A NOTE ON ANHOLONOMITY IN GEOMETRIC GEODESY

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1. Introduction

For computing geodetic networks on an ellipsoid of revolution the executed terrestrial observations need to be reduced to the reference surface. For this reduction gravityfield information is indispensable. In practice, however, the necessary gravityfield information is not always available meaning that the computed geodetic coordinates ϕ and λ are effected accordingly. In fact they become anholonomic. A description of the resulting misclosures can be made by using the general Stokes integral theorem.

2. Anholonomity and the Stokes integral theorem

Let us consider the differential one forms

$$w^\alpha = r_i^\alpha \, dx^i, \tag{1}$$

We assume the matrix r_i^α to be smooth, with det $r_i^\alpha \neq 0$ having a constant sign; its elements are functions of the coordinates x^i. The differential one form w^α is exact if there is a function f^α such that

$$w^\alpha = df^\alpha \tag{2}$$

Using the ∂_i notation for partial derivatives, it follows from (1) and (2) that

$$r_i^\alpha = \partial_i f^\alpha,$$

and since $\partial_i \partial_j f^\alpha = \partial_j \partial_i f^\alpha$, we get as necessary conditions for the differential one forms w^α to be exact:

$$\partial_j r_i^\alpha = \partial_i r_j^\alpha \tag{3}$$

A differential one form that satisfies (3) is called closed. Thus an exact differential form is always closed, but the converse, however, is generally false. An additional condition that will guarantee closed forms to be exact is that the domain of w^α is simply connected. Since we assume the domain to be an Euclidean space, which is simply connected, condition (3) is, in our case, a necessary and sufficient condition for the differential forms w^α to be exact.

After applying the exterior derivative operator d (see e.g. Flanders, 1963) to the differential forms (1), we get

$$dw^\alpha = d(r^\alpha_i\, dx^i) = \frac{1}{2}(\partial_j r^\alpha_i - \partial_i r^\alpha_j)\, dx^j \wedge dx^i, \qquad (4')$$

and substitution of $dx^i = r^i_\alpha w^\alpha$ (the inverse relation of (1)) gives

$$dw^\alpha = \frac{1}{2}\,(\partial_j r^\alpha_i - \partial_i r^\alpha_j) r^j_\beta r^i_\gamma w^\beta \wedge w^\gamma \overset{\text{def}}{=} \Omega^\alpha_{\beta\gamma} w^\beta \wedge w^\gamma. \qquad (4'')$$

where the quantity $\Omega^\alpha_{\beta\gamma}$ is known as the object of anholonomity (see e.g. Grafarend, 1975).

From (4) follows that, if the object of anholonomity $\Omega^\alpha_{\beta\gamma}$ does not vanish, $dw^\alpha \neq 0$, meaning that the so-called integrability conditions (3) are not fulfilled. In this case the differential forms w^α are inexact or anholonomic.

For exact differential one forms (see (2)) closed line integrals vanish. For anholonomic differential forms this is generally not the case. And the resulting misclosures can then be described with the aid of the general Stokes integral theorem, which reads

$$\int_{\partial G} w^\alpha = \int_G dw^\alpha \qquad \text{(see e.g. Flanders, 1963, p. 64)} \qquad (5)$$

Substitution of (4) into (5) gives

$$\int_{\partial G} w^\alpha = \int_G \Omega^\alpha_{\beta\gamma}\, w^\beta \wedge w^\gamma = \int_G \frac{1}{2}(\partial_j r^\alpha_i - \partial_i r^\alpha_j)\, dx^j \wedge dx^i \qquad (6)$$

In the geodetic literature already various examples of anholonomity are treated. In (Grafarend, 1975) the anholonomity of the natural orthonormal frame is considered and Frobinius type matrices of integrating factors are introduced which enable to transform anholonomic differentials into

holonomic ones. In (Leclerc, 1977) estimates are given for the misclosures in the local coordinates for a simple closed path and in (Doukakis, 1977) anholonomity caused by neglect of polar motion is studied.
Another case of anholonomity comes up in geometric geodesy. In classical geometric geodesy one is confronted with the problem of transforming the measured elements into the geodetic coordinates ϕ and λ. In order to execute this transformation one needs, among other things, a geodetic datum definition and the availability of gravityfield information for reducing the terrestrial observations to the reference surface, an ellipsoid of revolution.
In practice, the datum definition includes the fixing of one network point on the reference surface. But since this reference surface is an ellipsoid of <u>revolution</u>, one theoretically fixes one coordinate too many. Hence the possible occurence of anholonomity. In this paper we will, however, restrict ourselves to the case of anholonomity which follows when the gravityfield is not properly taken into account for the reduction of terrestrial observations to the reference ellipsoid. We will describe the resulting distortions in the geodetic coordinates ϕ and λ using the Stokes integral theorem.

3. Transforming the measured elements into the geodetic coordinate differentials

Let us consider the following four orthonormal triads:

The earth-fixed frame $\underline{e}_I$ with $\underline{e}_{I=3}$ toward the average terrestrial pole (CIO)
$\underline{e}_{I=1}$ toward the line of intersection of the plane of the average terrestrial equator and the plane containing the Greenwich vertical and parallel to $\underline{e}_{I=3}$
$\underline{e}_{I=2}$ completing the right-handed system.

The ellipsoid-fixed frame $\underline{e}_i$ with $\underline{e}_{i=3}$ parallel to the rotation axis of the ellipsoid of revolution

$\underline{e}_{i=1}$ lying in the ellipsoidal equator-plane.

$\underline{e}_{i=2}$ completing the right-handed system.

The astronomical frame $\underline{e}_A$ with $\underline{e}_{A=1}$ toward astronomical east

$\underline{e}_{A=2}$ toward astronomical north

$\underline{e}_{A=3}$ toward the local astronomical zenith.

The local geodetic frame $\underline{e}_\alpha$ with $\underline{e}_{\alpha=1}$ toward geodetic east

$\underline{e}_{\alpha=2}$ toward geodetic north

$\underline{e}_{\alpha=3}$ toward the local geodetic zenith.

These four frames are related by the following transformation formulae:

$$\underline{e}_A = R_{\Phi\Lambda}\,\underline{e}_I$$
$$\underline{e}_\alpha = R_{\phi\lambda}\,\underline{e}_i \qquad (7)$$
$$\underline{e}_i = (I+E)\underline{e}_I\ ,$$

where

$$R_{uv} = \begin{pmatrix} -\sin v & \cos v & 0 \\ -\sin u \cos v & -\sin u \sin v & \cos u \\ \cos u \cos v & \cos u \sin v & \sin u \end{pmatrix}$$

I: the identity matrix

$$E = \begin{pmatrix} 0 & -\varepsilon_z & \varepsilon_y \\ \varepsilon_z & 0 & -\varepsilon_x \\ -\varepsilon_y & \varepsilon_x & 0 \end{pmatrix}$$

Φ, ϕ are the astronomical and geodetic latitudes

Λ, λ are the astronomical and geodetic longitudes

ε_x, ε_y, ε_z are the small angles of rotation relating $\underline{e}_I$ to $\underline{e}_i$

When projecting the differential displacement vector $\underline{dx}$ onto the three axes of the frames $\underline{e}_A$, $\underline{e}_\alpha$ we get the differentials w^A, w^α respectively, with

$$\underline{dx} = w^A \underline{e}_A = w^\alpha \underline{e}_\alpha , \qquad (8)$$

and

$$w^A = \begin{pmatrix} dl \sin Z \sin A \\ dl \sin Z \cos A \\ dl \cos Z \end{pmatrix}, \quad w^\alpha = \begin{pmatrix} dl \sin \zeta \sin \alpha \\ dl \sin \zeta \cos \alpha \\ dl \cos \zeta \end{pmatrix} \qquad (9)$$

where

A, α are the astronomical and geodetic azimuths
Z, ζ are the astronomical and geodetic zenith angles
dl is the length of the displacement vector $\underline{dx}$.

From (7) and (8) follows:

$$w^\alpha = R_{\phi\lambda}(I+E)\, R^T_{\Phi\Lambda}\, w^A\ , \qquad (10)$$

and with a first order approximation, i.e. neglecting quantities like $(\Lambda-\lambda)^2$, $(\Lambda-\lambda)(\Phi-\phi)$, ε_x^2 etc., the transformation matrices occurring in (10) become:

$$R_{\phi\lambda}R^T_{\Phi\Lambda} = \begin{pmatrix} 1 & -(\Lambda-\lambda)\sin\phi & (\Lambda-\lambda)\cos \\ (\Lambda-\lambda)\sin\phi & 1 & (\Phi-\phi) \\ -(\Lambda-\lambda)\cos\phi & -(\Phi-\phi) & 1 \end{pmatrix} \qquad (11')$$

and

$$R_{\phi\lambda}ER^T_{\Phi\Lambda} = \begin{pmatrix} 0 & -[\varepsilon_x\cos\phi\cos\lambda+\varepsilon_y\cos\phi\sin\lambda + \varepsilon_z\sin\phi] & [-\varepsilon_x\sin\phi\cos\lambda-\varepsilon_y\sin\phi\sin\lambda + \varepsilon_z\cos\phi] \\ [\varepsilon_x\cos\phi\cos\lambda+\varepsilon_y\cos\phi\sin\lambda + \varepsilon_z\sin\phi] & 0 & -[\varepsilon_y\cos\lambda-\varepsilon_x\sin\lambda] \\ -[\varepsilon_x\sin\phi\cos\lambda-\varepsilon_y\sin\phi\sin\lambda + \varepsilon_z\cos\phi] & [\varepsilon_y\cos\lambda-\varepsilon_x\sin\lambda] & 0 \end{pmatrix}$$

For the purpose of obtaining the north-south component ξ and east-west component η of the deflection of the vertical θ, we temporarily assume $\underline{dx}$ to be parallel to the $\underline{e}_{A=3}$ axis (see figure 1). That is, we assume $Z=0$ for which $\zeta=\theta$ holds.

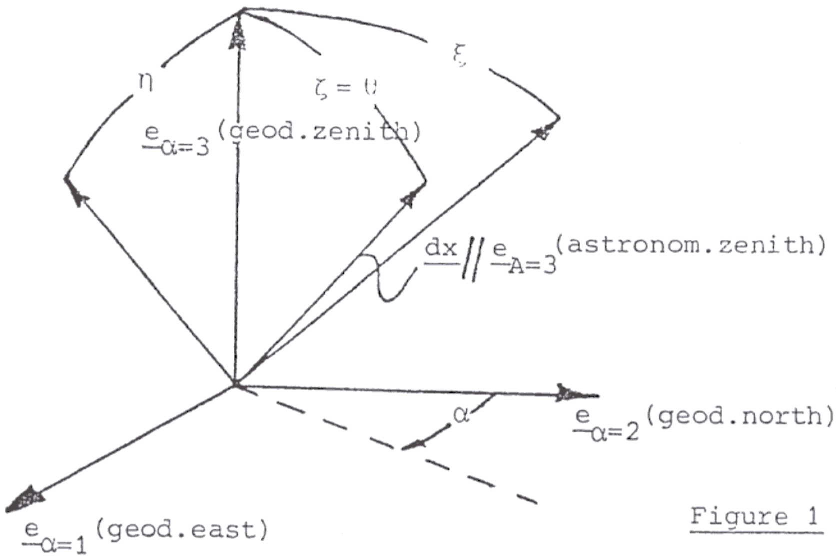

Figure 1

Substitution of Z = 0 and $\zeta = \theta$ into (9) gives with (10) and (11), and the approximation $\sin\theta \simeq \theta$:

$$\begin{pmatrix} \sin\theta\ \sin\alpha \\ \sin\theta\ \cos\alpha \\ \cos\theta \end{pmatrix} = \begin{pmatrix} \theta\ \sin\ \alpha \\ \theta\ \cos\ \alpha \\ 1 \end{pmatrix} = \begin{pmatrix} (\Lambda-\lambda)\cos\phi + [-\varepsilon_x \sin\phi\cos\lambda - \varepsilon_y \sin\phi\sin\lambda + \varepsilon_z \cos\phi] \\ (\Phi-\phi) \qquad + \quad [\varepsilon_x \sin\lambda - \varepsilon_y \cos\lambda] \\ 1 \end{pmatrix} \tag{12}$$

From the first two rows of (12) follows for the east-west component η and north-south component ξ respectively:

$$\eta = \theta\ \sin\ \alpha = (\Lambda-\lambda)\cos\phi + \varepsilon_z \cos\phi - \varepsilon_y \sin\phi\sin\lambda - \varepsilon_x \sin\phi\cos\lambda \qquad (13')$$

$$\xi = \theta\ \cos\ \alpha = (\Phi-\phi) \qquad + \varepsilon_x \sin\lambda - \varepsilon_y \cos\lambda \qquad (13'')$$

Note that the two last terms in (13') and (13") are polar migration-like correction terms (see e.g. Heiskanen and Moritz, p. 189).

Substitution of (13) into (11) gives with (9) and (10):

$$\begin{pmatrix} dl \sin\zeta \sin\alpha \\ dl \sin\zeta \cos\alpha \\ dl \cos\zeta \end{pmatrix} =$$

$$\begin{pmatrix} 1 & -[\eta\tan\phi+\cos^{-1}\phi(\varepsilon_x\cos\lambda+\varepsilon_y\sin\lambda] & \eta \\ [\eta\tan\phi+\cos^{-1}\phi(\varepsilon_x\cos\lambda+\varepsilon_y\sin\lambda)] & 1 & \xi \\ -\eta & -\xi & 1 \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} dl \sin Z \sin A \\ dl \sin Z \cos A \\ dl \cos Z \end{pmatrix} \tag{14}$$

This relation relates the measured elements ω^A to their geodetic counter parts ω^α. Consequently relation (14) should contain the classical reduction formulae for azimuth and zenith angles. This is seen as follows: Dividing the first row by the second row of relation (14) gives

$$\frac{\sin\alpha}{\cos\alpha} = \frac{\sin Z \sin A - [\eta\tan\phi+\cos^{-1}\phi\ (\varepsilon_x\cos\lambda+\varepsilon_y\sin\lambda)]\sin Z \cos A + \eta \cos Z}{[\eta\tan\phi+\cos^{-1}\phi(\varepsilon_x\cos\lambda+\varepsilon_y\sin\lambda)]\sin Z \sin A + \sin Z \cos A + \xi \cos Z} \tag{15}$$

The conversion formula for azimuth then follows from substitution of $A = \alpha + \Delta A$ into (15) using a first order approximation:

$$\Delta A = A-\alpha = \eta\tan\phi + [\xi\sin A-\eta\cos A]\cot Z + \cos^{-1}\phi[\varepsilon_x\cos\lambda+\varepsilon_y\sin\lambda] \tag{16}$$

The third row of (14) gives us

$$\cos\zeta = -\eta \sin Z \sin A - \xi \sin Z \cos A + \cos Z$$

Substitution of $Z = \zeta + \Delta Z$ and again using a first order approximation results in the reduction formula for zenith angles:

$$\Delta Z = Z - \zeta = -\eta \sin A - \xi \cos A \tag{17}$$

In an analogous way one can obtain the classical distance reduction formula from a relation like (14) (see e.g. Teunissen 1982).

The next step is to relate the differentials ω^A to the geodetic coordinate differentials $d\phi$, $d\lambda$ and dh (h is the geometric height above the reference ellipsoid).

With

$$\underline{dx} = dx^i \underline{e}_i = \omega^\alpha \underline{e}_\alpha \ , \tag{18}$$

and

$$\begin{pmatrix} x^{i=1} \\ x^{i=2} \\ x^{i=3} \end{pmatrix} = \begin{pmatrix} (N+h)\ \cos\phi\ \cos\lambda \\ (N+h)\ \cos\phi\ \sin\lambda \\ (\frac{b^2}{a^2} N+h)\ \sin\phi \end{pmatrix} \tag{19}$$

where

N is the principal radius of curvature perpendicular to the meridian
a is the semi major axis of the reference ellipsoid
b is the semi minor axis of the reference ellipsoid

we get, using (7):

$$\begin{pmatrix} d\lambda \\ d\phi \\ dh \end{pmatrix} = \begin{pmatrix} \cos^{-1}\phi(N+h)^{-1} & 0 & 0 \\ 0 & (M+h)^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \omega^{\alpha=1} \\ \omega^{\alpha=2} \\ \omega^{\alpha=3} \end{pmatrix} \tag{20}$$

where M is the principal radius of curvature in the meridian.

The combination of (14) and (20) finally gives us the relation which transforms to a first order approximation, the differentials ω^A, containing the measured elements, into the desired geodetic coordinate differentials $d\lambda$, $d\phi$ and dh:

$$\begin{pmatrix} d\lambda \\ d\phi \\ dh \end{pmatrix} =$$

$$\begin{pmatrix} \cos^{-1}\phi(N+h)^{-1} & -\cos^{-1}\phi(N+h)^{-1}[\eta\tan\phi+\cos^{-1}\phi(\varepsilon_x\cos\lambda+\varepsilon_y\sin\lambda)] & \eta\cos^{-1}\phi(N+h \\ (M+h)^{-1}[\eta\tan\phi + \cos^{-1}\phi(\varepsilon_x\cos\lambda+\varepsilon_y\sin\lambda)] & (M+h)^{-1} & \xi(M+h)^{-1} \\ -\eta & -\xi & 1 \end{pmatrix}$$

$$\cdot \begin{pmatrix} \omega^{A=1} \\ \omega^{A=2} \\ \omega^{A=3} \end{pmatrix} \tag{21}$$

To carry out this transformation properly, we see that we need information on the deflection components ξ, η and on the geometric height h above the ellipsoid. In practice, however, this information is not always available and consequently one will obtain anholonomic coordinate differentials $d\phi'$, $d\lambda'$ and dh'.

4. Anholonomity due to lack of gravityfield information

By setting η, ξ, h and the rotation angles ε equal to zero in (21) we obtain the anholonomic coordinate differentials:

$$\begin{pmatrix} d\lambda' \\ d\phi' \\ dh' \end{pmatrix} = \begin{pmatrix} \cos^{-1} N^{-1} & 0 & 0 \\ 0 & M^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dl \sin Z \sin A \\ dl \sin Z \cos A \\ dl \cos Z \end{pmatrix} \tag{22}$$

Combining (22) and (21) then gives us the relation between the exact coordinate differentials $d\phi$, $d\lambda$ and dh and the inexact or anholonomic geodetic coordinate differentials $d\phi'$, $d\lambda'$, dh':

$$\begin{pmatrix} d\lambda' \\ d\phi' \\ dh' \end{pmatrix} =$$

$$\begin{pmatrix} \frac{N+h}{N} & \frac{M+h}{N}\cos^{-1}\phi[\eta\tan\phi+\cos^{-1}\phi(\varepsilon_x\cos\lambda+\varepsilon_y\sin\lambda)] & -\eta\cos^{-1}\phi N^{-1} \\ -\frac{N+h}{M}\cos\phi[\eta\tan\phi+\cos^{-1}\phi(\varepsilon_x\cos\lambda+\varepsilon_y\sin\lambda)] & \frac{M+h}{M} & -\xi M^{-1} \\ \eta(N+h)\cos\phi & \xi(M+h) & 1 \end{pmatrix} .$$

$$\cdot \begin{pmatrix} d\lambda \\ d\phi \\ dh \end{pmatrix} \tag{23}$$

a relation which is of the form of (1) and for which (2) holds.

From the last row of relation (23) we get

$$\int_P^Q dh' - dh = \int_P^Q [\eta(N+h)\cos\phi d\lambda + \xi(M+h)d\phi] \,, \tag{24}$$

in which we recognize the well-known formula of astronomical levelling, in physical geodesy, for computing the height differences between the reference surface and the geoid.

In a similar way we can use the first two rows of (23) in geometric geodesy to describe the misclosures in ϕ and λ due to the neglect of gravityfield information. Assuming that the geodetic network points lie on a surface parametrized as $h = h(\lambda,\phi)$, it follows from (23), with $dh = \frac{\partial h}{\partial \phi} d\phi + \frac{\partial h}{\partial \lambda} d\lambda$, $ds_\lambda = N \cos\phi d\lambda$ and $ds_\phi = M\, d\phi$, that:

$$\oint ds'_\lambda - ds_\lambda = \oint [h\cos\phi - \frac{\partial h}{\partial \lambda}\eta\,]d\lambda + [(M+h)[\eta\tan\phi + \cos^{-1}\phi(\varepsilon_x\cos\lambda + \varepsilon_y\sin\lambda)] - \eta\frac{\partial h}{\partial \phi}]d\phi \quad (25')$$

$$\oint ds'_\phi - ds_\phi = \oint [-(N+h)\cos\phi[\eta\tan\phi + \cos^{-1}\phi(\varepsilon_x\cos\lambda + \varepsilon_y\sin\lambda)] - \xi\frac{\partial h}{\partial \lambda}]d\lambda + [h - \xi\frac{\partial h}{\partial \phi}]d\phi \quad (25'')$$

The Stokes integral theorem can now be applied to (25) in order to describe the misclosures in ϕ and λ. Since we are dealing here with differential one-forms, the Stokes integral becomes

$$\int c^\alpha = \int dc^\alpha .$$

with (26)

$$c^\alpha = c_1^\alpha(x,y)dx + c_2^\alpha(x,y)dy$$

$$dc^\alpha = \partial_y c_1^\alpha\, dy\wedge dx + \partial_x c_2^\alpha dx\wedge dy = (\partial_x c_2^\alpha - \partial_y c_1^\alpha)dx\wedge dy$$

As an example we apply (26) to (25'), assuming the rotation angles to be zero and approximating M and N by R the mean earth radius. The result is:

$$\oint ds'_\lambda - ds_\lambda = \iint [\frac{\partial \eta}{\partial s_\lambda}\tan\phi + \frac{\partial h}{\partial s_\lambda}\frac{\partial \eta}{\partial s_\phi} - \frac{\partial \eta}{\partial s_\lambda}\frac{\partial h}{\partial s_\phi} - \frac{1}{R}\frac{\partial h}{\partial s_\phi}]ds_\lambda \wedge ds_\phi \qquad (27)$$

Further simplifying (27) by assuming η,ξ to be constant for the region considered, gives:

$$\oint ds'_\lambda - ds_\lambda = \iint -\frac{1}{R}\frac{\partial h}{\partial s_\phi} ds_\lambda \wedge ds_\phi = -\frac{\xi}{R}\iint ds_\lambda \wedge ds_\phi \quad , \qquad (28)$$

from which follows that the misclosure in east-west direction is proportional to the enclosed surface.
In a similar way one will get from (25")

$$\oint ds'_\phi - ds_\phi = \frac{\eta}{R}\iint ds_\lambda \wedge ds_\phi \qquad (29)$$

Thus we can conclude that, under the simplifying assumptions made, expressions (28) and (29) enable one to estimate the misclosures in ϕ and λ, due to the neglect of gravityfield information, of closed loops in geodetic networks computed on the reference ellipsoid.

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