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Abstract

In this contribution we analyse the possible nonuniqueness in the least-squares solution of the GNSS carrier-phase compass model. It is shown that this lack of uniqueness may manifest itself in the fixed baseline estimator and therefore in the GNSS compass readings. We present the conditions under which nonuniqueness occurs and give explicit expressions for these nonunique least-squares solutions.

1 Introduction

Global Navigation Satellite System (GNSS) attitude determination is a field with a wide variety of challenging (terrestrial, air and space) applications, see e.g. Cohen (1992), Lu (1995), Tu et al. (1996), Montgomery et al. (1997), Park and Teunissen (2003), Simsky et al. (2005), Kuylen et al. (2006), Teunissen (2006), Hide and Pinchin (2007).

In the present contribution we consider the determination of heading and elevation (or yaw and pitch) and therefore restrict ourselves to the two-antenna, single baseline case. GNSS carrier phase data and integer ambiguity resolution are needed in order to determine the compass parameters with the highest possible precision. Short baseline, epoch-by-epoch, successful ambiguity resolution is however only possible if two or more frequencies are used, but not in the single-frequency case. In the single-frequency case, the underlying model is too weak to ensure a sufficiently high probability of correct integer estimation. Hence, for the single-frequency case, the corresponding GNSS model needs to be strengthened and this can be done by considering the length of the (small) baseline to be known. This model is referred to as the GNSS compass model and it differs from the standard GNSS single baseline model in that the known length of the baseline is added as a (weighted) constraint.

The inclusion of the baseline length constraint strengthens the model, thereby increasing the ambiguity success rates significantly, but at the same time it also complicates the least-squares estimation process. This is particularly true for short to very-short GNSS baselines (less than 1 m), as a reduction in baseline length increases the nonlinearity of the curved manifold. Related to the high-nonlinearity is another potential complication, namely the occurrence of singularities in the solution process. We will show that non-uniqueness in the attitude solutions may indeed

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occur and we will characterize them for different measurement scenarios. Singularities of different types may occur and we will identify them in relation to various data subspaces. The theoretical analysis presented also improves our understanding of the near-singular situations. Due to a lack of space, the theorems are presented without an extensive proof. These will be published elsewhere.

2   The GNSS Compass Model

In principle all GNSS baseline models can be cast in the following frame of linear(ized) observation equations,

$$\mathrm{E}(y) = Aa + Bb, \quad \mathrm{D}(y) = Q_{yy}, \quad a \in \mathbb{Z}^p, b \in \mathbb{R}^n \tag{14.1}$$

where y is the given GNSS data vector of order m, and a and b are the unknown parameter vectors of order p and n respectively. E(.) and D(.) denote the expectation and dispersion operator, and A and B are the given design matrices that link the data vector to the unknown parameters. Matrix A contains the carrier wavelengths and the geometry matrix B contains the receiver-satellite unit line-of-sight vectors. The variance matrix of y is given by the positive definite matrix Q_{yy}. The data vector y will usually consist of the 'observed minus computed' single- or multi-frequency double-difference (DD) phase and/or pseudorange (code) observations accumulated over all observation epochs. The entries of vector a are then the DD carrier phase ambiguities, expressed in units of cycles rather than range. They are known to be *integers*, $a \in \mathbb{Z}^p$. The entries of the vector b will consist of the remaining unknown parameters, such as baseline components (coordinates) and possibly atmospheric delay parameters (troposphere, ionosphere). They are known to be real-valued, $b \in \mathbb{R}^n$. Vectors a and b are referred to as the *ambiguity* vector and *baseline* vector, respectively.

Since we consider the GNSS-Compass application in the present contribution, we restrict attention to the case of satellite tracking with two near-by antennas. The short distance between the two antennas implies that we may neglect the (differential) atmospheric delays. Thus b consists then only of the three coordinates of the between baseline vector of the two antennas.

If we may assume that the two antennas are firmly attached to the body of the moving platform, the length

of the baseline vector may be determined a priori. In that case we can strengthen the GNSS model (14.1) by including the additional observation equation

$$\mathrm{E}(l) = \|b\|, \quad \mathrm{D}(l) = \sigma_l^2 \tag{14.2}$$

The required compass information (e.g. heading and pitch) follows from the baseline solution of the GNSS compass model (14.1) and (14.2). To obtain the most precise compass information, use needs to be made of the very precise carrier phase data. The inclusion of the carrier phase data into the model accounts for the presence of the unknown integer ambiguity vector a in (14.1).

3   The Least Squares Compass Solution

The least-squares (LS) objective function of the GNSS compass model (14.1) and (14.2) is given as $H(a, b) = \|y - Aa - Bb\|_{Q_{yy}}^2 + \sigma_l^{-2}(l - \|b\|)^2$, with $a \in \mathbb{Z}^p$, $b \in \mathbb{R}^n$, and $\|.\|_{Q_{yy}}^2 = (.)^T Q_{yy}^{-1}(.)$. The LS parameter solution is therefore given by the minimizers

$$\check{a} = \arg \min_{a \in \mathbb{Z}^p} [\min_{b \in \mathbb{R}^n} H(a, b)]$$

$$\check{b} = \arg \min_{b \in \mathbb{R}^n} H(\check{a}, b) \tag{14.3}$$

This can be worked out further if we let $\hat{a}$, with variance matrix $Q_{\hat{a}\hat{a}}$, denote the LS ambiguity solution of (14.1) without the integer constraint $a \in \mathbb{Z}^n$, and let $\hat{b}(a)$, with variance matrix $Q_{\hat{b}(a)\hat{b}(a)}$, denote the conditional LS baseline solution of (14.1) assuming a known. Then the LS solution (14.3) can be shown to work out as

$$\check{a} = \arg \min_{a \in \mathbb{Z}^p} \left(\|\hat{a} - a\|_{Q_{\hat{a}\hat{a}}}^2 + \min_{b \in \mathbb{R}^n} G(a, b) \right)$$

$$\check{b} = \arg \min_{b \in \mathbb{R}^n} G(\check{a}, b) \tag{14.4}$$

where $G(a, b) = \|\hat{b}(a) - b\|_{Q_{\hat{b}(a)\hat{b}(a)}}^2 + \sigma_l^{-2}(l - \|b\|)^2$.

Note that (14.4) reduces to the LS parameter solution of the GNSS-baseline model (14.1) in case $\sigma_l^2 = \infty$. Then $\min_{b \in \mathbb{R}^n} G(a, b) = 0$ and $\arg \min_{b \in \mathbb{R}^n} G(a, b) = \hat{b}(a)$, from which it follows that the minimizers of $H(a, b)$ are given as

$$\check{a} = \arg \min_{a \in \mathbb{Z}^n} \|\hat{a} - a\|_{Q_{\hat{a}\hat{a}}}^2 \quad \text{and} \quad \check{b} = \hat{b}(\check{a}) \tag{14.5}$$

This is the commonly used solution for real-time kinematic (RTK) GNSS baseline processing, see e.g. Strang and Borre (1997), Teunissen and Kleusberg (1998), Misra and Enge (2001), Hofmann-Wellenhoff and Lichtenegger (2001), Leick (2003).

The computational complexity of (14.5) resides in the computation of the integer least-squares (ILS) solution $\check{a}$. Its computation is based on an integer search inside an ellipsoidal search space, which can be efficiently executed by means of the standard LAMBDA method, see Teunissen (1994) and Teunissen (1995). The computation of $\check{b} = \hat{b}(\check{a})$, the so-called fixed baseline, is straightforward once $\check{a}$ is known.

In our present case, we have $\sigma_l^2 \neq \infty$. This increases the computational complexity considerably. First, the computation of the fixed baseline vector is more complicated; compare (14.4) with (14.5). Second, the computation of $\check{a}$ is now based on an integer search in a *non-ellipsoidal* search space. An efficient method for this search has been developed, see Teunissen (2006), Buist (2007), Park and Teunissen (2007), Giorgi et al. (2008).

4   Nonuniqueness of Compass Solution

Note that the minimization problem $\min_{b \in \mathbb{R}^n} G(a, b)$ of (14.4), is part of the ambiguity objective function. Thus for every evaluation of the ambiguity objective function, this minimization problem needs to be solved. A proper understanding of this minimization problem is therefore essential for the GNSS-compass ambiguity resolution problem. To simplify notation, we define

$$F(b) = \|b_0 - b\|_Q^2 + \sigma_l^{-2} (l - \|b\|)^2 \qquad (14.6)$$

Then, for $Q = Q_{\hat{b}(a)\hat{b}(a)}$, we have $\min_b G(a, b) = \min_b F(b)$ if $b_0 = \hat{b}(a)$, and $\check{b} = \arg\min_b F(b)$ if $b_0 = \hat{b}(\check{a})$, see (14.3) and (14.4). Since all the properties of the fixed baseline estimator can be derived from $F(b)$, we use from now on the simplified notation of (14.6).

The minimization of (14.6) is a nonlinear least-squares problem of which the manifold is highly curved if Q is large and l is small. This is the typical case for the GNSS compass, where the baseline is very short and the single-epoch solution is determined by the relative poor code data.

The problem of minimizing $F(b)$ can be described in geometric terms as the problem of finding a point of contact between the b_0-centred ellipsoid $\|b_0 - b\|_Q^2 =$ constant and the origin-centred sphere $\|b\|^2 =$ constant. These points of contact are easily determined in case Q is a scaled unit matrix, but not so in the general case.

We have the following theorem.

Theorem 1. *Let* $M_b = Q^{-1} + \sigma_l^{-2} (1 - l/\|b\|) I_n$. *Then* $F(\hat{b}) \leq F(b)$ *for all* $b \in \mathbb{R}^n$ *if and only if* $M_{\hat{b}}\hat{b} = Q^{-1}b_0$ *and* $M_{\hat{b}} \geq 0$.

This theorem formulates necessary and sufficiency conditions for $\hat{b}$ to be a *global* minimizer of $F(b)$. It also provides the conditions for having nonunique minimizers. Note that $M_b = M_{\hat{b}}$ for any $b \neq \hat{b}$ that has the same length as $\hat{b}$, $\|b\| = \|\hat{b}\|$. Thus for nonunique minimizers to exist, it is necessary that $M_{\hat{b}}$ is singular.

It can be shown that M_b is singular if and only if $\|b\| = l/(1 + \sigma_l^2 \lambda_1)$, where λ_1 is the smallest eigenvalue of Q^{-1}. Nonunique minimizers, if they exist, lie therefore all on the sphere with radius γ_1.

5   When do the Nonunique Solutions Exist?

Before we can determine the nonunique solutions, we first need to know whether they exist. The *consistency* requirement of the system of equations, $M_b b = Q^{-1}b_0$ and $\|b\| = \gamma_1$, results in two conditions that the data vector b_0 has to satisfy. The first condition is that b_0 must lie in the range space of matrix QM_b, $b_0 \in R(QM_b)$. The second condition comes into play when the first condition is satisfied. This second condition puts restrictions on the length of b_0. Not every b_0 that makes $M_b b = Q^{-1}b_0$ consistent, will namely produce a solution that satisfies $\|b\| = \gamma_1$.

We start with the first condition. Let the orthogonal matrix of eigenvectors of Q^{-1} be partitioned as $U = (U_1, U_2)$, with U_1 containing all eigenvectors having λ_1 as eigenvalue. Then the null space and range space of QM_b are spanned by the columns of U_1 and U_2, respectively. Hence, the first consistency condition can be formulated as

$$U_1^T b_0 = 0 \ \text{ or } \ b_0 \in R(U_2) \qquad (14.7)$$

If this condition is satisfied, then $M_b b = Q^{-1} b_0$ is solvable and its solutions can be expressed as

$$b = b_p + U_1 \beta_1 \qquad (14.8)$$

where $b_p = (QM_b)^+ b_0$ is a *particular* solution, with $(QM_b)^+$ the pseudo inverse of QM_b, and where $U_1 \beta_1$ is the *homogeneous* solution, with β_1 still undetermined.

Since b of (14.8) has to satisfy $\|b\| = \gamma_1$, the as yet undetermined β_1 cannot take on values freely, but has to satisfy

$$\|\beta_1\|^2 = \gamma_1^2 - \|(QM_b)^+ b_0\|^2 \geq 0 \qquad (14.9)$$

This shows that b_0 may not have an arbitrary length. Hence, the nonnegativity condition of (14.9) is the second consistency condition that b_0 has to satisfy.

If we take the two conditions, (14.7) and (14.9), together, we may summarize our result as follows.

Theorem 2a. *The function $F(b)$ (cf. (14.6)) has nonunique minimizers if and only if*

$$b_0 \in \mathbb{C} = \{x \in \mathbb{R}^n \mid \|x\|_{\Sigma}^2 \leq \gamma_1^2, \ x \in R(U_2)\} \qquad (14.10)$$

where $\Sigma^{-1} = U_1 U_1^T + (QM_b)^{+T}(QM_b)^+$.

Thus we now know, if the data vector b_0 lies in the intersection of $R(U_2)$ and the origin-centred ellipsoidal region $\|x\|_{\Sigma}^2 \leq \gamma_1^2$, that we will have more than one minimizer of $F(b)$.

6   The Nonunique Solutions Determined

It is now not difficult anymore to determine the nonunique solutions. We already know that the nonunique minimizers, if they exist, lie all on the sphere with radius γ_1, denoted as S_{γ_1}. This combined with the general solution of $M_b b = Q^{-1} b_0$ (cf. (14.8)), gives the following result.

Theorem 2b. *The nonunique minimizers of $F(b)$ (cf. (14.6)), if they exist, are given by the solution set*

$$\mathbb{S} = \{b_p + R(U_1)\} \cap S_{\gamma_1} \qquad (14.11)$$

where $b_p = (QM_b)^+ b_0$.

The consistency set $\mathbb{C}$ and solution set $\mathbb{S}$ can both be given a clear geometric interpretation. The set $\mathbb{C}$ describes the two consistency conditions (14.7) and (14.9). Geometrically this set describes the intersection of an origin-centred ellipsoidal region with the linear manifold $R(U_2)$. Since $R(U_1)$ and $R(U_2)$ are each others orthogonal complement, we have $\dim R(U_2) = n - \dim R(U_1)$. Thus if $\dim R(U_1) = n$, then $\dim R(U_2) = 0$ and $\mathbb{C} = \{0\}$, and if $\dim R(U_1) = n - 1$, then $\dim R(U_2) = 1$ and $\mathbb{C}$ reduces to an origin centred interval.

The solution set $\mathbb{S}$ is the intersection of the linear manifold $b_p + R(U_1)$ with the sphere S_{γ_1}. It consists of two points if the linear manifold is a straight line ($\dim R(U_1) = 1$) and it forms a circle if the linear manifold is a plane ($\dim R(U_1) = 2$). Since the dimension of the linear manifold is equal to the number of times the eigenvalues of Q^{-1} are equal to λ_1, we have $1 \leq \dim R(U_1) \leq n$. If $\dim R(U_1) = n$, then $b_p + R(U_1) = \mathbb{R}^n$ and $\mathbb{S} = S_{\gamma_1}$. This is the special case when all eigenvalues of Q^{-1} are equal (i.e. Q is a scaled unit matrix).

As an illustration, we now show for the cases $n = 1, 2, 3$ how the sets $\mathbb{C}$ and $\mathbb{S}$ may look like.

Case $n = 1$: If $n = 1$, then $U_1 = 1$, $U_2 = 0$, and $Q = \lambda_1^{-1}$. Therefore $\mathbb{C} = \{0\}$ and $\mathbb{S} = S_{\gamma_1} = \{b \in \mathbb{R} \mid b = \pm \gamma_1\}$. Since $F(b) = \sigma_l^{-2}(1 + \sigma_l^2 \lambda_1)(b^2 - 2\gamma_1 |b| + l\gamma_1)$ for $b_0 = 0$, it is readily verified that $\pm \gamma_1$ are indeed its two minimizers (see Fig. 14.1).

Case $n = 2$: We now have two cases: (a) $U_1 = I_2$, $U_2 = 0$ and (b) $U_1 = u_1$, $U_2 = u_2$. In case (a) we have $\mathbb{C} = \{0\}$ and $\mathbb{S} = S_{\gamma_1}$. In case (b), $\mathbb{C}$ is an origin-centred interval of length $2\gamma_1(1 - \lambda_1/\lambda_2)$ along the u_2 direction and $\mathbb{S}$ consists of the two intersection points of the line $b = b_p + u_1 \alpha$ with the circle S_{γ_1} (see Fig. 14.2). Note, if b_0 lies on the edge of $\mathbb{C}$ and thus
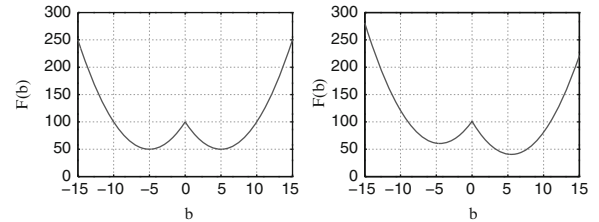

Fig. 14.1 The function $F(b) = \lambda_1(b_0 - b)^2 + \sigma_l^{-2}(l - |b|)^2$ for $b_0 = 0$ (*left*) and $b_0 \neq 0$ (*right*)
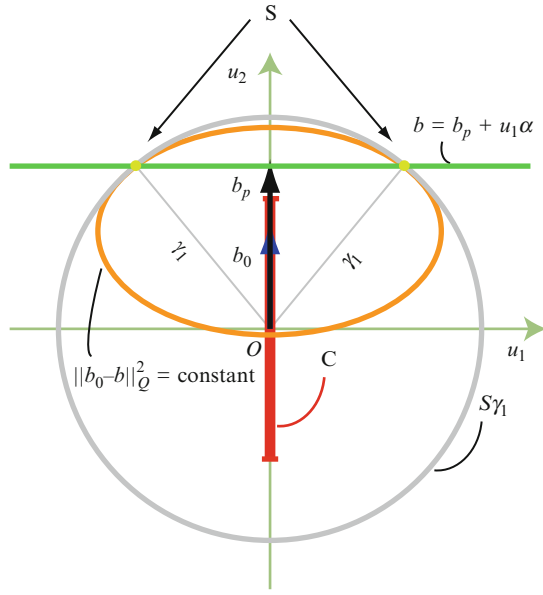
Fig. 14.2 The case $n = 2$ with $U_1 = u_1$, $U_2 = u_2$: Shown are the *circle* S_{γ_1} with axes u_1 and u_2, the interval $\mathbb{C}$, the two solution points of $\mathbb{S}$, the *line* $b = b_p + u_1\alpha$ and the *ellipse* $\|b_0 - b\|_Q^2 =$ constant

$\|b_0\| = \gamma_1(1 - \lambda_1/\lambda_2)$, that $\|b_p\| = \gamma_1$ and the two intersection points coincide in one point.

Case $n = 3$: We have the three cases: (a) $U_1 = I_3$, $U_2 = 0$, (b) $U_1 = (u_1, u_2)$, $U_2 = u_3$, and (c) $U_1 = u_1$, $U_2 = (u_2, u_3)$. In case (a) we have $\mathbb{C} = \{0\}$ and $\mathbb{S} = S_{\gamma_1}$. In case (b), $\mathbb{C}$ is an origin-centred interval of length $2\gamma_1(1 - \lambda_1/\lambda_3)$ along the u_3 direction and the solution space $\mathbb{S}$ is the circle with centre b_p and radius $\sqrt{\gamma_1^2 - \|b_p\|^2}$ that follows from intersecting the plane $b = b_p + u_1\alpha_1 + u_2\alpha_2$ with the sphere S_{γ_1}. In case (c), $\mathbb{C}$ is an origin-centred ellipse in the U_2-plane with principal axes $\gamma_1(1 - \lambda_1/\lambda_2)u_2$ and $\gamma_1(1 - \lambda_1/\lambda_3)u_3$. The solution space $\mathbb{S}$ is then the two point intersection of the line $b = b_p + u_1\alpha$ with the sphere S_{γ_1}.

The latter case ($n = 3$ (c)) is the one that is most likely to occur with GNSS, since the eigenvalues of the variance matrix $Q = Q_{\hat{b}(a)\hat{b}(a)}$ will usually all be different.

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