^{ot}Q_r^{-1}\Delta p - \bar{u}^{ot}Q_r^{-1}\Delta q}{\bar{u}^{ot}Q_r^{-1}\bar{u}^o + \bar{v}^{ot}Q_r^{-1}\bar{v}^o}\end{aligned}}$$

Note that the barred coordinates, such as $\bar{u}^o$, are centred coordinates using the metric Q_r^{-1}. The LLS-solution for the translational increments follows from (56) as

$$(\Delta\hat{t}_x\ \Delta\hat{t}_y)^t = (I_2\otimes e_n)^-[\Delta r - A^o(\Delta\hat{a}\ \Delta\hat{b})^t]$$

$$= (I_2\otimes(e_n^tQ_r^{-1}e_n)^{-1}e_n^tQ_r^{-1})[\Delta r - A^o(\Delta\hat{a}\ \Delta\hat{b})^t].$$

Hence,

$$(59)\quad \boxed{\begin{aligned}\Delta\hat{t}_x &= \Delta p_c - u_c^o\Delta\hat{a} - v_c^o\Delta\hat{b}\\ \Delta\hat{t}_y &= \Delta q_c - v_c^o\Delta\hat{a} + u_c^o\Delta\hat{b}\end{aligned}}$$

In order to derive the LLS-solution of the coordinate increments in the two coordinate systems we can follow the same procedure as used in section III. This gives then

$$(60)\quad \boxed{\begin{aligned}\Delta\hat{w} &= \Delta w + (B^{ot}\otimes Q_wQ_r^{-1})P^\perp_{S^o}\,\Delta r\\ \Delta\hat{z} &= \Delta z - (I_2\otimes Q_zQ_r^{-1})P^\perp_{S^o}\,\Delta r\end{aligned}}$$

The projector $P^\perp_{S^o}$ in (60) reads as

$$(61)\quad \boxed{\begin{aligned}&P^\perp_{S^o} = I_2\otimes P^\perp_{e_n} - P_{\bar{A}^o},\\ &\text{where}\quad P_{\bar{A}^o} = (\bar{u}^{ot}Q_r^{-1}\bar{u}^o + \bar{v}^{ot}Q_r^{-1}\bar{v}^o)^{-1}.\\ &\cdot\begin{pmatrix}(\bar{u}^o\bar{u}^{ot}+\bar{v}^o\bar{v}^{ot})Q_r^{-1} & (\bar{u}^o\bar{v}^{ot}-\bar{v}^o\bar{u}^{ot})Q_r^{-1}\\ (\bar{u}^o\bar{v}^{ot}-\bar{v}^o\bar{u}^{ot})^tQ_r^{-1} & (\bar{u}^o\bar{u}^{ot}+\bar{v}^o\bar{v}^{ot})Q_r^{-1}\end{pmatrix}\end{aligned}}$$

IV.2. Testing and Reliability

With (11) and (56) the overall model test becomes

$$(62)\quad \boxed{\begin{aligned}&M_1\ \text{invalid if}\\ &\frac{\Delta r^t(I_2\otimes Q_r^{-1})P^\perp_{S^o}\Delta r}{2n-4} > \frac{\chi^2(2n-4,0)_{\alpha_{2n-4}}}{2n-4}\end{aligned}}$$

For the b=2 dimensional test (12) we shall consider two cases. First we derive the test which tests whether M_1 is invalidated by an Affine transformation. The appropriate choice for C_b is then

$$(63)\quad C_b := \begin{pmatrix}u^o & v^o\\ -v^o & u^o\end{pmatrix}.$$

Since

$$(64)\quad \begin{cases}[(P^\perp_A C_b)^t Q_y^{-1}(P^\perp_A C_b)]^{-1} :=\\ := \dfrac{1}{4}\,\dfrac{\bar{u}^{ot}Q_r^{-1}\bar{u}^o + \bar{v}^{ot}Q_r^{-1}\bar{v}^o}{\bar{u}^{ot}Q_r^{-1}\bar{u}^o\,\bar{v}^{ot}Q_r^{-1}\bar{v}^o - (\bar{u}^{ot}Q_r^{-1}\bar{v}^o)^2}\, I_2\\ \\ ((P^\perp_A C_b)^t Q_y^{-1}y)^t((P^\perp_A C_b)^tQ_y^{-1}y) :=\\ := \Delta r^t P^{\perp t}_{S^o}(I_2\otimes Q_r^{-1})C_bC_b^t(I_2\otimes Q_r^{-1})P^\perp_{S^o}\Delta r,\end{cases}$$

it follows with,

$$(I_2\otimes Q_r^{-1})P_{(I_2\otimes P^\perp_{e_n})C_b} = (I_2\otimes Q_r)^{-1}P_{\bar{C}_b} =$$

$$= \frac{(I_2\otimes P^\perp_{e_n})^t(I_2\otimes Q_r^{-1})C_bC_b^t(I_2\otimes Q_r^{-1})(I_2\otimes P^\perp_{e_n})}{\bar{u}^{ot}Q_r^{-1}\bar{u}^o + \bar{v}^{ot}Q_r^{-1}\bar{v}^o}$$

and

$P_{\bar{C}_b}P^\perp_{S^o} = P_{\bar{C}_b}P^\perp_{\bar{A}^o}$, that

$$(65)\quad \begin{aligned}&((P^\perp_A C_b)^tQ_y^{-1}y)^t((P^\perp_A C_b)^tQ_y^{-1}y) :=\\ &:= (\bar{u}^{ot}Q_r^{-1}\bar{u}^o + \bar{v}^{ot}Q_r^{-1}\bar{v}^o)\Delta r^t(I_2\otimes Q_r^{-1})P^\perp_{\bar{A}^o}P_{\bar{C}_b}P^\perp_{\bar{A}^o}\Delta r.\end{aligned}$$

Hence, with (12), (64), (65) and $P_{\bar{C}_b} = (E\otimes I_n)P_{\bar{A}^o}(E\otimes I_n)$, where $E = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, the test becomes

$$(66)\quad \boxed{\begin{array}{l}M_1 \text{ is invalidated by an Affine transformation if} \\ \dfrac{(\bar{u}^{ot}Q_r^{-1}\bar{u}^{o}+\bar{v}^{ot}Q_r^{-1}\bar{v}^{o})^2}{4(\bar{u}^{ot}Q_r^{-1}\bar{u}^{o}\bar{v}^{ot}Q_r^{-1}\bar{v}^{o}-(\bar{u}^{ot}Q_r^{-1}\bar{v}^{o})^2)}\cdot \\ \cdot\dfrac{\Delta r^t(I_2\otimes Q_r^{-1})P^{\perp}_{\bar{A}o}(E\otimes I_n)P_{\bar{A}o}(E\otimes I_n)P^{\perp}_{\bar{A}o}\Delta r}{2} \\ >\dfrac{\chi^2(2,0)_{\alpha_2}}{2}\end{array}}$$

The with this test corresponding reliability ellipsoid follows from (22') and (64) as

$$(67)\quad \boxed{\begin{array}{l}(\nabla_2)_k=\frac{1}{2}d_k\sqrt{\dfrac{\lambda_2(\bar{u}^{ot}Q_r^{-1}\bar{u}^{o}+\bar{v}^{ot}Q_r^{-1}\bar{v}^{o})}{\bar{u}^{ot}Q_r^{-1}\bar{u}^{o}\bar{v}^{ot}Q_r^{-1}\bar{v}^{o}-(\bar{u}^{ot}Q_r^{-1}\bar{v}^{o})^2}}\ , \\ \text{with } d_k^t=(\cos\alpha_k\ \sin\alpha_k)\end{array}}$$

In order to derive the b=2 dimensional test which tests whether M_1 is invalidated by a misspecification in point i of the first coordinate system, we take

$$(68)\quad c_b^t := I_2\otimes c_i^t\ .$$

Then

$$(69)\quad \begin{cases}(P_A^{\perp}C_b)^tQ_y^{-1}y := (I_2\otimes\bar{c}_i^tQ_r^{-1})P^{\perp}_{\bar{A}o}\Delta r \\ [(P_A^{\perp}C_b)^tQ_y^{-1}(P_A^{\perp}C_b)] := (I_2\otimes\bar{c}_i^tQ_r^{-1})P^{\perp}_{\bar{A}o}(I_2\otimes\bar{c}_i)\end{cases}$$

Since the matrix

$$(70)\quad (I_2\otimes\bar{c}_i^tQ_r^{-1})P^{\perp}_{\bar{A}o}(I_2\otimes\bar{c}_i) = I_2\otimes\bar{c}_i^tQ_r^{-1}P^{\perp}_{\bar{A}o}\bar{c}_i\ ,$$

is diagonal, it follows that the with (68) corresponding b dimensional test can readily be computed from the one dimensional test corresponding with the choice

$$(71)\quad c_1^t := d^t\otimes c_i^t,\quad \text{with } d^t=(\cos\alpha\ \sin\alpha)\ .$$

From (71) follows that

$$(72)\quad \begin{cases}(P_A^{\perp}c_1)^tQ_y^{-1}y := d^t\otimes\bar{c}_i^tQ_r^{-1}P^{\perp}_{\bar{A}o}\Delta r \\ (P_A^{\perp}c_1)^tQ_y^{-1}(P_A^{\perp}c_1) := d^td\otimes\bar{c}_i^tQ_r^{-1}P^{\perp}_{\bar{A}o}\bar{c}_i\end{cases}$$

Hence, with (16) we have

$$(73)\quad \boxed{\begin{array}{l}M_1 \text{ is invalidated by a misspecification in} \\ \text{point i of the first coordinate system in the} \\ \text{direction } d^t=(\cos\alpha\ \sin\alpha) \text{ if} \\ \left|\dfrac{d^t\otimes\bar{c}_i^tQ_r^{-1}P^{\perp}_{\bar{A}o}\Delta r}{\sqrt{\bar{c}_i^tQ_r^{-1}P^{\perp}_{\bar{A}o}\bar{c}_i}}\right| > \sqrt{\chi^2(1,o)_{\alpha_1}}\end{array}}$$

The with this test corresponding reliability measure reads

$$(74)\quad \boxed{|\nabla_1| = \sqrt{\frac{\lambda_1}{\bar{c}_i^tQ_r^{-1}P^{\perp}_{\bar{A}o}\bar{c}_i}}\ ,}$$

where

$$(75)\quad \boxed{\begin{array}{l}\bar{c}_i^tQ_r^{-1}P^{\perp}_{\bar{A}o}\bar{c}_i = c_i^tQ_r^{-1}c_i - \dfrac{(c_i^tQ_r^{-1}e_n)^2}{e_n^tQ_r^{-1}e_n} \\ -\dfrac{(\bar{u}^{ot}Q_r^{-1}c_i)^2+(\bar{v}^{ot}Q_r^{-1}c_i)^2}{\bar{u}^{ot}Q_r^{-1}\bar{u}^{o}+\bar{v}^{ot}Q_r^{-1}\bar{v}^{o}}\end{array}}$$

The with (68) corresponding b=2 dimensional test statistic follows now simply from squaring and adding the test statistic of (73) for two mutually orthogonal directions.

V. Concluding Remarks

In this paper we have derived almost inversion-free formulae which are needed when adjusting and testing the Affine and Similarity transformation. The only two inversions needed are in case of the Affine transformation the inversion of the 2x2 matrix $(Q_z+B^oQ_wB^{ot})$ and in case of the Similarity transformation the inversion of the nxn matrix $(Q_z+\lambda^{o2}Q_w)$. Although we had to make some simplifying assumptions in the covariance structure of the observational variates, it is felt that these assumptions are sufficiently general for many practical applications. When digitizing maps, the covariance matrix of the digitized coordinates can often even be simplified to a scaled unit matrix.

The most important application of the Similarity transformation seems to be the connection of geodetic networks, although also here an application to the problem of transforming digitized maps is possible if one starts from the working hypothesis that no affine déformations are present. This hypothesis can then be tested with (66).

The assumption of the rotational invariant covariance structure is in many cases sufficient for geodetic networks. For instance, the Baarda-Alberda substitute matrix (see e.g. Brouwer et al., 1982; or Teunissen, 1984b), which is an example of a rotational invariant covariance matrix, describes the precision of many geodetic networks to a sufficient degree and can therefore be used in our formulae.

Our use of regular rotational invariant covariance matrices may seem to be contradictory at first sight with the fact that coordinates, as functions of geodetic observables, can only be operationally defined through the introduction of a coördinate reference system for which usually a priori non-stochastic values are adopted. Indeed, we know that the coordinate covariance matrices of free networks with an ordinary S-base definition (i.e. minimum constraints) are singular. Therefore it may seem that through our use of regular coordinates covariance matrices we identify z and w as absolute coordinates. This is however not true as the following remarks exemplify:

As said, the coordinate covariance matrices belonging to free networks with an ordinary S-base definition are singular. So, if we consider a part of a network in which the S-base lies then the corresponding covariance matrix will turn out to be singular. However, if the S-base does not lie in the part of the network considered, then the corresponding covariance matrix will be regular. Hence, if one connects overlapping networks one can have the situation of regular covariance matrices if the S-bases lie in the non-overlapping parts.

Secondly we note that it is not strictly necessary to follow the customary practise of introducing S-bases in a non-stochastic way. Stochastic S-bases (i.e. weighted minimum constraints) are allowed as long as one keeps good track of the estimable quantities in future manipulations. That is, no harm is done in using stochastic S-bases as long as only estimable functions are used in subsequent computations. This now is precisely the case in our connection problem. By using the similarity transformation as model one is actually only comparing the shape of the two overlapping networks. Therefore only estimable functions as angles and distance-ratios contribute to the adjustment.

Finally we note that one need not even has to start from the assumption that stochastic S-bases were introduced. For the connection problem one is namely allowed to regularize singular coordinate covariance matrices as was shown in (Teunissen, 1985b).

The conclusion must therefore be that it is permissable to use regular (rotational invariant) covariance matrices for the connection problem without identifying z and w as absolute coordinates.

In the previous sections we already mentioned that the functional models of the Affine and Similarity transformation are non-linear and therefore had to be linearized in order to apply linear estimation techniques. For the linearization approximate values are needed. Usually the approximate values are close enough to the least-squares solution so that no iteration is needed. However, if this is not the case, one should iterate the solution and this can be done by applying the so-called Gauss' method of iteration (see e.g. Teunissen, 1984c). One starts with an initial guess x_o about the unknown parameter vector x and proceeds to generate a sequence $x_o, x_1, x_2 ..$ which converges to the least-squares solution $\hat{x}$. Given x_k, the updated vector x_{k+1} is computed as $x_{k+1} = x_k + \Delta\hat{x}_k$, where $\Delta\hat{x}_k$ is the linear least-squares estimate of the parameter vector increment computed with our formulae using x_k as approximate value. As shown in (Teunissen, 1985b) the convergence behaviour of Gauss' method is mainly gouverned by the extrinsic curvature k_N of the non-linear manifold described by the functional model used. It can furthermore be shown that the two non-zero extrinsic curvatures of the manifold described by the non-linear Similarity transformation read

$$(76) \qquad k_N = \pm\, \alpha \sqrt{\frac{N^t Q_z^{-1} N}{\bar{u}^t Q_z^{-1} \bar{u} + \bar{v}^t Q_z^{-1} \bar{v}}}\ , \text{ if } Q_w = \alpha^2 Q_z \ ,$$

where the vector N has the first 2n elements of a unit vector normal to the manifold as its elements and the barred coordinates are centred using the metric Q_z^{-1}. Equation (76) shows that fortunately the manifold is only moderately curved. Hence, covergence will be fast in general. For more details on the geometric theory of non-linear adjustment as developed by the present author, we refer to (Teunissen, 1984c, 1985a and b).

Note that if the points in the second coordinate system are taken to be fixed, i.e. $Q_w=0$, both the model of the Affine and Similarity transformation become linear. In this case no linearization or iteration is needed and our results can be simplified by excluding the incremental sign "Δ" in the formulae derived. If $Q_w=0$, the linearity of the model of the Similarity transformation also follows from (76). With $\alpha=0$ all extrinsic curvatures are namely zero.

In the very special case that $Q_z=I_n$ *and* $Q_w=0$ our model of the Similarity transformation reduces to the well-known model of the Helmert transformation (see Helmert, 1893). And then of course our results

simplify to the well-known results which hold true for the Helmert transformation.
Finally we remark that if the variance factor of unit weight is not known a priori one has recourse to the test statistics $\tan^2\hat{\alpha}$, $\sin^2\hat{\alpha}$ or $\cos^2\hat{\alpha}$ (see figure 3). It is easily verified that in the notation of section II we have

$$(77)\quad \tan^2\hat{\alpha} = \frac{\|\hat{y}_1-\hat{y}_2\|^2}{\|y-\hat{y}_2\|^2},\quad \sin^2\hat{\alpha} = \frac{\|\hat{y}_1-\hat{y}_2\|^2}{\|y-\hat{y}_1\|^2}$$

$$\cos^2\hat{\alpha} = \frac{\|y-\hat{y}_2\|^2}{\|y-\hat{y}_1\|^2}.$$

From this follows then that under M_1:

$$(78)\quad \tan^2\hat{\alpha} \sim \frac{b}{m-n-b}F(b,m-n,o);\ \sin^2\hat{\alpha} \sim B(\frac{b}{2}, \frac{m-n-b}{2},0);$$

$$\cos^2\hat{\alpha} \sim B(\frac{m-n-b}{2},\frac{b}{2},0),$$

where $B(f_1,f_2,0)$ is the so-called central beta-distribution with degrees of freedom f_1 and f_2 (see e.g. Rao, 1973).

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