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GNSS Phase Ambiguity Validation: A Review

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Abstract. The LAMBDA method is very suitable for Multi-Carrier Ambiguity Resolution (MCAR), which is of interest with the coming of Galileo and the modernisation of GPS. Ambiguity resolution involves, however, not only integer estimation, but often also a decision on whether or not to accept the estimated integers. In this contribution, we show the relevance of this decision-step and show that integer aperture estimation is the proper theory for handling the combined integer estimation/acceptance problem. With this approach the user sets his/her own failure rate (irrespective of the strength of the underlying model), thus generating an aperture space which forms the basis of the decision process: the integer solution is chosen as output if the float solution resides inside the aperture space, otherwise the float solution is maintained. We show how the various approaches as used in the literature fit in the framework of integer aperture estimation and we also present the optimal integer aperture estimator. We also show how the popular ratio test fits in this framework. The weak point of this test, as it is currently used in practice, is the choice of the threshold value which determines whether or not the fixed solution will be accepted. Mostly a fixed value is used, not depending on the model under consideration. It will be shown here that this approach will often be too conservative, meaning that the fixed solution is rejected while the probability of being wrong is actually very low. This implies that the time to first fix may be unnecessarily long.

Keywords. LAMBDA method, integer aperture estimation, user-defined failure rate

1 The 4 steps of GNSS ambiguity resolution

As our point of departure we will take the following system of linear(ized) observation equations

$$y = Aa + Bb + e \tag{1}$$

where y is the given GNSS data vector of order m, a and b are the unknown parameter vectors respectively of order n and p, and where e is the noise vector. In principle all the GNSS models can be cast in this frame of observation equations. The data vector y will usually consist of the 'observed minus computed' double-difference (DD) phase and/or pseudorange (code) observations accumulated over all observation epochs. The entries of vector a are then the DD carrier phase ambiguities, expressed in units of cycles rather than range. They are known to be *integers*, $a \in Z^n$. The entries of the vector b will consist of the remaining unknown parameters, such as for instance baseline components (coordinates) and possibly atmospheric delay parameters (troposphere, ionosphere). They are known to be real-valued, $b \in R^p$.

The procedure which is usually followed for solving the GNSS model (1), can be divided into four steps. In the *first* step one simply disregards the integer constraints $a \in Z^n$ on the ambiguities and applies a standard least-squares adjustment, resulting in real-valued estimates of a and b, together with their variance-covariance (vc-) matrix

$$\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix} , \begin{bmatrix} Q_{\hat{a}} & Q_{\hat{a}\hat{b}} \\ Q_{\hat{b}\hat{a}} & Q_{\hat{b}} \end{bmatrix} \tag{2}$$

This solution is referred to as the 'float' solution. In the *second* step the 'float' ambiguity estimate $\hat{a}$ is used to compute the corresponding integer ambiguity estimate $\check{a}$. This implies that a mapping $S : R^n \mapsto Z^n$, from the n-dimensional space of reals to the n-dimensional space of integers, is introduced such that

$$\check{a} = S(\hat{a}) \tag{3}$$

The three best known integer estimators are integer rounding, integer bootstrapping and integer least-squares. The latter, given as

$$\check{a}_{LS} = \arg \min_{z \in Z^n} \| \hat{a} - z \|^2_{Q_{\hat{a}}} \tag{4}$$

with the squared norm $\|\cdot\|_Q^2 = (\cdot)^T Q^{-1}(\cdot)$, is known to be optimal, cf. (Teunissen, 1999), which means that the probability of correct integer estimation is maximized. In contrast to integer rounding and integer bootstrapping, an integer search is needed to compute $\check{a}_{LS}$. This ILS procedure is efficiently mechanized in the LAMBDA method, see e.g. (Teunissen, 1998).

Once the integer ambiguities are computed, they are used in the *third* step as input to decide whether or not to accept the integer solution. In the literature several such 'validation' tests have been proposed in order to decide whether or not to fix the ambiguities. A review and evaluation of the tests can be found in (Verhagen, 2004). Well-known examples are the ratio test, distance test and projector test. Among the most popular tests is the ratio test, see e.g. (Euler and Schaffrin, 1991; Wei and Schwarz, 1995; Han and Rizos, 1996; Leick, 2003), where the decision is made as follows:

1. Apply ILS to obtain $\check{a}$ and $\check{a}_2$
2. Evaluate ratio $\frac{\|\hat{a}-\check{a}\|^2_{Q_{\hat{a}}}}{\|\hat{a}-\check{a}_2\|^2_{Q_{\hat{a}}}}$
3. Decision: If $\frac{\|\hat{a}-\check{a}\|^2_{Q_{\hat{a}}}}{\|\hat{a}-\check{a}_2\|^2_{Q_{\hat{a}}}} \begin{cases} \leq \mu & \text{use } \check{a} \\ > \mu & \text{use } \hat{a} \end{cases}$

where $\check{a}_2$ is the second-best integer solution in the ILS sense. Note that in practice the reciprocal of the test statistic is mostly used.

The third step is often referred to in the literature as the 'validation' step. It should be noted, however, that this step is not designed to validate the underlying model. As will be explained in the context of integer aperture estimation, the third step is designed to ensure that a user-defined failure rate can be maintained. As another criticism of current practice, we note that the choice of the threshold value μ is often ad hoc or based on false theoretical grounds, see (Teunissen and Verhagen, 2004). Often a fixed value of $\frac{1}{2}$ or $\frac{1}{3}$ is used. In the next sections we will come back to these issues.

Once the integer solution is accepted, the *fourth* step consists of correcting the 'float' estimate of b. As a result one obtains the 'fixed' solution

$$\check{b} = \hat{b} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1}(\hat{a} - \check{a}) \tag{5}$$

2 Integer aperture estimation

2.1 Achieving user-defined failure rates

In practice, a user does not want to use the integer solution if the probability that it is wrong (the failure rate) is too large. Moreover, the user wants to have control over this failure rate, i.e. be able to set the value of the failure rate him/herself. For this purpose the class of integer aperture (IA) estimators was developed, see (Teunissen, 2003a; Teunissen, 2003b). With these estimators we are able to distinguish between three cases: *success* if the ambiguity is fixed correctly, *failure* if the ambiguity is fixed incorrectly, and *undecided* if the float solution is maintained.

Definition (*Integer aperture estimators*) The integer aperture estimator, $\bar{a}$, is defined as:

$$\bar{a} = \sum_{z \in \mathbb{Z}^n} z\omega_z(\hat{a}) + \hat{a}\Big(1 - \sum_{z \in \mathbb{Z}^n} \omega_z(\hat{a})\Big) \qquad \text{with} \quad \omega_z(x) = \begin{cases} 1 & \text{if } x \in \Omega_z \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

The Ω_z are the aperture pull-in regions, which have to fulfill the following conditions:

$$\begin{cases} \bigcup_{z \in \mathbb{Z}^n} \Omega_z = \Omega \\ \text{Int}(\Omega_u) \cap \text{Int}(\Omega_z) = \emptyset, \quad \forall u, z \in \mathbb{Z}^n, u \neq z \\ \Omega_z = z + \Omega_0, \qquad \forall z \in \mathbb{Z}^n \end{cases}$$

■

$\Omega \subset \mathbb{R}^n$ is called the aperture space. From the first condition follows that this space is built up of the Ω_z. The second and third conditions state that these aperture pull-in regions must be disjunct and translational invariant.

Figure 1 shows a two-dimensional example of aperture pull-in regions that fulfill the conditions of the Definition, together with the ILS pull-in regions, which are given as

$$S_z = \{x \in R^n \mid z = \arg\min_a ||\hat{a} - a||^2_{Q_{\hat{a}}}\}, \quad z \in Z^n \tag{7}$$

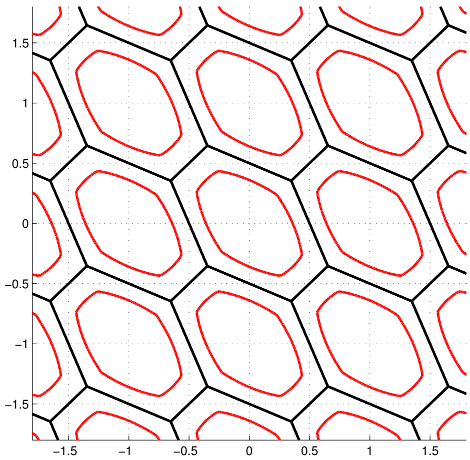

Figure 1: Two-dimensional example of aperture pull-in regions, together with the ILS pull-in regions (hexagons).

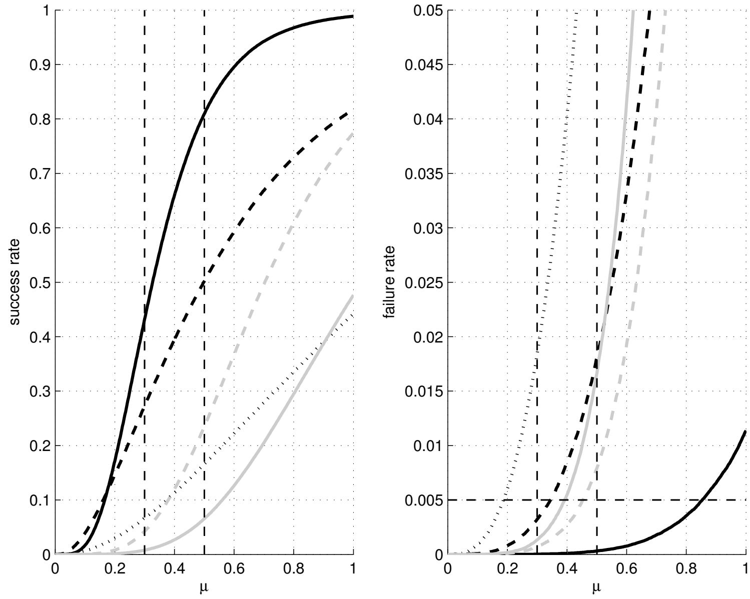

Figure 2: Success and failure rates as function of the threshold value δ for 5 GPS models.

So, when $\hat{a} \in \Omega$, then the ambiguity solution $\bar{a} = \check{a} \in Z^n$ will be accepted as integer vector, otherwise the float solution $\bar{a} = \hat{a} \in R^n$ is maintained. This means that indeed the following three cases can be distinguished:

$\hat{a} \in \Omega_a$	success: correct integer estimation
$\hat{a} \in \Omega \setminus \Omega_a$	failure: incorrect integer estimation
$\hat{a} \notin \Omega$	undecided: ambiguity not fixed

The corresponding probabilities of success (s), failure (f) and undecidedness (u) are given by:

$$\begin{aligned} P_s &= P(\bar{a} = a) = \int_{\Omega_a} f_{\hat{a}}(x)dx \\ P_f &= \sum_{z \in \mathbb{Z}^n \setminus \{a\}} \int_{\Omega_z} f_{\hat{a}}(x)dx \\ P_u &= 1 - P_s - P_f = 1 - \int_{\Omega_0} f_{\check{\epsilon}}(x)dx \end{aligned} \tag{8}$$

The first two probabilities are referred to as success rate and failure rate respectively. The expression for the undecided rate is obtained by using the probability density function of the ambiguity residuals $\check{\epsilon} = \hat{a} - \check{a}$, cf. (Verhagen and Teunissen, 2004):

$$f_{\check{\epsilon}}(x) = \sum_{z \in \mathbb{Z}^n} f_{\hat{a}}(x+z)s_0(x) \quad \text{with} \quad s_0(x) = \begin{cases} 1 & \text{if } x \in S_0 \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

2.2 Optimal integer aperture estimation

As mentioned in the beginning of this section, for a user it is especially important that the probability of failure, the failure rate, is below a certain limit. The approach of integer aperture estimation allows us now to choose a threshold for the failure rate, and then determine the *size* of the aperture pull-in regions such that indeed the failure rate will be equal to or below this threshold. So, applying this approach means that implicitly the ambiguity estimate is validated using a sound criterion. However, there are still several options left with respect to the choice of the *shape* of the aperture pull-in regions. We are therefore also able to search for an *optimal* integer aperture (OIA) estimator. As with integer estimation, the optimality property would be to maximize the success rate, but in this case for a fixed failure rate. So, the optimization problem is to determine the aperture space which fulfills:

$$\max_{\Omega_0 \subset S_0} P_s \quad \text{subject to:} \quad P_f = \beta \tag{10}$$

where β is a chosen fixed value for the failure rate. The solution of the optimization problem is given in (Teunissen, 2003b):

$$\Omega_0 = \{x \in S_0 \mid \sum_{z \in \mathbb{Z}^n} f_{\hat{a}}(x+z) \leq \mu f_{\hat{a}}(x+a)\} \tag{11}$$

with S_0 being the ILS pull-in region.

Using eqs.(9) and (11) the computational steps for the OIA-estimator are:

1. Apply ILS to obtain $\check{a}$
2. Evaluate the probability densities $f_{\check{\epsilon}}(\hat{a} - \check{a})$ and $f_{\hat{a}}(\hat{a} - \check{a})$
3. Determine μ based on the user-defined $P_f = \beta$
4. Decision: If $\frac{f_{\check{\epsilon}}(\hat{a}-\check{a})}{f_{\hat{a}}(\hat{a}-\check{a})} \begin{cases} \leq \mu & \text{use } \check{a} \\ > \mu & \text{use } \hat{a} \end{cases}$

Compare this result with the approach using the Ratio Test in section 1. In both approaches the test statistics are defined as a ratio. In the case of the Ratio Test, it only depends on $\|\hat{a} - \check{a}\|^2_{Q_{\hat{a}}}$ and $\|\hat{a} - \check{a}_2\|^2_{Q_{\hat{a}}}$, whereas from eq.(11) it follows that the optimal test statistic depends on all $\|\hat{a} - z\|^2_{Q_{\hat{a}}}$, $z \in \mathbb{Z}^n$ if it is assumed that the float solution is normally distributed.

It is very important to note that Integer Aperture estimation with a fixed failure rate is an *overall* approach of integer estimation and allows for an exact and overall probabilistic evaluation of the solution. With the traditional approaches, e.g. the Ratio Test applied with a fixed critical value, this is not possible.

3 ILS + Ratio Test is integer aperture estimator

3.1 Aperture space of ratio test

Although there is only one *optimal* integer aperture estimator, there exist many integer aperture estimators. Hence, one can in principle design an aperture estimator oneself. Despite the criticism on the Ratio Test given in section 1 it is possible to give a firm theoretical basis for this test, since it can be shown that the procedure underlying the Ratio Test is a member from the class of integer aperture estimators. The acceptance region or aperture space of the Ratio Test is given as:

$$\Omega = \{x \in \mathbb{R}^n \mid \|x - \check{x}\|^2_{Q_{\hat{a}}} \leq \delta \|x - \check{x}_2\|^2_{Q_{\hat{a}}},\ 0 < \delta \leq 1\} \tag{12}$$

with $\check{x}$ and $\check{x}_2$ the best and second-best ILS estimator of x. Let $\Omega_z = \Omega \cap S_z$, i.e. Ω_z is the intersection of Ω with the ILS pull-in region. Then all conditions of the Definition are fulfilled, since:

$$\begin{aligned} \Omega_0 &= \{x \in \mathbb{R}^n \mid \|x\|^2_{Q_{\hat{a}}} \leq \delta \|x - z\|^2_{Q_{\hat{a}}},\ \forall z \in \mathbb{Z}^n \setminus \{0\}\} \\ \Omega_z &= \Omega_0 + z,\ \forall z \in \mathbb{Z}^n \\ \Omega &= \bigcup_{z \in \mathbb{Z}^n} \Omega_z \end{aligned} \tag{13}$$

The acceptance region of the Ratio Test consists thus of an infinite number of regions, each one of which is an integer translated copy of $\Omega_0 \subset S_0$ (see Figure 1). The acceptance region plays the role of the aperture space, and δ plays the role of the aperture parameter since it controls the size of the aperture pull-in regions.

Compared to the approach in section 1 an important difference is that δ is now based on the user-defined failure rate. Hence, before the decision can be made, it is required to determine δ based on the user-defined choice $P_f = \beta$.

3.2 Optimal aperture estimation and ratio test compared

In (Teunissen and Verhagen, 2004) the performance of the ratio test and optimal integer aperture estimator were compared. It followed that often the ratio test performs close to optimal, provided our fixed failure rate approach is used. A shorter time to first fix can be obtained as compared to the traditional ratio test with fixed critical value δ, while at the same time it is guaranteed that the failure rate is below a user-defined threshold.

As an illustration of the difference between the traditional and the Integer Aperture approach with the Ratio Test, five dual-frequency GPS models are considered. Based on Monte-Carlo simulations the success and failure rates as function of δ are determined for each of the models, see figure 2. It can be seen that with a fixed value of $\delta = 0.3$ (close to the value of $\frac{1}{3}$ often used in practice) for most of the models considered here a very low failure rate is obtained, but that this is not guaranteed. This seems good, but at the same time also the corresponding success rate is low. If the threshold value would have been based on a fixed failure rate of e.g. 0.005, the corresponding δ would have been very different for each of the models, and in most cases larger than 0.3, and thus a higher success rate and higher probability of a fix ($= P_s + P_f$) would be obtained. Hence, the integer aperture approach with fixed failure rate is to be preferred.

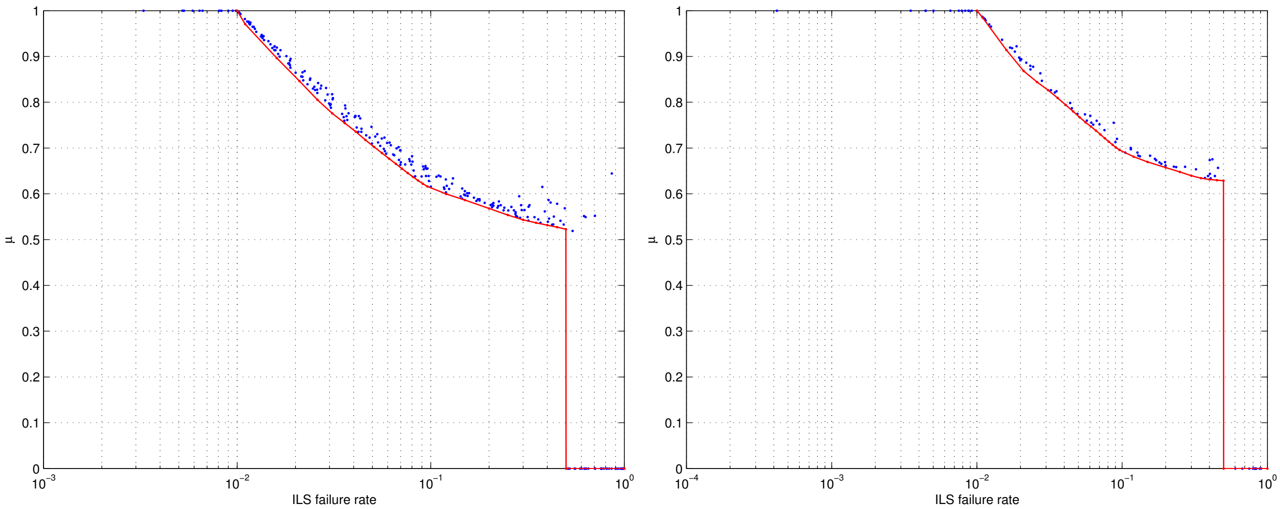

Figure 3: Threshold values as function of the integer least-squares failure rate for $n = 12$ (left) and $n = 16$ (right). The points depict true values from simulations; the line depicts the values obtained with interpolation using the look-up table.

$P_{f,ILS}$	$n = \ldots$	$n = 6$	$n = 7$	$n = 8$	$n = 9$	$n = 10$	$n = \ldots$
0	$\cdots$	1	1	1	1	1	$\cdots$
0.01	$\cdots$	1	1	1	1	1	$\cdots$
0.011	$\cdots$	0.978	0.977	0.979	0.980	0.982	$\cdots$
0.016	$\cdots$	0.878	0.880	0.884	0.891	0.897	$\cdots$
0.021	$\cdots$	0.879	0.800	0.819	0.829	0.839	$\cdots$
0.026	$\cdots$	0.730	0.741	0.775	0.787	0.800	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 1: Example of a part of the look-up table (values are indicative).

3.3 Determining the aperture parameter

To determine the aperture parameter μ, one needs simulations based on the variance matrix of the floating ambiguities. This may lead to a high computational burden. It is now shown how one can approximate the threshold value accurately in an efficient manner. The idea is that look-up tables are created from which the appropriate aperture parameter can be determined based on the variance matrix of the floating ambiguities. This means that the only input for the complete ambiguity resolution kernel would consist of the floating ambiguities and their variance matrix.

The aperture parameter μ was analysed as function of the integer least-squares failure rate by collecting the simulation results of a whole range of GNSS models together. In this case, the maximum allowable failure rate was set to 0.01. It turned out that there is a clear relation between μ, the ILS failure rate P_f, and the dimension n of the integer ambiguity vector.

The following approach was then applied in order to exploit this relation. First, the results of all models with the same dimension n were combined. From this a look-up table was created with conservative values of μ as function of the ILS failure rate, an example is shown in Table 1. For a given variance matrix of the float ambiguities the appropriate value of μ can be determined by interpolation after computing the ILS failure rate and selecting the column from the table corresponding to the dimension n in that case.

In order to demonstrate this approach, the results for models with $n = 12$ and $n = 16$ are shown in Fig.3. Different GNSS models (GPS and Galileo) were used. If the ILS failure rate is larger than approximately 0.5, the $\mu-$value is set to zero, since it turned out that then the fix probability will be close to zero anyway. Shape-preserving piecewise cubic interpolation was applied to obtain values of the threshold value for any given ILS failure rate. The interpolated values are shown with the solid line.

One problem with this approach is the computation of the ILS failure rate. Exact computation is not possible, so that an approximation is needed. Several approximations are available, most of which are known to be either an upper bound or lower bound. Obviously, an upper bound should be used in order to guarantee that the actual failure rate is lower than the maximum allowable value. For each of the models used in this study, the $\mu-$values obtained with this approach are then used to determine the corresponding failure rates and fix probabilities based on the simulated data. Ideally, the failure rates should be very close to the fixed value 0.01.

The results are shown in Fig.4 for two models. The actual failure rate as function of time is shown in the left panels. The fix probability is shown in the right panels. The black solid line shows the failure rate obtained by using the approximated $\mu-$value with the look-up table. The grey lines show the failure rates obtained by using a fixed $\mu-$value of 0.5. The dashed black lines show the true values if the $\mu-$value

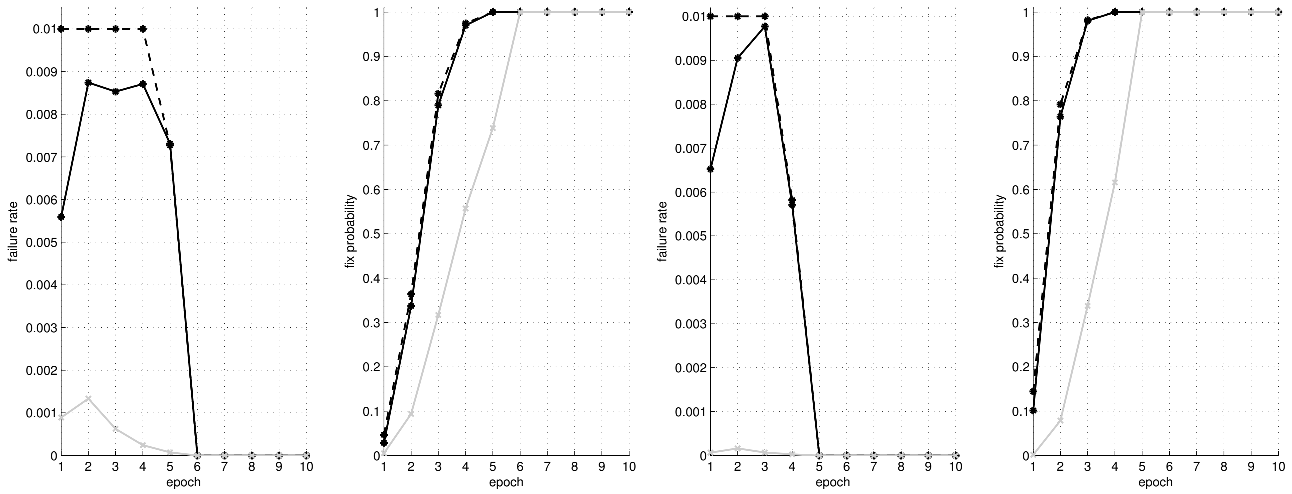

Figure 4: Failure rates and fix probabilities with approximated threshold value using a look-up table (black solid). True values (black dashed) and values with fixed threshold value of 0.5 (grey) are also shown. Left panels: 3-frequency GPS, 15 ambiguities. Right panels: 2-frequency Galileo/GPS, 20 ambiguities.

corresponding to a fixed failure rate of 0.01 was used. Note that as soon as the ILS failure rate is smaller than 0.01, the threshold value becomes equal to 1, and hence the failure rate becomes equal to the ILS failure rate.

It follows that the approximation of the $\mu-$value using the look-up table works very well, even though the upper bound of the ILS failure rate was used. In general the failure rates are somewhat lower than the required value, which is good. This implies that also the fix probabilities are somewhat lower, since a smaller failure rate means that the acceptance region is smaller. However, the difference compared to the probabilities obtained with the 'true' $\mu-$value is small. Obviously, using the model-driven determination of the $\mu-$value gives much better performance as compared to using a fixed value, as is done with the traditional Ratio Test.

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References

Euler, H. J. and Schaffrin, B. (1991). On a measure for the discernibility between different ambiguity solutions in the static-kinematic GPS-mode. *IAG Symposia no.107, Kinematic Systems in Geodesy, Surveying, and Remote Sensing, Springer-Verlag, New York*, pages 285–295.

Han, S. and Rizos, C. (1996). Integrated methods for instantaneous ambiguity resolution using new-generation GPS receivers. *Proc. of IEEE PLANS'96, Atlanta GA*, pages 254–261.

Leick, A. (2003). *GPS Satellite Surveying*. John Wiley and Sons, New York, 3rd edition.

Teunissen, P. J. G. (1998). *GPS carrier phase ambiguity fixing concepts*. In: PJG Teunissen and Kleusberg A, *GPS for Geodesy*, Springer-Verlag, Berlin.

Teunissen, P. J. G. (1999). An optimality property of the integer least-squares estimator. *Journal of Geodesy*, 73(11):587–593.

Teunissen, P. J. G. (2003a). Integer aperture GNSS ambiguity resolution. *Artificial Satellites*, 38(3):79–88.

Teunissen, P. J. G. (2003b). Towards a unified theory of GNSS ambiguity resolution. *Journal of Global Positioning Systems*, 2(1):1–12.

Teunissen, P. J. G. and Verhagen, S. (2004). On the foundation of the popular ratio test for GNSS ambiguity resolution. *Proc. of ION GNSS-2004, Long Beach CA*, pages 2529–2540.

Verhagen, S. (2004). Integer ambiguity validation: an open problem? *GPS Solutions*, 8(1):36–43.

Verhagen, S. and Teunissen, P. J. G. (2004). *PDF evaluation of the ambiguity residuals*. In: F Sansò (Ed.), *V. Hotine-Marussi Symposium on Mathematical Geodesy*, International Association of Geodesy Symposia, Vol. 127, Springer-Verlag.

Wei, M. and Schwarz, K. P. (1995). Fast ambiguity resolution using an integer nonlinear programming method. *Proc. of ION GPS-1995, Palm Springs CA*, pages 1101–1110.