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Geodetic Boundary Value Problem and Linear Inference

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The geodetic boundary value problem (b.v.p.) is discussed in the context of the definition of a complete physical model, consisting of observations, deterministic and stochastic model. The linearized classical b.v.p.'s in spherical approximation, scalar and vectorial, are solved by linear inference. Introducing a minimum principle the same approach is applied to the solution of the overdetermined geodetic b.v.p. The result of the overdetermined problem is worked out for a combination of the observables potential, scalar gravity, vertical gradient and vertical geometry. It is discussed together with a number of specializations and with a suggestion concerning the non-linear problem. The overdetermined case is treated in parametric form as well as in the form of condition equations. The stochastic interpretation of the method provides a means to analyse the propagation of observation and discretization errors.

keywords: physical geodesy, geodetic boundary value problem, overdetermined geodetic boundary value problem, least squares adjustment.

1. Introduction.

With the large increase during recent years of the number of available observations, with the advent of various new, very precise, measurement techniques, such as satellite altimetry and positioning by GPS, and with the development of very efficient processing algorithms, physical geodesy shall much more intrude into everyday practise in the coming years. In such a situation it is of fundamental importance that adequate physical models are available. Under physical model we understand, starting from a given problem, the complete sequence from the definition of the required precision and the resulting choice of observations, choice of the stochastic and deterministic model, linking of observations to the models, estimation process, quality control and mapping backwards (prediction) from the model to the "real world".

In such a context the classical work on the geodetic boundary value problem (b.v.p.) by Stokes (1849) and Molodenskii et al. (1962), as well as the modern work, reviewed for example in (Holota, 1978) or (Sansó, 1981), can be viewed as the derivation of the deterministic model in linearized form, respectively as investigation of the existence, uniqueness, and stability of the deterministic and in general, non-linear geodetic b.v.p., of course, having in mind a certain idealized set-up in terms of observable quantities. On the other hand, one could claim that least squares collocation, (Krarup, 1969) and (Moritz, 1972), and Krarup's integrated geodesy (Eeg and Krarup, 1975) represent already physical models in the above described sense. This is true, at least to a large extent. However, in these works emphasis is given especially to a proper treatment of the most fundamental complication in physical geodesy, its intrinsic underdeterminancy, for a field function, the gravitational field, is to be derived from the always finite number of observations.

Our intention is to take one step backwards and to treat the linearized unique and overdetermined geodetic b.v.p. by linear inference. This approach seems not only to have the advantage of being familiar to all geodesists and surveyors, it should also permit to solve in a straightforward manner the overdetermined geodetic b.v.p. in all possible configurations, just by applying the least squares principle. In a second step, a stochastic interpretation shall be given to the outcomes and observation error propagation and discretisation error can be discussed. Hence, the main purpose of this paper can be described as being threefold:

1. The introduction of the principle of linear inference as a means to solve the global linearized geodetic b.v.p., uniquely determined as well as overdetermined.
2. The explicit solution of the overdetermined geodetic b.v.p. in various configurations.
3. The stochastic interpretation of the results in terms of observation and discretization errors.

This way we also try to emphasize, that despite of the general intrinsic underdeterminancy in physical geodesy, it is very well possible in practise to achieve redundancy at all or many sampling points. The present work should be seen as one step further on our intended way to derive a complete physical model, the previous steps being (Rummel & Teunissen, 1982) and (Rummel, 1984).

2. Physical Model.

Our notion of a physical model is strongly influenced by Baarda's work (1967 and 1973). In short, it can be described as follows. Assume a practical problem is posed, the division of some real estate, the construction of a bridge, or e.g. the prediction of a satellite trajectory. In addition, in order the problem to be meaningful, the precision requirement is defined. Only the description of the required precision permits the selection of the proper observations and deterministic (mathematical) model, a point somewhat neglected in physical geodesy. The observations can be viewed as a mapping from "a piece of real world" to the world of numbers. They are associated with mathematical concepts (distance, velocity, force etc.) and compared with the corresponding quantities of the deterministic model. Depending on the problem and the kind of selected observables the deterministic model could in our case be either purely geometrical, kinematical, or dynamical. Typical to the observables in physical geodesy is that they are considered functionals of the earth's gravitational field. The process of linking observations (numbers) with a deterministic model (mathematical abstraction) could be denoted connection. It is the key to everything, for it means that it is possible to describe a piece of nature by a mathematical model. Without it physics would be disconnected from mathematics. The observed numbers shall never perfectly agree with their model counterparts. The discrepancies are due to imperfections of the measurement process as well as of the chosen deterministic model. They are described on an average basis by the chosen stochastic model. The observations are assumed to belong to a larger ensemble of stochastic samples, taken under similar circumstances. If simple second-order statistics cannot adequately describe the discrepancies, it is common practise to either bring the observations closer to the deterministic model by means of reductions, or to extend the deterministic model, so as to describe better the taken observations. For a more detailed discussion it is referred to (Rummel, 1984). After connection of the observed numbers with the deterministic model and after introduction of a proper stochastic model, the estimation process and the internal quality control is carried out. Finally, the estimated parameters are mapped back into the physical environment, which means they are given a physical interpretation, and controlled externally by independent experiments. In case the external control meets the beforehand defined standard the process is completed. In Figure 1 the basic ingredients of the physical model are shown schematically.

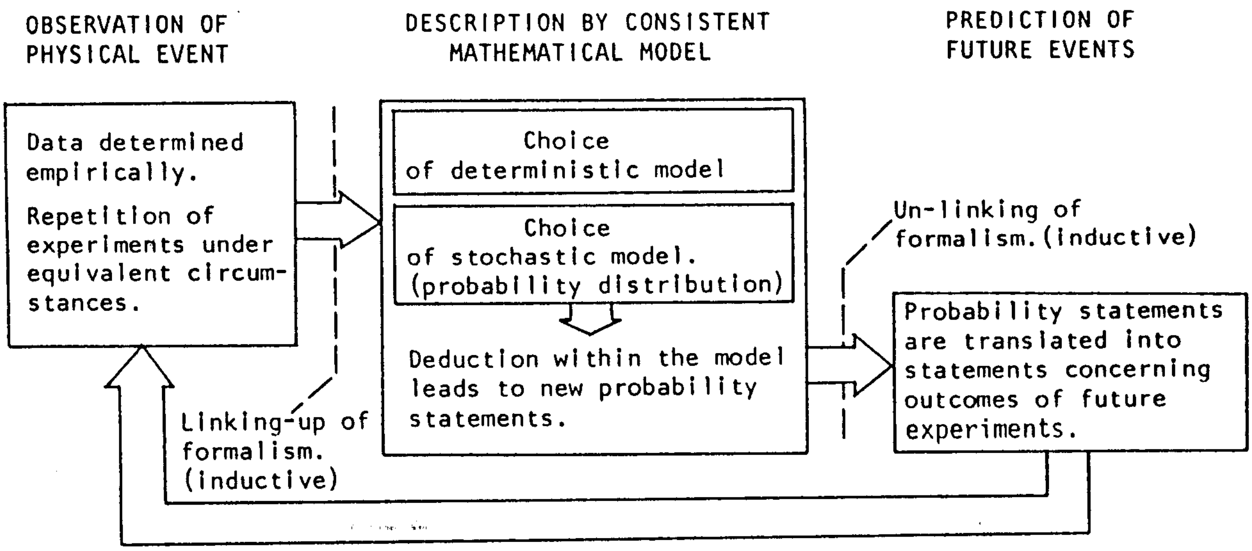

Figure 1

In case the results do not meet the desired standard, the cause is in general either an inadequate choice of the observations or of the deterministic model. In physical geodesy, where a field quantity is to be reconstructed from a finite number of observations, the model error could be divided into the discretization error and the model error in the narrow sense (the error of the continous model). Only the latter, e.g. the error of linearization, is usually treated in theoretical studies of the geodetic b.v.p.

In summary, any application case in physical geodesy requires

- the proper problem definition, that has to include the definition of the quality standard to be achieved,
- the selection of observation types and spatial distribution, stochastic and deterministic model in such a manner, that observation, discretization, and model (in the narrow sense) error meet the required standard,
- internal as well as external quality control.

3. Geodetic B.V.P. and Linearization.

In this chapter the selection of the deterministic model is discussed for the case of the classical geodetic b.v.p. No observation and discretization errors are considered. Hence the problem definition is: Given at all points of the earth's surface S, the gravity potential differences C and the gravity vector $\underline{g}$, expressed by the scalar gravity g, and the astronomical latitude and longitude, Φ and Λ, determine S and the gravitational potential in the space outside S. Thereby it is assumed that V, the gravitational potential, is a harmonic function in the space outside S and regular at infinity.

This problem is aimed to be solved ourdays with a relative precision of 10^{-8} (uncertainty of S relative to the earth's radius R). This implies for example that the earth's topography and atmosphere have to be carefully taken into account, (see (Moritz, 1974) and (Mather, 1974)), and that the observables are, as usual, corrected for the luni-solar tidal effect. On the other hand, it also implies that a large variety of time variable effects, such as pole tide, crustal deformation, groundwater variations a.s.o. can be neglected.

The problem of determining S, as defined above, represents a free, oblique-derivative b.v.p., compare e.g. (Grafarend, 1975). The first step towards a solution is a <u>linearization</u>. The process of linearization shall be shown explicitly for the potential difference $C(\underline{x},\underline{x}_0) = W(\underline{x}) - W(\underline{x}_0)$, where W is the gravity potential at an arbitrary point, expressed by its Cartesian coordinate vector $\underline{x}$ and the gravity potential at the datum point $\underline{x}_0$. The gravity potential W consists of the centrifugal part $Z(\underline{x})$, which is assumed to be known, and the gravitational part $V(\underline{x})$:

$$W(\underline{x}) = Z(\underline{x}) + V(\underline{x}) \tag{3.1}$$

The gravitational potential V can be expressed in a chosen parameter form by

$$V(x) = u^0 \sum_{k=1}^{\infty} c_k e_k(\underline{x}) \tag{3.2}$$

with scale factor $u^0 = \frac{GM_0}{R}$, c_k dimensionless coefficients and $e_k(\underline{x})$ base functions fulfilling Laplace' partial differential equation outside S. For example, the base functions could be the fully normalized complex spherical harmonics

$$\left(\frac{R}{r}\right)^{n+1}\bar{Y}_{nm}(\varphi,\lambda) = \left(\frac{R}{r}\right)^{n+1}\bar{P}_n^m(\varphi)e^{im\lambda} = \left(\frac{R}{r}\right)^{n+1}[\bar{R}_{nm}(\varphi,\lambda) + i\bar{S}_{nm}(\varphi,\lambda)] \tag{3.3}$$

of degree n and order m, where

$$\left.\begin{matrix}\bar{R}_{nm}(\varphi,\lambda) \\ \\ \bar{S}_{nm}(\varphi,\lambda)\end{matrix}\right\} = \bar{P}_{nm}(\sin\varphi)\left\{\begin{matrix}\cos m\lambda \\ \\ \sin m\lambda\end{matrix}\right. \tag{3.4}$$

In this case the c_k are easily related to the spherical harmonic coefficients $\bar{C}_{nm}$ and $\bar{S}_{nm}$. Furtheron we define for later use, that the base functions $e_n(\underline{x})$ are orthonormal on the unit sphere $\sigma\{0,1\}$. Hence for r = R, it holds

$$\frac{1}{4\pi}\int_\sigma e_k(\underline{x})e_\ell(\underline{x})d\sigma = \delta_{k\ell} \tag{3.5}$$

With eq. (3.2) the gravity potential can now be written

$$W(\underline{x}) = Z(\underline{x}) + u^0 \sum_k c_k\, e_k(\underline{x}) \tag{3.6}$$

with unknowns c_k and $\underline{x}$. With approximate values c_k^0 and $\underline{x}^0$ linearization gives

$$W(\underline{x}) = [Z(\underline{x}^0) + u^0 \sum_k c_k^0 e_k(\underline{x}^0)] + [\partial_j Z(\underline{x}^0) + u^0 \sum_k c_k^0 \partial_j c_k(\underline{x}^0)]\Delta x^j$$
$$+ u^0 \sum_k \Delta c_k e_k(\underline{x}^0) \qquad (3.7)$$

where $\Delta\underline{x} = \underline{x} - \underline{x}^0$ and $\Delta c_k = c_k - c_k^0$. The neglected second and higher order terms become part of the model errors. With the definitions

$$U(\underline{x}) = Z(\underline{x}) + u^0 \sum_k c_k^0 e_k(\underline{x}) \qquad (3.8)$$

for the normal gravity potential and

$$T(\underline{x}) = u^0 \sum_k \Delta c_k e_k(\underline{x}) \qquad (3.9)$$

for the disturbing potential, eq. (3.7) becomes

$$W(\underline{x}) = U(\underline{x}^0) + \partial_j U(\underline{x}^0)\Delta x^j + T(\underline{x}^0)$$

or with the potential anomaly

$$\Delta W(\underline{x}) = \partial_j U(\underline{x}^0)\Delta x^j + T(\underline{x}^0) \qquad (3.10)$$

Carrying out the same linearization at the datum point $\underline{x}_0$, yields the observable potential difference anomaly

$$\Delta C(\underline{x},\underline{x}_0) = -\Delta W(\underline{x}_0) + \partial_j U(\underline{x}^0)\Delta x^j + T(\underline{x}^0) \qquad (3.11)$$

If we consider now $\underline{x}$ and $\underline{x}^0$ to represent discrete points P_i and P_i^0 with $i = 1,\dots,I$ and the base functions $e_n(\underline{x})$ run from $k = 1,\dots,K$, eq. (3.11) becomes a linear system of equations of dimension $I\cdot(3I+K)$ in the unknowns Δx_i^j, where $j = 1,2,3$, and Δc_k and with one additional column for the datum parameter ΔW_0. It is assumed that the points P_i represent a uniform coverage on the sphere σ, arranged in a so-called ε-net; for a definition we refer to (Moritz, 1975; ch. 2). For $I \to \infty$ the ε-net shall converge towards a global, continous coverage and $K \to \infty$ (a complete, infinite countable base).

The linearized model, written for all four observable anomalies, ΔC, Δg, $\Delta\Phi$, and $\Delta\Lambda$, results - apart from the datum parameter ΔW_0 - in a system of dimension 4I·(3I+K), compare also (Knickmeyer, 1985). It is the discrete analogon of the linearized global geodetic b.v.p. The problem is still oblique derivative. At each point P_i the coordinate corrections Δx^i could be eliminated, by forming certain linear combinations of the four anomalies, as shown in (Grafarend, 1979). This results in the classical boundary conditions. On purpose this step is not taken here. Finally, it should be mentioned that (besides ΔW_0) the system has a rank defect of three. It is invariant with respect to three translations, see (Krarup, 1973).

For finite dimension, the anomalies arranged e.g. in an equi-angular grid, the system could be solved by least-squares methods, depending on the size of I and K, with methods very similar to (Colombo, 1981). However it is very difficult to derive an analytical solution for the global limit ($I \rightarrow \infty$ and $K \rightarrow \infty$) case. Because of this difficulty further approximations are introduced. First, the coefficients of the unknowns Δx_i^j and Δc_k are computed in <u>spherical approximation</u>. In our example, this means that $\partial_j U(\underline{x}^0)$ in eq. (3.11) is approximated by $\partial_j (\frac{GM}{r})|_{\underline{x}_0}$ with $r = [x_0^2+y_0^2+z_0^2]^{\frac{1}{2}}$. With Δx^j expressed in the local geodetic triad, eq. (3.11) reads now

$$\Delta C(\underline{x},\underline{x}_0) = -\Delta W(\underline{x}_0) - \gamma(\underline{x}^0)\Delta z + T(\underline{x}^0) \tag{3.12}$$

where $\gamma(\underline{x}) = \frac{GM}{r^2}$. To stress the point, the linearized model still refers to the normal potential defined in eq. (3.8), only its coefficients are computed in spherical approximation. This type of approximation underlies, for example, the solution of the Molodenskii problem by surface layer given in (Heiskanen & Moritz, 1967; ch. 8-6 and 8-7). It is this step that results in a significantly simplified structure of the coefficient matrix, on the expense that the unknowns can only be determined with a relative precision of the order of the flattening ($\approx \frac{1}{300}$).

Finally, the coefficient matrix can be further simplified by computing all coefficients of the linear model at one mean radius $r = R$ instead of at the individual spherical distances r of the points $\underline{x}_i^0$. We could call it <u>constant radius approximation</u>. With this further simplification the Stokes problem is obtained, formulated for the telluroid. In our example $\gamma(\underline{x}^0)$ is replaced by $\gamma = \frac{GM}{R^2}$. Hence eq. (3.12) becomes

$$\Delta C(\underline{x},\underline{x}_0) = -\Delta W(\underline{x}_0) - \gamma\Delta z + T(\underline{x}^0) \tag{3.13}$$

The approximations applied to the coefficients can be interpreted geometrically as a slight change of the base vectors, that span the tangent space in $\{\underline{x}^0, U\}$.

Carrying out the same type of linearization for the scalar gravity, and the astronomical latitude and longitude for an arbitrary point P_i we obtain the well-known equations, compare e.g. (Rummel, 1984)

$$\begin{bmatrix} \Delta C \\ \Delta g \\ \Delta \Phi \\ \Delta \Lambda \end{bmatrix}_i = \begin{bmatrix} -\Delta W \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -\gamma \\ 0 & 0 & -\frac{2\gamma}{R} \\ \frac{1}{R} & 0 & 0 \\ 0 & \frac{1}{R\cos\varphi} & 0 \end{bmatrix}_i \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} + \begin{bmatrix} T \\ -\frac{\partial T}{\partial r} \\ -\frac{1}{R\gamma}\frac{\partial T}{\partial \varphi} \\ -\frac{1}{R\gamma\cos^2\varphi}\frac{\partial T}{\partial \lambda} \end{bmatrix} \tag{3.14}$$

<u>Telluroid.</u>

We did not discuss how the approximate coordinates $x^{0,j}$ are defined. In principle, the solution of the linearized problem has to be independent of the choice of the Taylor points $\underline{x}^0$, as long as the $\underline{x}^0$ lie inside the radius of convergence of the problem, and as long as one is prepared to compute the solution in several iterations. Hence it does not matter whether the $\underline{x}^0$ are derived simply from topographic maps or in a systematic manner from the observations themselves. The latter case implies that the $\underline{x}^0$ are derived from the so-called observable natural coordinates $\{W,\Phi,\Lambda\}$ or $\{g,\Phi,\Lambda\}$, by means of one of the so-called mappings, compare the discussion in (Bakker, 1983). The mapping $\{W,\Phi,\Lambda\}_{\underline{x}} = \{U,\varphi,\lambda\}_{\underline{x}^0}$ is denoted Marussi mapping, $\{g,\Phi,\Lambda\}_{\underline{x}} = \{\gamma,\varphi,\lambda\}_{\underline{x}^0}$ gravity mapping. The complete set of approximate points defines the telluroid. If either gravity or Marussi mapping is applied, the numericals value of the corresponding anomalies shall be zero, see the left hand side of eq. (3.14). As discussed in (van Es, 1983) the distance $\underline{\Delta x}$ between the telluroid derived from the gravity mapping and S, shall be greater by a factor $\frac{\Delta W}{\gamma}$ than that of the classical "Marussi" telluroid.

Solution of the Non-linear B.V.P.

The model errors introduced by the linearization and by the spherical and constant radius approximations can be corrected for by careful backward substitution in an iterative solution, see (Rummel & Teunissen, 1982).

Start with initial values $\underline{x}^0$ and c_n^0. Solution of the linearized b.v.p. gives $\Delta\underline{x}$ and Δc_n. Hence it is

$$\underline{x}^1 = \underline{x}^0 + \Delta\underline{x} \qquad \& \qquad c_n^1 = c_n^0 + \Delta c_n$$

The non-linear expressions, e.g. eq. (3. 6) are then linearized using $\underline{x}^1$ and c_n^1, which means in fact a careful up-date of all anomalies. Then the second iteration loop can be started. No difference seems to exist between the convergence measures known from non-linear adjustment problems and the problem at hand. The former is treated in (Teunissen, 1985).

4. Linearized Geodetic B.V.P. and Linear Inference.

The general scope of our paper is a discussion of the definition and proper use of a closed physical model in physical geodesy. In this chapter the technique is introduced that is applied throughout for our derivations. The idea is to write the linear geodetic b.v.p. in spherical approximation in the form of a linear system and to apply to it the procedures well-known from least-squares adjustment. It shall be possible this way to address a whole class of uniquely determined and overdetermined geodetic b.v.p., formulate the overdetermined cases in the form of observation and condition equations, analyze rank deficiencies and study the error propagation and discretization errors.

In order to attain maximal symmetry in the linear system, we define all quantities dimensionless:

$$dW = \frac{\Delta C + \Delta W_0}{U^0}, \quad dg = \frac{\Delta g}{\gamma^0}, \quad d\Phi = \Delta\Phi, \quad d\Lambda = \Delta\Lambda,$$

$$dx = \frac{\Delta x}{R}, \quad dy = \frac{\Delta y}{R}, \quad dz = \frac{\Delta z}{R},$$

where $U^0 = \frac{GM_0}{R}$, and $\gamma^0 = \frac{GM_0}{R^2}$, and

$$dT = \frac{T}{U^0} = \sum_k \Delta c_k e_k = \sum_{n,m} \beta_{nm} Y_{nm} \qquad (4.1)$$

We see that two types of base function representations are defined for dT. Although, in principle the first one would suffice, the second one is convenient in cases where lateral derivatives occur, for it provides an immediate connection to the spherical harmonics, eq. (3.3) and their coefficients. Furtheron we notice, that for convenience the datum anomaly ΔW_0 has been merged with ΔC. We shall return to this point later. The coordinate corrections dx^i are understood to be defined in the local geodetic triad. With these definitions the linear system (3.14) becomes

$$\begin{pmatrix} dW \\ dg \\ d\Phi \\ \cos\varphi d\Lambda \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} + \begin{pmatrix} \sum_k \Delta c_k \, e_k \\ -\sum_k \Delta c_k \, \partial_r \, e_k \\ -\sum_k \Delta c_k \, \partial_\varphi \, e_k \\ -\sum_k \frac{\Delta c_k}{\cos\varphi} \partial_\lambda \, e_k \end{pmatrix} \tag{4.2}$$

or analogous expressions with spherical harmonic base functions Y_{nm}. For simplicity it is assumed, that the translational invariance is removed by setting the three first degree coefficients β_{10}, β_{1-1}, β_{11} zero.
In the sequel four cases of the uniquely defined geodetic b.v.p. shall be treated.

4.1 Classical Stokes.

Since the classical Stokes solution is known very well, it is used to explain our approach. It starts from solely dW and dg. Hence with eq. (4.2) in point P_i it is

$$\begin{aligned} dW_i &= -dz_i + \sum_k \Delta c_n \, e_k(P_i) \\ dg_i &= -2dz_i + \sum_k \Delta c_n (k+1) e_k(P_i) \end{aligned} \tag{4.3a,b}$$

The respective weights of dW and dg are denoted p_W and p_g. They can later be interpreted stochastically as inverse variances σ_W^2 and σ_g^2. With eqs. (4.3) formulated at each point of the ε-net a linear system of type

$$\underline{y} = \underline{A}\,\underline{x} \tag{4.4}$$

is obtained with a priori weight matrix $\underline{P}$. The dimensions are dim $\underline{y}$ = (2I·1), dim $\underline{A}$ = 2I·(I+K), dim $\underline{x}$ = (I+K)·1 and dim $\underline{P}$ = (2I·2I). Alternatively to the matrix notation we use[x)]

$$\underline{y} = \begin{pmatrix} [dW]_i \\ \\ [dg]_i \end{pmatrix} \tag{4.5}$$

$$\underline{A} = \begin{pmatrix} [-\delta_{ij}]_{ii} & [e_k(P_i)]_{ik} \\ \\ [-2\delta_{ij}]_{ii} & [(k+1)e_k(P_i)]_{ik} \end{pmatrix} \tag{4.6}$$

$$\underline{x} = \begin{pmatrix} [dz]_i \\ \\ [\Delta c]_k \end{pmatrix} \tag{4.7}$$

$$\underline{P}_y = \begin{pmatrix} [p_W\delta_{ij}]_{ii} & [0]_{ii} \\ \\ [0]_{ii} & [p_g\delta_{ij}]_{ii} \end{pmatrix} \tag{4.8}$$

The notation is self-explanatory. Inside the brackets the matrix elements are stated, where δ_{ij} is the Kronecker delta. Outside the brackets, the first index denotes the rows, the second the columns. Throughout we assume $\{i|1 \le i \le I\}$, $\{k|1 \le k \le K\}$, or when the basis Y_{nm} is used degree $\{n|0 \le n \le N\}$ and order $\{m|-n \le m \le n\}$ where in any case elimination of the first degree is assumed.

It is common practise to eliminate dz_i from eqs. (4.3), which results in the so-called fundamental equation of physical geodesy. This step shall not be taken here, to show that this step is not essential in obtaining a solution of the geodetic b.v.p. Actually, since $\underline{A}$ of eq. (4.4) is regular for $I \to \infty$ and $K \to \infty$, a solution of (4.4) would simply require its inversion. Since the inversion could be rather cumbersome, we compute it by $\underline{A}^{-1} = (\underline{A}^T\underline{A})^{-1}\underline{A}^T$. Hence, formally the classical least-squares approach is followed.

It is

$$\hat{\underline{x}} = \underline{Q}_x \, \underline{A}^T \, \underline{P}_y \, \underline{y} \tag{4.9}$$

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x) In order to keep the formulas compact k is to be understood at the same time as index of the base functions and as (spherical harmonic) degree.

where

$$\underline{Q}_x = \underline{N}^{-1} = (\underline{A}^T \underline{P}_y \underline{A})^{-1} \tag{4.10}$$

Estimated quantities are denoted by "^", but at this point no stochastic interpretation is given to the results. Insertion of eqs. (4.6) and (4.8) yields

$$\underline{N} = \begin{pmatrix} [(p_w+4p_g)\delta_{ij}]_{ii} & [-(p_w+2(k+1)p_g)e_k(P_i)]_{ik} \\ & [(p_w+(k+1)^2 p_g)\delta_{k\ell}]_{kk} \end{pmatrix} \tag{4.11}$$

For the derivation of element N_{22} of $\underline{N}$ the global limit of the ε-net together with the orthogonal relationship of the $e_k(P)$ has been used:

$$\lim_{I\to\infty} \sum_i e_k(P_i)e_\ell(P_i) = \frac{1}{4\pi} \int_\sigma e_k(P)e_\ell(P)d\sigma_P = \delta_{k\ell} \tag{4.12}$$

With the well-known relations

$$Q_{22} = (N_{22}-N_{21}N_{11}^{-1}N_{12})^{-1}$$

$$Q_{12} = -N_{11}^{-1}N_{12}Q_{22} \tag{4.13a-c}$$

$$Q_{11} = N_{11}^{-1} + N_{11}^{-1}N_{12}Q_{22}N_{21}N_{11}^{-1}$$

we find for the elements of Q_x

$$Q_{11} = [\frac{1}{p_w+4p_g} (\delta_{ij} + \sum_k \frac{(p_w+2(k+1)p_g)^2}{(k-1)^2 p_g p_w} e_k(P_i)e_k(P_j))]_{ii}$$

$$Q_{12} = [\frac{p_w+2(k+1)p_g}{(k-1)^2 p_g p_w} e_k(P_i)]_{ik} \tag{4.14a-c}$$

$$Q_{22} = [\frac{p_w+4p_g}{(k-1)^2 p_w p_g} \delta_{k\ell}]_{kk}$$

Furtheron it is

$$\underline{A}^T \underline{P}_y \underline{y} = \begin{pmatrix} -[p_w dW + 2p_g dg]_i \\ \\ [p_w dW_k + p_g(k+1)dg_k]_k \end{pmatrix} \quad , \tag{4.15}$$

where we used

$$\lim_{I\to\infty} \sum_i dW_i \, e_k(P_i) = \frac{1}{4\pi} \int_\sigma dW(P)e_k(P)d\sigma_P = dW_k \tag{4.16}$$

and analogously

$$\lim_{I\to\infty} \sum_i dg_i \, e_k(P_i) = dg_k \quad . \tag{4.17}$$

The inverse relations to eqs. (4.16) and (4.17) are

$$\begin{aligned} dW(P) &= \sum_k dW_k \, e_k(P) \quad , \\ dg(P) &= \sum_k dg_k \, e_k(P) \quad . \end{aligned} \tag{4 18a-b}$$

Eq. (4.9) gives with (4.14) and (4.15)

$$\Delta\hat{c}_k = \frac{1}{k-1} (dg_k - 2dW_k) \quad , \tag{4.19}$$

the well-known solution of the Stokes problem (in the spectral domain). Note the solution does not depend on the weights p_g and p_w.

For $d\hat{z}_i$ (Bruns' equation) we find

$$\begin{aligned} d\hat{z}_i &= \frac{1}{p_w+4p_g} [-(p_w dW+2p_g dg)_i + \sum_k (p_w+2(k+1)p_g)\Delta\hat{c}_k \, e_k(P_i)] \\ &= -\sum_k dW_k \, e_k(P_i) + \sum_k \frac{1}{k-1} (dg_k - 2dW_k) e_k(P_i) \end{aligned} \tag{4.20}$$

Finally, eqs. (4.19) and (4.20) can be brought into the usual - but still dimensionless - form

$$\begin{aligned} d\hat{T}(P) &= \sum_k \Delta\hat{c}_k \, e_k(P) \\ &= d(GM) + \frac{1}{4\pi} \int_\sigma St(\psi_{PQ})(dg-2dW)_Q d\sigma_Q \end{aligned} \tag{4.21}$$

and

$$d\hat{z}(P) = -dW(P) + d(GM) + \frac{1}{4\pi} \int_\sigma St(\psi_{PQ})(dg-2dW)_Q d\sigma_Q \quad , \qquad (4.22)$$

or when splitting back dW into relative potential difference $dC = \frac{\Delta C}{U^0}$ and height datum effect $dW_0 = \frac{\Delta W_0}{U^0}$:

$$d\hat{z}(P) = -dW_0 - dC(P) + d(GM) + \frac{1}{4\pi} \int_\sigma St(\psi_{PQ})(dg-2dC)_Q d\sigma_Q \quad . \qquad (4.23)$$

In eq. (4.21) the zero-order term has been interpreted as a relative scale-error $d(GM) = \frac{\delta(GM)}{GM_0}$, $St(\psi)$ is the Stokes integral kernel.

We see, that our approach leads to the classical integral formulas in physical geodesy. In most textbooks, implicitely the Marussi mapping, $dW \overset{def.}{=} 0$ is assumed. One feature of our approach is that it yields automatically the error propagation, if $\underline{Q}_x$ is interpreted statistically.

4.2 Stokes and Vening-Meinesz.

Although traditionally, when solving Stokes problem, the point of departure is the four eqs. (4.2), only the two for dW and dg are employed for the solution. From them one solves for dT, and after insertion into the Bruns' equation dz. Independently dT is used to derive the deflections of the vertical $\xi = -\frac{1}{r\gamma^0}\frac{\partial T}{\partial \varphi}$ and $\eta = -\frac{1}{r\gamma^0 \cos\varphi}\frac{\partial T}{\partial \lambda}$ from dT, the Vening-Meinesz equations.

We want to show that the four equations (4.2) can be treated in one step, the so-called vectorial Stokes problem. Since the derivation runs analogous to the previous one, details are left out.

We use now the representation of dT

$$dT = \sum_{n,m} \beta_{nm} Y_{nm} \qquad (4.1)$$

It is

$$\underline{y} = \begin{pmatrix} [dW]_i \\ [dg]_i \\ [d\Phi]_i \\ [\cos\varphi \, d\Lambda]_i \end{pmatrix} \qquad (4.24)$$

$$\underline{A} = \begin{pmatrix} [0]_{ii} & [0]_{ii} & [-\delta_{ij}]_{ii} & [Y_{nm}(P_i)]_{i(nm)} \\ [0]_{ii} & [0]_{ii} & [-2\delta_{ij}]_{ii} & [(n+1)Y_{nm}(P_i)]_{i(nm)} \\ [\delta_{ij}]_{ii} & [0]_{ii} & [0]_{ii} & [-\partial_\varphi\, Y_{nm}(P_i)]_{i(nm)} \\ [0]_{ii} & [\delta_{ij}]_{ii} & [0]_{ii} & [\frac{-1}{\cos\varphi}\,\partial_\lambda\, Y_{nm}(P_i)]_{i(nm)} \end{pmatrix} \tag{4.25}$$

$$\underline{x} = \begin{pmatrix} [dx]_i \\ [dy]_i \\ [dz]_i \\ [\beta]_{(nm)} \end{pmatrix} \tag{4.26}$$

$$\underline{P}_y = \begin{pmatrix} [p_w\delta_{ij}]_{ii} & [0]_{ii} & [0]_{ii} & [0]_{ii} \\ & [p_g\delta_{ij}]_{ii} & [0]_{ii} & [0]_{ii} \\ & & [p_\Phi\delta_{ij}]_{ii} & [0]_{ii} \\ \text{symmetric} & & & [p_\Lambda\delta_{ij}]_{ii} \end{pmatrix} \tag{4.27}$$

It is assumed, that $p_\Phi = p_\Lambda$.

Before we derive $\underline{N}$ some preparations are needed. For Laplace's surface spherical harmonics (r=R) holds, compare (Heiskanen & Moritz, 1967; p. 20)

$$\partial_\varphi^2\, Y_{nm} - \tan\varphi\,\partial_\varphi\, Y_{nm} + \cos^{-2}\varphi\,\partial_\lambda^2\, Y_{nm} = -n(n+1)Y_{nm}$$

Hence it is $\cos^{-2}\varphi[\cos\varphi\,\partial_\varphi(\cos\varphi\,\partial_\varphi) + \partial_\lambda^2]Y_{nm} = -n(n+1)Y_{nm}$ or with the surface gradient operator (see (Meissl, 1971))

$$\text{Grad} = \cos^{-1}\varphi \begin{pmatrix} \cos\varphi\,\partial_\varphi \\ \partial_\lambda \end{pmatrix}$$

we find that

$$\text{Lap}\; Y_{nm} = -n(n+1)Y_{nm} \quad , \tag{4.28}$$

where the Laplacean Lap is the surface divergence of the surface gradient Grad. As a consequence of Green's first identity we have for twice differentiable functions f and g on the unit sphere :

$$\int_\sigma (\text{Grad } f, \text{ Grad } g)d\sigma = -\int_\sigma f \text{ Lap } g \; d\sigma = -\int_\sigma g \text{ Lap } f \; d\sigma \quad . \qquad (4.29)$$

With f and g = Y_{nm} eqs. (4.28) and (4.29) become

$$\begin{aligned}\frac{1}{4\pi}\int_\sigma (\text{Grad } Y_{nm}, \text{ Grad } Y_{k\ell})d\sigma &= \\ &= -\frac{1}{4\pi}\int_\sigma Y_{nm} \text{ Lap } Y_{k\ell} \; d\sigma = \\ &= n(n+1)\frac{1}{4\pi}\int_\sigma Y_{nm} Y_{k\ell} \; d\sigma = \\ &= n(n+1)\delta_{nk}\delta_{m\ell} \qquad (4.30)\end{aligned}$$

The orthogonality relationship and (4.30) shall be used to derive the normal matrix $\underline{N}$.

It is

$$\underline{A}^T\underline{P}_y\underline{y} = \begin{pmatrix} [p_\phi d\Phi]_i \\ [p_\Lambda \cos\varphi \; d\Lambda]_i \\ -[2p_g dg + p_W dW]_i \\ [p_W dW_{nm} + p_g(n+1)dg_{nm}]_{(nm)} - \frac{1}{4\pi}\int_\sigma [d\Phi p_\phi \partial_\varphi Y_{nm} + d\Lambda p_\Lambda \partial_\lambda Y_{nm}]d\sigma \end{pmatrix} \qquad (4.31)$$

where again

$$\lim_{I\to\infty} \sum_i dW(P_i)Y_{nm}(P_i) = \frac{1}{4\pi}\int_\sigma dW(P)Y_{nm}(P)d\sigma = dW_{nm}$$

and

$$N_{11} = \begin{pmatrix} [p_\phi\delta_{ij}]_{ii} & [0]_{ii} & [0]_{ii} \\ & [p_\Lambda\delta_{ij}]_{ii} & [0]_{ii} \\ \text{symmetric} & & [(p_W+4p_g)\delta_{ij}]_{ii} \end{pmatrix} \qquad (4.32a)$$

$$N_{12} = \begin{pmatrix} [-p_\Phi \; \partial_\varphi \; Y_{nm}(P_i)]_{i(nm)} \\ [-p_\Lambda \; \cos^{-1} \; \varphi_i \; \partial_\lambda \; Y_{nm}(P_i)]_{i(nm)} \\ [-(p_w+2p_g(n+1))Y_{nm}(P_i)]_{i(nm)} \end{pmatrix} \tag{4.32b}$$

$$N_{22} = [[p_w + p_g(n+1)^2 + p_\Phi n(n+1)]\delta_{nk}\delta_{m\ell}]_{(nm)(nm)} \tag{4.32c}$$

Inversion of $\underline{N}$ yields the elements of $\underline{Q}$:

$$Q_{11}^{(11)} = [\frac{\delta_{ij}}{p_\Phi}]_{ii} + [\sum_{nm} \frac{p_w+4p_g}{p_w p_g(n-1)^2} \partial_\varphi Y_{nm}(P_i)\partial_\varphi Y_{nm}(P_j)]_{ii}$$

$$Q_{11}^{(12)} = [\sum_{nm} \frac{p_w+4p_g}{p_w p_g(n-1)^2} \partial_\varphi Y_{nm}(P_i)\cos^{-1} \varphi_j \partial_\lambda Y_{nm}(P_j)]_{ii}$$

$$Q_{11}^{(13)} = [\sum_{nm} \frac{p_w+2p_g(n+1)}{p_w p_g(n-1)^2} \partial_\varphi Y_{nm}(P_i) Y_{nm}(P_j)]_{ii} \tag{4.33e-f}$$

$$Q_{11}^{(22)} = [\frac{\delta_{ij}}{p_\Lambda}]_{ii} + [\sum_{nm} \frac{p_w+4p_g}{p_w p_g(n-1)^2} \cos^{-1} \varphi_i \partial_\lambda Y_{nm}(P_i)\cos^{-1} \varphi_j \partial_\lambda Y_{nm}(P_j)]_{ii}$$

$$Q_{11}^{(23)} = [\sum_{nm} \frac{p_w+2p_g(n+1)}{p_w p_g(n-1)^2} \cos^{-1} \varphi_i \partial_\lambda Y_{nm}(P_i) Y_{nm}(P_j)]_{ii}$$

$$Q_{11}^{(33)} = [\frac{\delta_{ij}}{(p_w+4p_g)}]_{ii} + [\sum_{nm} \frac{[p_w+2p_g(n+1)]^2}{(p_w+4p_g)(p_w p_g(n-1)^2)} Y_{nm}(P_i) Y_{nm}(P_j)]_{ii}$$

$$Q_{12} = \begin{pmatrix} \frac{p_w+4p_g}{p_w p_g(n-1)^2} \partial_\varphi Y_{nm}(P_i) \\ \frac{p_w+4p_g}{p_w p_g(n-1)^2} \cos^{-1} \varphi_i \partial_\lambda Y_{nm}(P_i) \\ \frac{p_w+4p_g}{p_w p_g(n-1)^2} Y_{nm}(P_i) \end{pmatrix} \tag{4.34}$$

$$Q_{22} = [\frac{p_w+4p_g}{p_w p_g(n-1)^2} \delta_{nk}\delta_{m\ell}]_{(nm)(nm)} \tag{4.35}$$

After a lengthy derivation we find

$$\hat{\beta}_{nm} = \frac{dg_{nm}-2dW_{nm}}{n-1} \qquad \text{(Stokes)} \qquad (4.36)$$

and

$$d\hat{x}_i = d\Phi_i + \sum_{nm} \partial_\varphi Y_{nm}(P_i) \frac{dg_{nm}-2dW_{nm}}{n-1} \qquad \text{(Vening-Meinesz)} \quad (4.37a\text{-}b)$$

$$d\hat{y}_i = \cos\varphi_i \, d\Lambda_i + \sum_{nm} \cos^{-1}\varphi_i \, \partial_\lambda Y_{nm}(P_i) \frac{dg_{nm}-2dW_{nm}}{n-1}$$

$$d\hat{z}_i = -dW_i + \sum_{nm} Y_{nm}(P_i) \frac{dg_{nm}-2dW_{nm}}{n-1} \qquad \text{(Bruns-Stokes)} \qquad (4.38)$$

These expressions can very easily be transformed into the corresponding well-known integral formulas. For eqs. (4.36) and (4.38) the result is eqs. (4.21) and (4.23), eqs. (4.37) become

$$\begin{aligned} d\hat{x}(P) &= d\Phi(P) + \frac{1}{4\pi}\int_\sigma \frac{dSt(\psi)}{d\psi} \cos\alpha (dg-2dW)d\sigma \\ d\hat{y}(P) &= \cos\varphi_P \, d\Lambda + \frac{1}{4\pi}\int_\sigma \frac{dSt(\psi)}{d\psi} \sin\alpha (dg-2dW)d\sigma \end{aligned} \qquad (4.39a\text{-}b)$$

4.3 Hotine.

The classical Hotine integral (Hotine, 1969) assumes that besides scalar gravity the geometry of the boundary surface S is given. As a consequence the corresponding b.v.p. is fixed, not free.

We shall treat, in short, two versions of this problem, where we assume one time potential differences to be given, the other time scalar gravity, besides the surface coordinates x, y, z in the local geodetic triad, obtained for example from GPS and altimeter measurements.

CASE ONE: Given C and x, y, z of S.

In analogy to eq. (4.2) we write for an arbitrary point

$$\begin{pmatrix} dW \\ dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} + \begin{pmatrix} \sum_k \Delta c_k e_k \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad (4.40)$$

or for the complete system

$$\begin{pmatrix} [dW]_i \\ [dx]_i \\ [dy]_i \\ [dz]_i \end{pmatrix} = \begin{pmatrix} [0]_{ii} & [0]_{ii} & [-\delta_{ij}]_{ii} & [e_k(P_i)]_{ik} \\ [\delta_{ij}]_{ii} & [0]_{ii} & [0]_{ii} & [0]_{ik} \\ [0]_{ii} & [\delta_{ij}]_{ii} & [0]_{ii} & [0]_{ik} \\ [0]_{ii} & [0]_{ii} & [\delta_{ij}]_{ii} & [0]_{ik} \end{pmatrix} \begin{pmatrix} [dx]_i \\ [dy]_i \\ [dz]_i \\ [\Delta c]_k \end{pmatrix} \quad (4.40)$$

Following the approach explained in chapter 4.1 we find for Q_x

$$Q_{11} = \begin{pmatrix} [\frac{\delta_{ij}}{p_x}]_{ii} & [0] \\ [0] & [\frac{\delta_{ij}}{p_y}]_{ii} \end{pmatrix} \quad (4.42a)$$

$$Q_{12} = [0] \quad (4.42b)$$

$$Q_{22} = \frac{1}{p_w p_z} \begin{pmatrix} [p_w \delta_{ij}]_{ii} & [p_w e_k(P_i)]_{ik} \\ \cdot & [(p_w + p_z)\delta_{k\ell}]_{kk} \end{pmatrix} \quad (4.42c)$$

The estimated parameters become, as to be expected,

$$\Delta\hat{c}_k = dz_k + dW_k \quad (4.43)$$

or

$$d\hat{T}(P) = dz(P) + dW(P) = dz(P) + dC(P) + dW_0 \quad (4.44)$$

and

$$\begin{aligned} d\hat{z}(P) &= dz(P) \\ d\hat{y}(P) &= dy(P) \\ d\hat{x}(P) &= dx(P) \end{aligned} \quad (4.45)$$

CASE TWO: Given g and x, y, z of S.

In this case eq. (4.45) remains valid and

$$\Delta\hat{c}_k = \frac{1}{k+1}\,(dg_k + 2dz_k)$$

which, when transformed back from the spectral domain, gives

$$dT(P) = \frac{1}{4\pi} \int_\sigma Ho(\psi_{PQ})(dg(Q)+2dz(Q))d\sigma_Q \qquad \underline{\text{Hotine}} \qquad (4.46)$$

where the integral kernel

$$Ho(\psi) = \sum_n \frac{2n+1}{n+1} P_n(\cos\psi)$$

4.4 Gradiometric B.V.P.

For completeness also the (vertical) gradiometric b.v.p., discussed in (Heck, 1982) or (Rummel, 1985), is included.

CASE ONE: Given dC and dΓ.

The linear model becomes for this case

$$\begin{aligned} dW_i &= -dz_i + \sum_k \Delta c_k\, e_k \\ d\Gamma_i &= -3dz_i + \sum_k \tfrac{1}{2}(k+1)(k+2)\Delta c_k\, e_k \end{aligned} \qquad (4.47)$$

where the dimensionless vertical gradient anomaly is defined as $d\Gamma = \frac{\Delta\Gamma}{\Gamma^0}$ and $\Gamma^0 = \frac{2\gamma_0}{r}$. The elements of Q_x are

$$Q_{11} = [\frac{1}{p_W+9p_\Gamma} \{\delta_{ij} + \sum_k \frac{(p_W + \frac{3}{2} p_\Gamma(k+1)k+2))^2}{(k-1)^2 E_k} e_k(P_i)e_k(P_j)\}]_{ij}$$

$$Q_{12} = [\frac{p_W + \frac{3}{2} p_\Gamma(k+1)(k+2)}{(k-1)^2 E_k} e_k(P_i)]_{ik} \qquad (4.48a\text{-}c)$$

$$Q_{22} = [\frac{p_W+9p_\Gamma}{(k-1)^2 E_k}]_{kk}$$

where $E_k = p_\Gamma p_W(-\frac{1}{2}(k+4))^2$. The solution is

$$\Delta\hat{c}_k = \frac{1}{\frac{1}{2}(k-1)(k+4)} (-3dW_k+d\Gamma_k) \qquad (4.49)$$

or

$$d\hat{T} = d(GM) + \frac{1}{4\pi} \int_\sigma H(\psi_{PQ})2(-3dW(Q)+d\Gamma(Q))d\sigma_Q \qquad (4.50)$$

where

$$H(\psi) = \sum_n \frac{2n+1}{(n-1)(n+4)} P_n(\cos\psi)$$

and the factor 2 inside the integral is due to the chosen normalization of $\Delta\Gamma$. For dz we find

$$d\hat{z}(P) = -d\hat{C}(P) - dW_0 + d(GM) + \frac{1}{4\pi}\int_\sigma H(\psi_{PQ})2(-3dW(Q)+d\Gamma(Q))d\sigma_Q \tag{4.51}$$

CASE TWO: Given dg and dΓ.

Now the elements of Q_x become

$$Q_{11} = [\frac{1}{4p_g+9p_\Gamma}\{\delta_{ij} + \sum_k \frac{(2p_g(k+1) + \frac{3}{2}p_\Gamma(k+1)(k+2))^2}{(k-1)^2E_k} e_k(P_i)e_k(P_j)\}]_{ii}$$

$$Q_{12} = [-\frac{2p_g(k+1) + \frac{3}{2}p_\Gamma(k+1)(k+2)}{(k-1)^2E_k} e_k(P_i)]_{ik} \tag{4.52a-c}$$

$$Q_{22} = [\frac{4p_g+9p_\Gamma}{(k-1)^2E_k}\delta_{k\ell}]_{kk}$$

with $E_k = p_g p_\Gamma(1+k)^2$.

Then it is

$$\Delta\hat{c}_k = \frac{1}{(k-1)(k+1)}(2d\Gamma_k - 3dg_k) \tag{4.53}$$

or

$$d\hat{T}(P) = \frac{1}{4\pi}\int_\sigma G(\psi_{PQ})(2d\Gamma(Q)-3dg(Q))d\sigma_Q \tag{4.54}$$

with

$$G(\psi) = \sum_n \frac{2n+1}{(n-1)(n+1)} P_n(\cos\psi)$$

The derivation of $d\hat{z}$ is straightforward.

Conclusions.

Although the purpose of this chapter was just to demonstrate our approach for a number of classical cases of the uniquely determined geodetic b.v.p. some conclusions can be drawn with respect to the interpretation of the results:

1. In the treatment of the potential differences one datum unknown ΔW_0 was introduced. In reality ΔW_0 cannot be assumed to be constant all over the earth. For each height datum zone a separate constant ΔW_0 (zone) has to be introduced. For a brief discussion it is referred to (Rummel, 1984). An analysis (by the authors) of the height datum problem and its proper treatment is in preparation.
2. In the so-called Bruns' equation, eq. (4.23) of the classical Stokes solution the anomaly dC at the point of computation appears outside the integral. Due to the implicit use of the Marussi mapping the numerical value of dC may usually be assumed zero. However this does not imply that this term has not to be included in the error propagation. Actually this means, that when estimating the variances of computed geoid heights (or height anomalies) the variance of the potential differences (orthometric height, normal height) of the computation point has to be added. This fact is to our knowledge neglected in practise.
3. Throughout constant weights were assumed for all observables. Especially the assumption p_W = const. is not realistical. This problem needs additional consideration.
4. The linear model, eq. (4.2) shows that $\Delta\Phi$ and $\Delta\Lambda \cos\varphi$ have to be strictly distinguished from the deflections of the vertical, ξ and η, at least in theory, the differences being $\frac{\Delta x}{r}$ and $\frac{\Delta y}{r}$, respectively, The neglect of these differences leads to the anholonomity in astronomical levelling, discussed in (Teunissen, 1982).
5. The discussion in chapter 4.2 shows, that the coordinate corrections Δx and Δy result from a combination of $\Delta\Phi$, $\Delta\Lambda$ and the Vening-Meinesz integrals, eqs. (4.37) and (4.38). We see that the Stokes and Vening-Meinesz integrals together represent in a consistent manner the complete solution of the vectorial Stokes problem in spherical approximation.
6. The step from the spherical approximation to the solution of the originally non-linear or linearized geodetic b.v.p. can be taken in the "backward" computation (up-date) step of an iterative scheme.

5. Overdetermined Geodetic B.V.P. and Linear Inference.

All problems treated in the previous chapter were one-to-one. Now we shall apply our solution method to the class of overdetermined geodetic b.v.p.'s. This requires the introduction of some minimum principle. We choose the least-squares principle, where even at this point we are still not forced to give a stochastic interpretation to the outcomes. With other words the observables may still be seen as non-stochastic, and the p's still be interpreted as weights. It is a small step to go from here to a stochastic model.

Three overdetermined problems shall be discussed, a gradio-gravimetric b.v.p. with dC, dg, and dΓ given, an altimetric-gravimetric one with dC, dg, and dz, and a combination of the two with dC, dg, dΓ, and dz. Any other version can easily be derived along the same line.

5.1 Gradio-Gravimetric B.V.P.

The linear model for this case is with eqs. (4.2) and (4.47):

$$\begin{aligned} dW &= -dz + \sum_k \Delta c_k \, e_k \\ dg &= -2dz + \sum_k (k+1)\Delta c_k \, e_k \\ d\Gamma &= -3dz + \sum_k \tfrac{1}{2}(k+1)(k+2)\Delta c_k \, e_k \end{aligned} \tag{5.1}$$

Then we obtain for the elements of the linear system $\{\underline{y}, \underline{A}\,\underline{x}, \underline{P}_y\}$:

$$\underline{y} = \begin{pmatrix} [dW]_i \\ [dg]_i \\ [d\Gamma]_i \end{pmatrix} \tag{5.2}$$

$$\underline{A} = \begin{pmatrix} [-\delta_{ij}]_{ii} & [e_k(P_i)]_{ik} \\ [-2\delta_{ij}]_{ii} & [(k+1)e_k(P_i)]_{ik} \\ [-3\delta_{ij}]_{ii} & [\tfrac{1}{2}(k+1)(k+2)e_k(P_i)]_{ik} \end{pmatrix} \tag{5.3}$$

$$\underline{x} = \begin{pmatrix} [dz]_i \\ [\Delta c]_k \end{pmatrix} \tag{5.4}$$

$$\underline{P}_y = \begin{pmatrix} [p_W \delta_{ij}]_{ii} & [0]_{ii} & [0]_{ii} \\ & [p_g \delta_{ij}]_{ii} & [0]_{ii} \\ \text{symmetric} & & [p_\Gamma \delta_{ij}]_{ii} \end{pmatrix} \tag{5.5}$$

Following the line of chapter 4 we find for $\underline{Q}_x = \underline{N}^{-1} = (\underline{A}^T \underline{P}_y \underline{A})^{-1}$:

$$Q_{11} = [\frac{1}{p_W+4p_g+9p_\Gamma} \{\delta_{ij} + \sum_k \frac{(p_W+2p_g(k+1) + \frac{3}{2} p_\Gamma(k+1)(k+2))^2}{(k-1)^2 E_k} e_k(P_i) e_k(P_j)\}]_{ii}$$

$$Q_{12} = [\frac{p_W+2p_g(k+1) + \frac{3}{2} p_\Gamma(k+1)(k+2)}{(k-1)^2 E_k} e_k(P_i)]_{ik} \tag{5.6a-c}$$

$$Q_{22} = [\frac{p_W+4p_g+9p_\Gamma}{(k-1)^2 E_k} \delta_{k\ell}]_{kk}$$

with $E_k = \{p_W p_g + p_g p_\Gamma (1+k)^2 + p_\Gamma p_W (-\frac{1}{2}(k+4))^2\}$.
Since

$$\underline{A}^T \underline{P}_y \underline{y} = \begin{pmatrix} -[p_W dW + 2p_g dg + 3p_\Gamma d\Gamma]_i \\ [p_W dW_k + p_g(k+1) dg_k + p_\Gamma \frac{1}{2}(k+1)(k+2) d\Gamma_k]_k \end{pmatrix} \tag{5.7}$$

one arrives after some lengthy derivation at:

$$\Delta\hat{c}_k = \frac{1}{(k-1)E_k} \{p_W p_g (dg_k - 2dW_k) + p_g p_\Gamma (k+1)(2d\Gamma_k - 3dg_k) + p_\Gamma p_W \tfrac{1}{2}(k+4)(-3dW_k + d\Gamma_k)\} \tag{5.8}$$

or alternatively

$$\Delta\hat{c}_k = \frac{1}{(k-1)E_k} \{-(2p_g + \tfrac{3}{2}(k+4)p_\Gamma) p_W dW_k + (p_W - 3(k+1)p_\Gamma) p_g dg_k + (2(k+1)p_g + \tfrac{1}{2}(k+4)p_W) p_\Gamma d\Gamma_k\} \tag{5.9}$$

with E_k as defined above. Eqs. (5.8) and (5.9) are the core of the solution of this type of overdetermined geodetic b.v.p. No closed integral formulas are derived.

With (5.8) or (5.9) it is

$$d\hat{T}(P) = \frac{\hat{T}}{U^0} = \sum_k \hat{\beta}_k \, e_k(P)$$

$$d\hat{T}_r(P) = \frac{1}{\gamma^0} \frac{\partial \hat{T}}{\partial r} = -\sum_k \hat{\beta}_k (k+1) e_k(P) \qquad (5.10a\text{-}c)$$

$$d\hat{T}_{rr}(P) = \frac{1}{\Gamma^0} \frac{\partial^2 \hat{T}}{\partial r^2} = \sum_k \hat{\beta}_k \, \tfrac{1}{2}(k+1)(k+2) e_k(P)$$

Finally, it follows the overdetermined "Bruns equation":

$$d\hat{z}(P) = \frac{1}{p_W + 4p_g + 9p_\Gamma} [p_W(d\hat{T}(P) - dW(P)) + 4p_g(d\hat{T}_r(P) - dg(P)) + 9p_\Gamma(d\hat{T}_{rr}(P) - d\Gamma(P))] \qquad (5.11)$$

The three weights p_W, p_g, and p_Γ are assumed constant. From an appropriate definition of the weights, the uniquely determined b.v.p. are rederived:

CASE ONE: $p_\Gamma = 0$ (Stokes).

Eq. (5.8) gives, compare also eq. (4.19):

$$\Delta\hat{c}_k = \frac{1}{k-1} (dg_k - 2dW_k)$$

CASE TWO: $p_g = 0$ (gradiometric b.v.p. version A).

Again from eq. (5.8)

$$\Delta\hat{c}_k = \frac{1}{\frac{1}{2}(k-1)(k+4)} (-3dW_k + d\Gamma_k)$$

compare eq. (4.49).

CASE THREE: $p_W = 0$ (gradiometric b.v.p. version B).

From eq. (5.8)

$$\Delta\hat{c}_k = \frac{1}{(k-1)(k+1)} (2d\Gamma_k - 3dg_k)$$

compare eq. (4.53).

5.2 Altimetric-Gravimetric B.V.P.

Observables are potential differences C, scalar gravity g, and the geometric surface S (from GPS and/or altimetry). The linear model for this case becomes with eqs. (4.2) and (4.40)

$$dW = -dz + \sum_k \Delta c_k \, e_k$$

$$dg = -2dz + \sum_k (k+1)\Delta c_k \, e_k \qquad (5.12)$$

$$dz = dz$$

We find for the elements of Q_x:

$$Q_{11} = [\frac{1}{p_w+4p_g+p_z} \{\delta_{ij} + \sum_k \frac{(p_w+2p_g(k+1))^2}{E_k} e_k(P_i)e_k(P_j)\}]_{ii}$$

$$Q_{12} = [-\frac{(p_w+2p_g(k+1))}{E_k} e_k(P_i)]_{ik} \qquad (5.13a\text{-}c)$$

$$Q_{22} = [\frac{p_w+4p_g+p_z}{E_k} \delta_{k\ell}]_{kk}$$

with

$$E_k = (k-1)^2 p_w p_g + p_z(p_w+p_g(k+1)^2) \quad . \qquad (5.14)$$

Then the gravity field parameters become:

$$\Delta\hat{c}_k = \frac{1}{E_k} \{p_w p_g(k-1)(dg_k-2dW_k) + p_g p_z(k+1) \cdot$$

$$\cdot (2dz_k+dg_k) + p_z p_w(dW_k+dz_k)\} \qquad (5.15)$$

or alternatively

$$\Delta\hat{c}_k = \frac{1}{E_k} \{(-2p_g(k-1)+p_z)p_w dW_k + (p_w(k-1) + p_z \cdot$$

$$\cdot (k+1))p_g dg_k + (p_w+2p_g(k+1))p_z dz_k\} \quad . \qquad (5.16)$$

This is the solution of the <u>overdetermined "altimetric-gravimetric" b.v.p.</u>, where $d\hat{T}$ and $d\hat{z}$ are obtained completely analogous to chapter 5.1.

Again the special cases of uniquely determined b.v.p., treated in chapter 4 are derived from either p_w, p_g or $p_z = 0$.

5.3 Overdetermined B.V.P. of Type C, g, Γ, and z.

Naturally at this point the derivations become a formal exercise. Only for reasons of completeness the overdetermined case with four observables, namely potential differences C, gravity g, vertical gradient Γ and known boundary S is included.

The linear model for this case becomes

$$
\begin{aligned}
dW &= -dz + \sum_k \Delta c_k \, e_k \\
dg &= -2dz + \sum_k (k+1)\Delta c_k \, e_k \\
d\Gamma &= -3dz + \sum_k \tfrac{1}{2}(k+1)(k+2)\Delta c_k \, e_k \\
dz &= dz \quad .
\end{aligned}
\qquad (5.17)
$$

The elements of Q_x are

$$
Q_{11} = \left[\frac{1}{p_W+4p_g+9p_\Gamma+p_z}\left\{\delta_{ij} + \sum_k \frac{(p_W+2p_g(k+1) + \frac{3}{2}p_\Gamma(k+1)(k+2))^2}{E_k} \cdot \right.\right.
$$

$$
\left.\left. \cdot \, e_k(P_i)e_k(P_j)\right\}\right]_{ii}
$$

$$
Q_{12} = \left[\frac{p_W+2p_g(k+1) + \frac{3}{2}p_\Gamma(k+1)(k+2)}{E_k} e_k(P_i)\right]_{ik} \qquad (5.18a\text{-}c)
$$

$$
Q_{22} = \left[\frac{p_W+4p_g+9p_\Gamma+p_z}{E_k}\delta_{k\ell}\right]_{kk} \quad ,
$$

and

$$
\begin{aligned}
E_k = \{&(k-1)^2(p_Wp_g+p_gp_\Gamma(k+1)^2 + \tfrac{1}{4}p_\Gamma p_W(k+4)^2) + \\
&+ p_z(p_W+p_g(k+1)^2 + p_\Gamma(\tfrac{1}{2}(k+1)(k+2))^2\}
\end{aligned}
\qquad (5.19)
$$

We derive for $\Delta\hat{c}_k$:

$$\Lambda\hat{c}_k = \frac{1}{E_k}\{p_W p_g (k-1)(dg_k - 2dW_k) + p_g p_\Gamma (k-1)(k+1)$$

$$(2d\Gamma_k - 3dg_k) + p_\Gamma p_z \tfrac{1}{2}(k+1)(k+2)(3dz_k + d\Gamma_k)$$

$$+ p_W p_z (dz_k + dW_k) + p_z p_g (k+1)(dg_k + 2dz_k)$$

$$+ p_g p_\Gamma \tfrac{1}{2}(k-1)(k+4)(-3dW_k + d\Gamma_k)\} \qquad (5.20)$$

or alternatively

$$\Lambda\hat{c}_k = \frac{1}{E_k}\{(-2p_g(k-1) - 3p_\Gamma \tfrac{1}{2}(k-1)(k+4) + p_z)p_W dW_k +$$

$$+ (p_W(k-1) - 3p_\Gamma(k-1)(k+1) + p_z(k+1))p_g dg_k +$$

$$+ (p_W \tfrac{1}{2}(k-1)(k+4) + 2p_g(k+1)(k-1) + p_z \tfrac{1}{2}(k+1)(k+2))p_\Gamma d\Gamma_k +$$

$$+ (p_W + 2p_g(k+1) + 3p_\Gamma(k+1)(k+2))p_z dz_k\} \qquad (5.21)$$

Again $d\hat{\Gamma}$ and $d\hat{z}$ are derived analogously to chapter 5.1. From eq. (5.20) or (5.21) follow all cases treated so far, uniquely as well as overdetermined, by defing either one (overdetermined) or two (uniquely determined) weights to be zero. The additional special cases, not treated so far, $\{W,\Gamma,z\}$, $\{g,\Gamma,z\}$, and $\{\Gamma,z\}$, result from, respectively, $p_g = 0$, $p_W = 0$, and $p_W = p_g = 0$.

Conclusions.

1. The proposed - least squares - approach allows to solve the overdetermined geodetic b.v.p. in spherical approximation. The relatively simple result depends however on the assumption that each individual weight (or variance) is taken constant all over the earth.
2. From the result a variety of uniquely and overdetermined b.v.p.'s can be deduced by specializing the weights.
3. The observables astronomical latitude Φ and Λ, or alternatively the gradients Γ_{xz} and Γ_{yz} could be included in a straightforward manner, the way it was shown for the vectorial Stokes problem in chapter 4.2.

4. The geodetic b.v.p. with more than one observable being a function of Δx or Δy has not been treated so far.
5. The matrix $\underline{Q}_x$ can be interpreted as a posteri variance-covariance matrix.
6. In the overdetermined cases the weights (or inverse variances) permit a comparison of the required relative precisions of the involved observables.

6. Overdetermined B.V.P. - Dual Formulation.

It is well known from adjustment theory, that adjustment by parameters and by condition equations are dual to each other. In the previous chapter the overdetermined b.v.p.'s have been solved in parametric form, by least-squares. In the sequel the dual formulation in the form of condition equations shall be discussed.

6.1 Condition Equations for dW, dg, and dΓ.

We had, eqs. (5.1) to (5.4)

$$\begin{pmatrix} [dW]_i \\ [dg]_i \\ [d\Gamma]_i \end{pmatrix} = \begin{pmatrix} [-\delta_{ij}]_{ii} & [e_k(P_i)]_{ik} \\ [-2\delta_{ij}]_{ii} & [(k+1)e_k(P_i)]_{ik} \\ [-3\delta_{ij}]_{ii} & [\tfrac{1}{2}(k+1)(k+2)e_k(P_i)]_{ik} \end{pmatrix} \begin{pmatrix} [dz]_i \\ [\Delta c]_k \end{pmatrix} \qquad (6.1)$$

or rewritten

$$\begin{pmatrix} [dg-2dW]_i \\ [2d\Gamma-3dg]_i \end{pmatrix} = \begin{pmatrix} [(k-1)e_k(P_i)]_{ik} \\ [(k-1)(k+1)e_k(P_i)]_{ik} \end{pmatrix} \begin{pmatrix} [\Delta c]_k \end{pmatrix} \qquad (6.2)$$

Transforming the left-hand side to the spectral domain condition equations per degree (= k) are obtained:

$$\frac{1}{k-1}(dg_k-2dW_k) - \frac{1}{(k-1)(k+1)}(2d\Gamma_k-3dg_k) = 0 \quad \text{for } k \neq 1$$

$$dg_k-2dW_k = 0 \quad \& \quad 2d\Gamma_k-3dg_k = 0 \quad \text{for } k = 1 \qquad (6.3a\text{-}b)$$

In integral form eqs. (6.3) become:

$$\frac{1}{4\pi}\int_{\sigma}\Big(\sum_{\substack{k=0\\k\neq 1}}^{\infty}\frac{2k+1}{k-1}P_k(\cos\psi_{PQ})\Big)(dg(Q)-2dW(Q))d\sigma_Q =$$

$$= \frac{1}{4\pi}\int_{\sigma}\Big(\sum_{\substack{k=0\\k\neq 1}}^{\infty}\frac{2k+1}{(k-1)(k+1)}P_k(\cos\psi_{PQ})\Big)(2d\Gamma(Q)-3dg(Q))d\sigma_Q \tag{6.4}$$

$$\frac{1}{4\pi}\int_{\sigma}Y_{1m}(Q)(dg(Q)-2dW(Q))d\sigma_Q = 0, \qquad m=-1,0,+1 \tag{6.5}$$

and

$$\frac{1}{4\pi}\int_{\sigma}Y_{1m}(Q)(2d\Gamma(Q)-3dg(Q))d\sigma_Q = 0\ , \qquad m=-1,0,1 \tag{6.6}$$

One should not confuse the elimination of the translational rank defect in the parametric model with condition eqs. (6.5) and (6.6). Eqs. (6.5) and (6.6) merely state that the observable combinations (dg-2dW) and (2dΓ-3dg) must not contain first degree terms.

Multiplying both sides of eq. (6.3a) by k-1 gives

$$(dg_k-2dW_k) - \frac{1}{k+1}(2d\Gamma_k-3dg_k) = 0\ , \qquad \text{for } k\neq 1 \tag{6.7}$$

which results in an integral formula alternative to eq. (6.4)

$$dg(P)-2dW(P) = \frac{1}{4\pi}\int_{\sigma}\Big(\sum_{\substack{k=0\\k\neq 1}}^{\infty}\frac{2k+1}{k+1}P_k(\cos\psi_{PQ})\Big)(2d\Gamma(Q)-3dg(Q))d\sigma_Q\ . \tag{6.8}$$

Eqs. (6.5) and (6.6) remain valid.

6.2 Condition Equations for dW, dg, and dz.

The same reasoning leads in the case of dW, dg, and dz to the condition equations per degree

$$\frac{1}{k-1}(dg_k-2dW_k) - \frac{1}{k+1}(dg_k+2dz_k) = 0\ , \qquad \text{for } k\neq 1$$

$$dg_k-2dW_k = 0\ , \qquad \text{for } k=1 \tag{6.8a-b}$$

or in terms of integral formulas to

$$\frac{1}{4\pi}\int_\sigma \Big(\sum_{\substack{k=0\\k\neq 1}}^{\infty} \frac{2k+1}{k-1} P_k(\cos\psi_{PQ})\Big)(dg(Q)-2dW(Q))d\sigma_Q = \tag{6.9}$$

$$= \frac{1}{4\pi}\int_\sigma \Big(\sum_{\substack{k=0\\k\neq 1}}^{\infty} \frac{2k+1}{k+1} P_k(\cos\psi_{PQ})\Big)(dg(Q)+2dz(Q))d\sigma_Q \quad ,$$

$$\frac{1}{4\pi}\int_\sigma Y_{1m}(Q)(dg(Q)-2dW(Q))d\sigma_Q = 0 \quad , \qquad m = -1,0,+1 \tag{6.10}$$

Alternatively one could derive from the parametric form

$$(dW_k+dz_k) - \frac{1}{k+1}(dg_k+2dz_k) = 0 \quad . \tag{6.11}$$

In this case the integral formula connecting the observables becomes

$$dW(P) + dz(P) = \frac{1}{4\pi}\int_\sigma \Big(\sum_k \frac{2k+1}{k+1} P_k(\cos\psi_{PQ})(dg(Q)+2dz(Q))d\sigma_Q \tag{6.12}$$

This equation has been derived by Baarda (1979; p. 63).
Still another condition in the spectral domain would be

$$\begin{aligned} &\frac{1}{k-1}(dg_k-2dW_k) - (dz_k+dW_k) = 0 \quad , && \text{for } k \neq 1 \\ &dg_k-2dW_k = 0 \quad , && \text{for } k = 1 \quad , \end{aligned} \tag{6.13}$$

with the integral conditions

$$dz(P) + dW(P) = \frac{1}{4\pi}\int_\sigma \Big(\sum_{\substack{k=0\\k\neq 1}}^{\infty} \frac{2k+1}{k-1} P_k(\cos\psi_{PQ})\Big)(dg(Q)-2dW(Q))d\sigma_Q \tag{6.14}$$

and

$$\frac{1}{4\pi}\int_\sigma Y_{1m}(Q)(dg(Q)-2dW(Q))d\sigma_Q = 0 \quad , \quad m = -1,0,+1 \quad , \tag{6.15}$$

compare again (Baarda, ibid; p. 63). With eq. (6.14) altimetric information can for example be merged with gravimetric information to one system of condition equations. Note however, that at the same time condition (6.15) is to be taken into account.

Conclusion.

In this chapter a series of possible condition equations has been derived, dual to the parametric forms of chapter 5. With these equations, various kinds of observables are linked to self-controlled systems, similar to the condition equations in geometric adjustment problems.

7. Observation and Discretization Errors.

In the previous three chapters the deterministic models related to the geodetic b.v.p. have been discussed. Our initial intention was however, to discuss the complete physical model, especially the error situation. We divided the errors into

- model errors, e.g. incomplete model, linearization errors, errors due to spherical approximation, where the latter two classes can be taken care of in an iterative approach,
- discretization errors, due to the fact that a field quantity, the gravitational field, is to be reconstructed from the always discrete measurements, and
- measurement errors.

In this paper we give only a very short and preliminary discussion of the analysis of discretization and measurement errors.

We assume at this stage the observables to be stochastic variates, e.g. $d\tilde{C}$, $d\tilde{g}$, $d\tilde{\Gamma}$, and $d\tilde{z}$, with the corresponding measurement errors being normal distributed with zero expectation and variance σ^2, e.g. $\sigma_W^2 = \frac{1}{p_W}$, $\sigma_g^2 = \frac{1}{p_g}$, $\sigma_\Gamma^2 = \frac{1}{p_\Gamma}$, and $\sigma_z^2 = \frac{1}{p_z}$. Then the computed parameters of ch. 5 become statistical estimates and the matrices Q_x a posteriori variance-covariance matrices.

We can now look into a number of questions of immediate practical relevance:

1. The required relative precisions of the various observables become immediately comparable. We see for example from eq. (5.8), that in case dg is given with a relative precision σ_g of 10^{-6} ($\hat{=}$ 1 mgal), the required relative precision of the potential differences has to be $\sigma_W = \frac{1}{2} \cdot 10^{-6}$ ($\hat{=}$ 3 m) and that of the vertical gradients $\sigma_\Gamma = \frac{2}{3} \cdot 10^{-6}$ ($\hat{=} 2 \cdot 10^{-3}$ E.U.). We should be very cautious with this statement. It only suggests that the stated relations should hold in order to obtain a comparable input precision for the observables dC, dg, and dΓ. It certainly does not

imply, that for example topographic heights are only required with a precision of 3 m, it also does not imply that these relative precisions would result in the same contribution coming from these three observables, as will be discussed below.

2. From the elements of the various $\underline{Q}_x$ matrices given in chapter 4 and 5, we see that our approach gives a complete error picture of the estimated parameters, variances and covariances. Of course, things remain simple only, if all a priori variances are assumed to be constant all over the earth.

   Hence from the $\underline{Q}_x$ matrix a rather complete insight can be deduced about e.g. the variances in geoid heights (or height anomalies) and in the disturbing potential under various assumptions on the attainable measurement precisions of the observables, prior to any experiment.

3. From the element Q_{22} of the various $\underline{Q}_x$ in chapter 6 it is seen that it gives the a posteriori variance per degree of the estimated Δc_n (spherical harmonic coefficients). It is the a posteriori error spectrum propagated from the white noise on a sphere of the input observables.

   Again it can be discussed under various assumptions about the variance of the observables. It also makes the error spectrum behaviour of the observables comparable. We see, for example, from Q_{22} of eq. (5.6c) observing E_k, that the error contribution of the vertical gradient $d\Gamma$ converges much faster for increasing degree, than e.g. that of dW.

   Even more, with these formulas a complete signal-noise spectrum analysis can be set up for the various geodetic b.v.p.'s, very much alike those described for satellite-to-satellite tracking and/or satellite gradiometry in (Jekeli & Rapp, 1980) and (Rummel, 1980). It permits a prior to experiment analysis of the resolution (maximum resolvable frequency), commission error and omission error, to be anticipated.

4. The error behaviour among the observables differs also as a function of the spherical distance from the computation point, e.g. gradiometry has a more local behaviour, than gravity. This can be studied by including into the spectral error analysis Q_n-type truncation coefficients, well known from geoid computations, compare (Heiskanen & Moritz, 1967; ch. 7.4).

5. In practise we work in physical geodesy with point observations or with mean values of a certain block size. The solutions of the geodetic

b.v.p.'s are derived under the assumption of a global continuous coverage with observable functions. Therefore all practical methods, whether collocation type or based upon numerical integration result in what may be called discretization error, although from method to method very different in character.

In our approach discretization errors can be studied either by introducing β_n moving average functions (Meissl, 1971) into the derivations, or by comparing the error spectra for $K \to \infty$ with the finite spectrum up to degree K or by introducing into the propagation of the error spectrum collocation type filters, the way described in (Gerstl & Rummel, 1981).

Finally it is hoped to come along this line to a complete theory that provides a complete insight into the physical model related to the geodetic boundary value problem and that permits a judgement of its quality.

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