HORIZONTAL TYPE BOUNDARY VALUE PROBLEM, LEAST-SQUARES COLLOCATION AND ASTRONOMICAL LEVELLING.

by

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ABSTRACT

It is possible to solve the geodetic boundary value problem (b.v.p.) in spherical and constant radius approximation employing horizontal type observables, such as astronomical latitude and longitude, horizontal positions and horizontal gravity gradients. The solution is overdetermined when using all three types of observables and uniquely determined when using any combination of two. For the case of vertical deflections, i.e. the combination of astronomical latitude and longitude with horizontal position, it is shown that if the number of observations on the boundary surface goes to infinite the least-squares collocation solution converges toward the solution of the boundary value problem. The basic difference of the horizontal type b.v.p. with astronomical levelling is that the former yields T on the surface and in its exterior, whereas the latter is of local character.

1. INTRODUCTION

In a recent work on the overdetermined geodetic boundary value problem (1987) the authors derived, among others, a solution of horizontal type. Horizontal is understood here in the sense, that the observation model of the three considered pairs of observables after linearization and in spherical approximation contains the "horizontal" coordinate unknowns, Δx and Δy, but not the vertical one. It was rather interesting to us that a solution of this type exists, however several questions remained open as to its proper interpretation. The purpose of this contribution is therefore to present this solution and discuss its relationship with unique versions of horizontal type boundary value problems (b.v.p), with collocation and with the astronomical levelling.

2. OVERDETERMINED GEODETIC B.V.P. OF HORIZONTAL TYPE

Given one of the three pairs of observables $\{\Phi, \Lambda\}$ astronomical latitude and longitude, $\{x, y\}$ local horizontal positions in North-South and East-West direction e.g. from G.P.S. and $\{\Gamma_{zx}, \Gamma_{zy}\}$ the mixed vertical-horizontal gradients. The observables are given on the Earth's surface. The linearized observation model of the anomalies of the observables becomes in spherical and constant radius approximation, compare (ibid).

$$\begin{bmatrix} \begin{bmatrix} \Delta\Phi \\ \Delta\Lambda\cos\varphi \end{bmatrix} \\ \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \\ \begin{bmatrix} \Delta\Gamma_{xz} \\ \Delta\Gamma_{yz} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \frac{1}{r} & 0 \\ 0 & \frac{1}{r} \\ 1 & 0 \\ 0 & 1 \\ \frac{3\Gamma o}{r} & 0 \\ 0 & \frac{3\Gamma o}{r} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} + \begin{bmatrix} -\frac{1}{Uo} \text{Grad } T \\ 0 \\ \frac{1}{r} \text{Grad } \frac{\partial T}{\partial r} - \frac{1}{r^2} \text{Grad } T \end{bmatrix} \quad (1)$$

In (1) it is $T = W - U$, the distubing potential, Grad denotes the surface gradient $\left[\frac{\partial}{\partial\varphi}, \cos^{-1}\varphi \frac{\partial}{\partial\lambda} \right]^T$ on the unit sphere σ (0,1) and $Uo = \frac{GMo}{R}$ and

$\Gamma o = \frac{\gamma}{R} = \frac{GMo}{R^3}$. We assume further:

$$\Delta T = \nabla^2 T = 0 \quad \text{(T is harmonic) outside } \sigma \qquad (2)$$

$$\lim_{r \to \infty} T \Rightarrow 0 \quad \text{(T is regular).}$$

In this case T can be expressed as a series of fully normalized spherical harmonics:

$$T(r,\varphi,\lambda) = U_o \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} \left(\frac{R}{r}\right)^{n+1} \Delta C_{nm} Y_{nm}(\varphi,\lambda). \qquad (3)$$

The ΔC_{nm} are the unknown dimensionless coefficients of degree n and order m and Y_{nm} the orthogonal spherical harmonics. (The reader should not get confused by the indexing of m from -n to + n, since the explicit form of the harmonics shall not be needed).

The surface gradient $\frac{1}{Uo}$ Grad T in (1), a vector field on σ with components ξ and η, the N-S and E-W components of the deflection of the vertical, respectively, becomes with (3) (cf. (Meissl, 1971)):

$$\underline{\epsilon} = \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \frac{1}{Uo} \text{Grad } T =$$

$$= \sum_{n,m} \left[\frac{R}{r}\right]^{n+1} \Delta C_{nm} \text{ Grad } Y_{nm}(\varphi,\lambda) \qquad (4)$$

$$= \sum_{n,m} \left[\frac{R}{r}\right]^{n+1} \Delta C_{nm} \begin{bmatrix} \partial_\varphi Y_{nm}(\varphi,\lambda) \\ \cos^{-1}\varphi \, \partial_\lambda Y_{nm}(\varphi,\lambda) \end{bmatrix}$$

Eq.(1) can now be written in constant radius approximation (r = R) as:

$$
\begin{bmatrix} \begin{bmatrix} \Delta\Phi \\ \Delta\Lambda \cos\varphi \end{bmatrix} \\ \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \\ \begin{bmatrix} \Delta\Gamma_{xz} \\ \Delta\Gamma_{yz} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \frac{1}{R} & 0 \\ 0 & \frac{1}{R} \\ 1 & 0 \\ 0 & 1 \\ 3\frac{\Gamma o}{R} & 0 \\ 0 & 3\frac{\Gamma o}{R} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} + \begin{bmatrix} -\sum_{n,m} \Delta C_{nm} \text{ Grad } Y_{nm} \\ 0 \\ -\Gamma o \sum_{n,m} (n+2) \Delta C_{nm} \text{ Grad } Y_{nm} \end{bmatrix} \tag{5}
$$

Each combination of two of the three vectors of anomalies in (5) represents together with (2) a uniquely determined b.v.p. The combination $\{\Delta\Phi, \Delta\Lambda \cos\varphi\}$ together with $\{\Delta\Gamma_{xz}, \Delta\Gamma_{yz}\}$ requires, in addition, the elimination of the three translational degrees of freedom. The full set of three pairs together with (2) defines an overdetermined b.v.p. where, in addition, a weight function has to be assigned to each pair.

The overdetermined problem can be solved by least squares. For the solution of the inner products of type $\frac{1}{4\pi}\int_\sigma (\text{Grad } Y_{nm}, \text{ Grad } Y_{kl})d\sigma$, that appear in the normal equations, use can be made of Green's first identity, as shown in (Meissl, ibid):

$$
\begin{aligned}
\frac{1}{4\pi}\int_\sigma (\text{Grad } Y_{nm}, \text{ Grad } Y_{kl})d\sigma &= \\
-\frac{1}{4\pi}\int_\sigma Y_{nm} \text{ Lap } Y_{kl} \, d\sigma &= \\
n(n+1) \frac{1}{4\pi}\int_\sigma Y_{nm} Y_{kl} \, d\sigma &= \\
n(n+1)\, \delta_{nm}\, \delta_{kl}, &
\end{aligned} \tag{6}
$$

where Lap denotes the surface Laplace operator. From (6) follows that the vector functions $U_{nm} = \frac{1}{\sqrt{n(n+1)}}$ Grad Y_{nm} can be viewed as a normalized and orthogonal basis in a space of differentiable surface vector functions $\underline{v}$ for which Rot $\underline{v}$ = 0.

The least-squares solution is divided into two steps. First the geometric unknowns Δx and Δy are eliminated by forming linear combinations of the observables:

$$
\begin{bmatrix} \begin{bmatrix} \Delta\Phi - \Delta\varphi \\ (\Delta\Lambda - \Delta\lambda)\cos\varphi \end{bmatrix} \\ \begin{bmatrix} \Delta\Gamma_{xz} - 3\Gamma o\Delta\varphi \\ \Delta\Gamma_{yz} - 3\Gamma o\Delta\lambda\cos\varphi \end{bmatrix} \end{bmatrix} = \begin{bmatrix} -\sum_{n,m} \Delta C_{nm} \text{Grad } Y_{nm} \\ -\Gamma o \sum_{n,m} (n+2)\, \Delta C_{nm} \text{ Grad } Y_{nm} \end{bmatrix} \tag{7}
$$

Thereby use is made of $\frac{\Delta x}{R} = \Delta\varphi$ and $\frac{\Delta y}{R\cos\varphi} = \Delta\lambda$.

We observe that $\Delta\Phi - \Delta\varphi = (\Phi_p - \varphi_{p'}) - (\varphi_p - \varphi_{p'}) = \Phi_p - \varphi_p = \delta\Phi$ and analogously $(\Delta\Lambda - \Delta\lambda)\cos\varphi = (\Lambda_p - \lambda_p)\cos\varphi = \delta\Lambda\cos\varphi$ are the components of the deflection of the vertical ξ and η. They are disturbance quantities.

With p_ϕ the dimensionless weight of $\{\Delta\Phi, \Delta\Lambda\cos\varphi\}$, p_x that of $\{\Delta x, \Delta y\}$ and p_Γ that of $\{\Delta\Gamma_{xz}, \Delta\Gamma_{yz}\}$, the solution of the overdetermined b.v.p. of horizontal type becomes, compare (Rummel, Teunissen & V. Gelderen, 1987):

$$
\begin{aligned}
\Delta C_{nm} = \; & \frac{p_\phi(3p_\Gamma(n-1)-p_x)}{F_n} \frac{1}{4\pi}\int_\sigma \left(\text{Grad } Y_{nm}, \begin{bmatrix} \xi \\ \eta \end{bmatrix}\right)d\sigma \\
& - \frac{p_\Gamma(p_\phi(n-1)+p_x(n+2)}{F_n} \frac{1}{4\pi\Gamma o}\int_\sigma \left(\text{Grad } Y_{nm}, \begin{bmatrix} \Delta\Gamma_{xz} - 3\Gamma_0\Delta\varphi \\ \Delta\Gamma_{yz} - 3\Gamma_0\Delta\lambda\cos\varphi \end{bmatrix}\right)d\sigma
\end{aligned} \tag{8}
$$

where $F_n = n(n+1)((9p_\phi p_\Gamma + p_x p_\phi - 6p_\phi p_\Gamma(n+2) + (p_\phi p_\Gamma + p_\Gamma p_x)(n+2)^2)$

Finally, with eq. (3), a closed formula for T can be derived:

$$
T(P) = \frac{Uo}{4\pi}\int_\sigma \frac{dKg(\psi)}{d\psi} \{\cos\alpha\, \xi + \sin\alpha\, \eta\}_Q \delta\sigma_Q \tag{9}
$$

$$+\frac{R^2}{4\pi}\int\frac{dKh(\psi)}{d\psi}\cos\alpha\;(\Delta\Gamma_{xz}-3\Gamma_0\Delta\varphi)+\sin\alpha\;(\Delta\Gamma_{yz}-3\Gamma_0\Delta\lambda\cos\varphi)\}_Q\,d\sigma_Q$$

We observe that the integral kernels $\frac{dKg(\psi)}{d\psi}$ and $\frac{dKh(\psi)}{d\psi}$ are isotropic and that the arguments $\{\cos\alpha\xi+\sin\alpha\eta\}$ and $\{\cos\alpha\;(\Delta\Gamma_{xz}-3\Gamma_0\Delta\varphi)+\sin\alpha\;(\Delta\Gamma_{yz}-3\Gamma_0\Delta\lambda\cos\varphi)\}$ are the projections of the observable anomalies on the line joining P and Q. The expansion of the kernels in a Legendre series becomes:

$$Kg(\psi)=\sum_n\frac{p_\phi(3p_\Gamma(n-1)-p_x)}{F_n}(2n+1)P_n(\cos\psi)\qquad(10a,b)$$

$$Kh(\psi)=\sum_n\frac{p_\Gamma(p_\phi(n-1)+p_x(n+2))}{F_n}(2n+1)P_n(\cos\psi).$$

3. SPECIAL CASES AND INTERPRETATION

If instead of all three pairs of observable anomalies only two are assumed to be given, three types of uniquely determined geodetic b.v.p. emerge. They are simply derived by setting in (8) either p_ϕ, p_x, or p_Γ = 0 to zero. The results are :

$$p_x=0:\ \Delta C_{nm}=-\frac{1}{n\,(n+1)(n-1)}\frac{1}{4\pi}\int_\sigma(\mathrm{Grad}\ Y_{nm},\begin{pmatrix}\frac{\Delta\Gamma_{xz}}{\Gamma_0}-3\Delta\Phi\\ \frac{\Delta\Gamma_{yz}}{\Gamma_0}-3\Delta\Lambda\cos\varphi\end{pmatrix})d\sigma$$

$$p_\Gamma=0:\ \Delta C_{nm}=-\frac{1}{n\,(n+1)}\frac{1}{4\pi}\int_\sigma(\mathrm{Grad}\ Y_{nm},\begin{pmatrix}\xi\\ \eta\end{pmatrix})d\sigma\qquad(11\ a\text{-}c)$$

$$p_\Phi=0:\ \Delta C_{nm}=-\frac{1}{n\,(n+1)(n+2)}\frac{1}{4\pi}\int_\sigma(\mathrm{Grad}\ Y_{nm},\begin{pmatrix}\frac{\Delta\Gamma_{xz}}{\Gamma_0}-3\Delta\varphi\\ \frac{\Delta\Gamma_{yz}}{\Gamma_0}-3\Delta\lambda\cos\varphi\end{pmatrix})d\sigma$$

The corresponding expressions of the disturbing potential become:

$$p_x=0:\ T(P)=\frac{R^2}{4\pi}\int_\sigma\frac{dKi(\psi)}{d\psi}\{\cos\alpha(\Delta\Gamma_{xz}-3\Gamma_0\Delta\Phi)+\sin\alpha(\Delta\Gamma_{yz}-3\Gamma_0\Delta\Lambda\cos\varphi)\}d\sigma$$

$$p_\Gamma=0:\ T(P)=\frac{Uo}{4\pi}\int_\sigma\frac{dKj(\psi)}{d\psi}\{\cos\alpha\xi+\sin\alpha\eta\}d\sigma\qquad(12\ a\text{-}c)$$

$$p_\Phi=0:\ T(P)=\frac{R^2}{4\pi}\int_\sigma\frac{dKk(\psi)}{d\psi}\{\cos\alpha(\Delta\Gamma_{xz}-3\Gamma_0\Delta\varphi)+\sin\alpha(\Delta\Gamma_{yz}-3\Gamma_0\Delta\lambda\cos\varphi)\}d\sigma.$$

In the uniquely determined case closed expressions can be derived for the integral kernels Ki, Kj, and Kk and their derivatives with respect to ψ using the method described in (Tscherning and Rapp, 1976) and (Moritz, 1980). The series expansions are :

$$Ki(\psi)=-\sum_{n=2}^{\infty}\frac{2n+1}{n(n+1)(n-1)}P_n(\cos\psi)$$

$$Kj(\psi)=-\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}P_n(\cos\psi)\qquad(13\ a\text{-}c)$$

$$Kk(\psi) = - \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)(n+2)} P_n(\cos\psi)$$

All three integral formulas (12) require some type of surface gradient function to be observable. The integration is carried out over the projection of the surface gradient function on the line connecting integration and computation point, the so-called longitudinal part of the surface gradient. Hence the formulas require the complete surface gradient to be available all over the globe. In the future, when gradiometers in combination with satellite positioning become operational eq. (12c) could gain practical importance. However (12a) and (12b) bear hardly any practical relevance, since Φ and Λ will always have to be derived from a global coverage with astronomical observations, a very laborious and time-consuming endeavour. We did not come across these integral formulas in earlier literature. Only integral kernel Kj is discussed in (Meissl, 1971).

Krarup (1969) developed least-squares collocation, one of the really great achievements in physical geodesy. As is well known, this method represents a natural generalisation of least-squares prediction. It determines in the least-squares sense an optimal approximation of the disturbing potential in a chosen reproducing kernel Hilbert space from an arbitrary number of discretely distributed observables. The observables are arbitrary linear functionals of the disturbing potential. The observables are perfectly reproduced by the approximated function or its functionals. If the observables are assumed to be contaminated by noise approximation can be combined with smoothing. Also this type of collocation is well known in geodesy, compare e.g.(Moritz, 1973), and is called here least-squares smoothing collocation.

A very obvious question that arises when thinking about collocation is what happens if the number and precision of the observations goes to infinite. This question has been addressed in (Krarup, 1981) but must still be considered an open case. Naturally one expects that for this global limit collocation converges towards the classical solution of the geodetic boundary value problem. Furthermore it is our expectation that if the observables are different from those of the classical quadruple of the

geodetic b.v.p., i.e. different from W,g,Φ,Λ, this global limit becomes one of the solutions of the uniquely or overdetermined geodetic b.v.p. discussed in (Rummel, Teunissen & Van Gelderen, 1987) or (Sacerdote & Sanso, 1985). In view of the solutions discussed there one could take a view about the place of collocation and of the geodetic b.v.p. in our geodetic building different from that taken in (Moritz, 1980, ch. 26). There the distinction is made between the operational approach, which starts from the observables and derives the desired quantities from them and the model approach. The geodetic b.v.p. would be an example of the latter, least-squares collocation and integrated geodesy (Eeg & Krarup, 1975) of the former. Could one not as well argue that the solutions of the geodetic b.v.p.'s., uniquely and over-determined, as well as collocation and integrated geodesy belong to the same building, in which the solutions of the various b.v.p.'s represent the global limit of collocation for the number and precision of the observations going to infinite? Without such an intimate connection collocation would rest on loose ground.

The relationship of discrete and stochastic observables with the functional model, as obtained from the solution of the geodetic b.v.p. is of fundamental importance. For a discussion we refer to (Baarda, 1967), (Rummel & Teunissen, 1986), or (Sanso, 1987).

As already mentioned, the proof that collocation converges to the solution of the various b.v.p.'s. in the limit is rather difficult. For the case of least-squares collocation the limit for the number (not the precision) of the observations towards infinite has been treated in (Moritz, 1976) or (Sjöberg, 1979).

In order to establish the connection between the solution of the uniquely determined b.v.p. of horizontal type, e.g. for ξ and η and the corresponding collocation case, the following procedure can be followed. The solutions of the uniquely and overdetermined b.v.p. were derived from the global limit case of the least-square problem $\|y - Ax\|^2_{D^{-1}} = \min.$ with D the inverse weight matrix or a priori variance matrix defining the metric of this quadratic norm. This approach can be followed irrespective of y being element of R(A) or not. The elements of y, A, x and D are given in (Rummel et al., 1987). It is well known that least-squares collocation is identical to a regularisation solution of the form

$$\|y - Ax\|^2_{D^{-1}} + \alpha\|x\|^2_{K^{-1}} = \min. \tag{14}$$

with $x \in H$, a Hilbert space with reproducing kernel K. The solution of eq. (14) can be written in two ways:

$$\hat{x} = (A^T D^{-1} A + \alpha K^{-1})^{-1} A^T D^{-1} y \tag{15a}$$

$$= KA^T(AKA^T + \alpha D)^{-1} y \tag{15b}$$

In (Rummel et al., ibid) form (15a), without the term αK^{-1}, has been derived for the problem under discussion. Least-squares smoothing collocation takes form (15b) and can easily be derived from (15a).

We have $\underline{y}$ the global surface gradient function containing pairwise the deflection components $\begin{bmatrix}\xi\\ \eta\end{bmatrix}$, A consisting of pairs of $-\begin{bmatrix}\partial_\varphi Y_{nm}(P)\\ \cos^{-1}\varphi\partial_\lambda Y_{nm}(P)\end{bmatrix}$, the solution vector x containing all coefficients ΔC_{nm} and K for this case being infinite dimensional and diagonal

$$K = \frac{c_n}{2n+1}\,\delta_{nk}\delta_{ml} \tag{16}$$

Where c_n are the degree variances defining the reproducing kernel. Then it can easily be shown that the elements of an arbitrary 2 x 2 sub-matrix of AKA^T, i.e. the covariances between ξ and η at two points P and P' become

$$K_{\xi\xi} = \sum_{n,m} \frac{c_n}{2n+1}\,\partial_\varphi Y_{nm}(P)\partial_\varphi Y_{nm}(P')$$

$$= \sum_n c_n\left[\partial_\varphi t\partial_{\varphi'} t P_n''(t) + \partial^2_{\varphi\varphi'} t P_n'(t)\right]$$

$$K_{\xi\eta} = \sum_{n,m} \frac{c_n}{2n+1}\cos^{-1}\varphi\partial_\lambda Y_{nm}(P)\partial_\varphi Y_{nm}(P') \tag{17a-c}$$

$$= \sum_n c_n \cos^{-1}\varphi\left[\partial_\varphi t\partial_{\lambda'} t P_n''(t) + \partial^2_{\varphi\lambda'} t P_n'(t)\right]$$

$$K_{\eta\eta} = \sum_{n,m} \frac{c_n}{2n+1}\cos^{-1}\varphi\cos^{-1}\varphi'\partial_\lambda Y_{nm}(P)\partial_\lambda Y_{nm}(P')$$

$$= \sum_n c_n\left[\cos^{-1}\varphi\cos^{-1}\varphi'\partial_\lambda t\partial_{\lambda'} t P_n''(t) + \partial^2_{\lambda\lambda'} t P_n'(t)\right]$$

with $t = \cos\psi_{PP'}$, in agreement with (Tscherning & Rapp, 1974) and (Moritz, 1972). Similarily a ξ and η element of KA^T becomes

$$K_{\Delta C_{nm}\xi} = \sum_{n,m} \frac{c_n}{2n+1}\,\partial_\varphi Y_{nm}(P) \tag{18a,b}$$

$$K_{\Delta C_{nm}\eta} = \sum_{n,m} \frac{c_n}{2n+1}\cos^{-1}\varphi\partial_\lambda Y_{nm}(P).$$

With eqs.(17) and (18) collocation formula (15b) can be established. The global limit - ξ and η given pairwise globally - is derived using eq. (15a). The result for this case is eq.(8) with F_n in the denominator replaced by

$$F_n + \alpha\,\frac{(2n+1)(p_\Phi + 9P_\Gamma + P_x)}{c_n}$$

The limit $\alpha \Rightarrow 0$ results in the solution of the geodetic b.v.p. of horizontal type, eq.(8). Hence a consistent building is obtained.

Finally one might ask what the relationship is between eq.(12b) and the very simple astronomical levelling :

$$\Delta N = N_P - N_Q = - \int_Q^P \epsilon ds$$

$$= - \int_Q^P \{\cos\alpha\xi + \sin\alpha\eta\} ds \qquad (19)$$

In both cases the observable, that goes into the integral is $\epsilon = \cos\alpha\xi + \sin\alpha\eta$, the output in (19) is the geoid difference ΔN between P and Q, in (12b) it can be made ΔN by considering $N = \frac{T}{\gamma}$ and evaluating (12b) at P and Q. However, whereas (19) requires ϵ only to be given along an arbitrary profile connecting P and Q, eq.(12b) requires global coverage. At first sight this seems a paradox situation. But (19) is derived from the differential $\frac{dN}{ds} = - \epsilon$ and ϵ contains only horizontal derivatives of T. Therefore ΔN can be computed locally, requiring for instance no continuation into outer space. Eq(12), on the other hand, has been derived from a solution of a b.v.p.. This implies that in this case by applying (12) T or any functional of T is not only derived on the boundary surface, but as well in its exterior. Thus we can conclude that the astronomical levelling is very convenient in many practical situations, eq.(12) involving the same observable, but given globally, is much more general.

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