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THE LAMBDA METHOD FOR THE GNSS COMPASS

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ABSTRACT Global Navigation Satellite System carrier phase ambiguity resolution is the key to high precision positioning and attitude determination. In this contribution we consider the GNSS compass model. We derive the integer least-squares estimators and discuss the various steps involved in the ambiguity resolution process. This includes the method that has successfully been used in (*Park and Teunissen*, 2003). We emphasize the unaided, single frequency, single epoch case, since this is considered the most challenging mode of GNSS attitude determination.

1 INTRODUCTION

Global Navigation Satellite System (GNSS) ambiguity resolution is the process of resolving the unknown cycle ambiguities of the carrier phase data as integers. The sole purpose of ambiguity resolution is to use the integer ambiguity constraints as a means of improving significantly on the precision of the remaining model parameters. Apart from the current Global Positioning System (GPS) models, carrier phase ambiguity resolution also applies to the future modernized GPS and the future European Galileo GNSS. An overview of GNSS models, together with their applications in surveying, navigation, geodesy and geophysics, can be found in textbooks such as (*Hofmann-Wellenhof et al.*, 1997), (*Leick*, 1995), (*Parkinson and Spilker*, 1996), (*Strang and Borre*, 1997) and (*Teunissen and Kleusberg*, 1998).

In this contribution we will consider the problem of ambiguity resolution for GNSS attitude determination, see e.g. (*Peng et al.*, 1999), (*Park and Teunissen*, 2003), (*Moon and Verhagen*, 2006). Attitude determination based on GNSS is a rich field of current studies, with a wide variety of challenging (terrestrial, air and space) applications. In the present contribution we restrict ourselves to the single baseline case (two antennas) and we therefore only consider the determination of heading and elevation (or yaw and pitch). The corresponding model is referred to as the GNSS compass model. Despite this restriction, the GNSS compass model in itself already has a wide range of important applications, as is also evidenced by the commercial products available, see e.g. (*Furuno*, 2003) or (*Simsky et al.*, 2005), both of which make use of the LAMBDA (Least-squares AMBiguity Decorrelation Adjustment) method. Moreover, a proper understanding of the intricacies of ambiguity resolution for the GNSS compass is considered essential for attitude determination based on an array of antennas.

The GNSS compass model differs from the standard single GNSS baseline model in that the length of the baseline is assumed known. Hence, the GNSS compass model is the standard GNSS model with an additional baseline length constraint. When considering ambiguity resolution for the GNSS compass model, one should keep in mind that the difficulty of computing and estimating the integer ambiguities depends very much on the strength of the underlying model. It will be easier when GNSS is externally aided with additional sensors (e.g. inertial measurement unit IMU). It will also be easier when multiple epochs and/or multiple frequencies are used. The ultimate challenge is therefore to be able to perform successful and efficient attitude ambiguity resolution for the *unaided, single frequency, single epoch* case. In (*Park and Teunissen*, 2003) it was shown that this is indeed possible with the use of the LAMBDA method. This result, however, seems to have passed relatively unnoticed. One of the goals of the present contribution is therefore to provide a more detailed methodological description of how these results are obtained. Two other goals of this contribution are to compare the standard GNSS model with the GNSS compass model and to provide a description of some variations on the integer estimation process.

This contribution is organised as follows. In Section 2, we give a very brief review of ambiguity resolution for the standard GNSS model. In Section 3, we introduce the GNSS compass model and describe a first, albeit approximate, approach to ambiguity resolution. In Sections 4 and 5, we describe the ambiguity resolution building blocks for the GNSS compass model. In Section 4, we first derive the integer least-squares estimators of the ambiguities and the constrained baseline. The search space for the integer ambiguities is introduced and it is shown that it needs the solution of a quadratically constrained least-squares problem. This least-squares problem is particularly computational intensive if it needs to be solved many times. A relaxation of this part is therefore introduced, which results in the use of bounding search spaces. Section 4 includes the method of Park and Teunissen (2003). In Section 5, we introduce an orthogonal decomposition of the objective function that differs from the one used in Section 4. This alternative is based on the constrained float solution and is therefore expected to improve efficiency. Since it leads to a search space which is similar in structure to the one given in Section 4, the same LAMBDA based search approach can be used.

2 INTEGER AMBIGUITY RESOLUTION

2.1 THE UNCONSTRAINED GNSS BASELINE MODEL

In principle all the GNSS baseline models can be cast in the following frame of linear(ized) observation equations,

$$E(y) = Aa + Bb \ , \ \ D(y) = Q_y \tag{1}$$

where $E(.)$ and $D(.)$ denote the expectation and dispersion operator, y is the given GNSS data vector of order m, a and b are the unknown parameter vectors of order n and p, and where A and B are the given design matrices that link the data vector to the unknown parameters. The geometry matrix B contains the unit line-of-sight vectors. The variance matrix of y is given by the positive definite matrix Q_y, which is assumed known. The data vector y will usually consist of the 'observed minus computed' single- or multiple-frequency double-difference (DD) phase and/or pseudorange (code) observations accumulated over all observation epochs. The entries of vector a are then the DD carrier phase ambiguities, expressed in units of cycles rather than range. They are known to be *integers*, $a \in Z^n$. The entries of the vector b will consist of the remaining unknown parameters, such as for instance baseline components (coordinates) and possibly atmospheric delay parameters (troposphere, ionosphere). They are known to be real-valued, $b \in R^p$.

When solving the GNSS model (1), one usually applies the least-squares principle. This amounts to solving the following minimization problem,

$$\min_{a,b} \parallel y - Aa - Bb \parallel^2_{Q_y} \quad , a \in Z^n \ , b \in R^p \tag{2}$$

with the weighted squared norm $||.||^2_{Q_y} = (.)^T Q_y^{-1}(.)$. Note that the minimization is taken over the space $Z^n \times R^p$. Problem (2) was introduced in (*Teunissen*, 1993) and has been called a (mixed) *integer least-squares* (ILS) problem by the author.

2.2 AN ORTHOGONAL DECOMPOSITION

To gain insight into the ILS-problem (2), it is helpful to first apply an orthogonal decomposition to the objective function. Hence, we write the objective function as a sum of three squares,

$$\parallel y - Aa - Bb \parallel^2_{Q_y} = \parallel \hat{e} \parallel^2_{Q_y} + \parallel \hat{a} - a \parallel^2_{Q_{\hat{a}}} + \parallel \hat{b}(a) - b \parallel^2_{Q_{\hat{b}(a)}} \tag{3}$$

with

$$\begin{array}{rcl} \hat{e} & = & y - A\hat{a} - B\hat{b} \\ \hat{a} & = & (\bar{A}^T Q_y^{-1} \bar{A})^{-1} \bar{A}^T Q_y^{-1} y \\ \hat{b} & = & (\bar{B}^T Q_y^{-1} \bar{B})^{-1} \bar{B}^T Q_y^{-1} y \\ \hat{b}(a) & = & (B^T Q_y^{-1} B)^{-1} B^T Q_y^{-1} (y - Aa) \end{array} \tag{4}$$

where $\bar{A} = P_B^{\perp} A$, $\bar{B} = P_A^{\perp} B$, with the orthogonal projectors $P_B^{\perp} = I - P_B$, $P_B = B(B^T Q_y^{-1} B)^{-1} B^T Q_y^{-1}$, $P_A^{\perp} = I - P_A$ and $P_A = A(A^T Q_y^{-1} A)^{-1} A^T Q_y^{-1}$. The matrix P_B is the orthogonal projector that projects orthogonally onto the range of B (with respect to the metric of Q_y^{-1}). Similarly, P_A is the orthogonal projector that projects orthogonally onto the range of A.

The variance matrices of $\hat{a}$, $\hat{b}$ and $\hat{b}(a)$ are given as

$$Q_{\hat{a}} = (\bar{A}^T Q_y^{-1} \bar{A})^{-1} \ , \ Q_{\hat{b}} = (\bar{B}^T Q_y^{-1} \bar{B})^{-1} \ , \ Q_{\hat{b}(a)} = (B^T Q_y^{-1} B)^{-1} \tag{5}$$

The vectors $\hat{a}$ and $\hat{b}$ are referred to as the float ambiguity solution and the float baseline solution, respectively. They follow when one solves (2) *without* the integer constraints $a \in Z^n$. The vector $\hat{e}$ is the least-squares residual vector that corresponds with this float solution.

The vector $\hat{b}(a)$ is the least-squares solution for b, assuming that a is known. It is therefore a conditional least-squares solution of b. Note that the conditional least-squares solution $\hat{b}(a)$ and its variance matrix $Q_{\hat{b}(a)}$, can also be written as

$$\hat{b}(a) = \hat{b} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} (\hat{a} - a) \ , \ Q_{\hat{b}(a)} = Q_{\hat{b}} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} Q_{\hat{a}\hat{b}} \tag{6}$$

with the covariance matrix $Q_{\hat{b}\hat{a}} = -(B^T Q_y^{-1} B)^{-1} B^T Q_y^{-1} A Q_{\hat{a}}$.

2.3 INTEGER LEAST-SQUARES SOLUTION

With the help of the orthogonal decomposition (3), we can now show how the solution of the ILS-problem (2) is obtained. We have

$$\begin{aligned} &\min_{a\in Z^n, b\in R^p} \parallel y - Aa - Bb \parallel^2_{Q_y} = \\ &= \parallel \hat{e} \parallel^2_{Q_y} + \min_{a\in Z^n, b\in R^p} \left(\parallel \hat{a} - a \parallel^2_{Q_{\hat{a}}} + \parallel \hat{b}(a) - b \parallel^2_{Q_{\hat{b}(a)}} \right) \\ &= \parallel \hat{e} \parallel^2_{Q_y} + \min_{a\in Z^n} \left(\parallel \hat{a} - a \parallel^2_{Q_{\hat{a}}} + \min_{b\in R^p} \parallel \hat{b}(a) - b \parallel^2_{Q_{\hat{b}(a)}} \right) \end{aligned} \tag{7}$$

Note that the first term, $||\hat{e}||^2_{Q_y}$, is irrelevant for the minimization, since it does not depend on a and b. Also note that the last term can be made zero for any a. Hence, the sought for ILS-solution is given as

$$\begin{aligned} \check{a} &= \arg\min_{a\in Z^n} ||\hat{a} - a||^2_{Q_{\hat{a}}} \\ \check{b} &= \hat{b}(\check{a}) = \hat{b} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1}(\hat{a} - \check{a}) \end{aligned} \tag{8}$$

The vectors $\check{a}$ and $\check{b}$ are often referred to as the fixed ambiguity solution and the fixed baseline solution, respectively.

2.4 THE INTEGER SEARCH

An integer search is needed to compute $\check{a}$. The LAMBDA method provides an efficient way of obtaining $\check{a}$ (*Teunissen*, 1993). The main steps are as follows. One starts by defining the ambiguity search space

$$\Omega_{\hat{a}} = \{a \in Z^n \mid \; ||\hat{a} - a||^2_{Q_{\hat{a}}} \; \leq \; \chi^2\} \tag{9}$$

with χ^2 a suitable chosen positive constant. In order for the search space not to contain too many integer vectors, a small value for χ^2 is required, but one that still guarantees that the search space contains at least one integer grid point.

The boundary of the search space $\Omega_{\hat{a}}$ is ellipsoidal. It is centred at $\hat{a}$ and its shape is governed by the variance matrix $Q_{\hat{a}}$. In case of GNSS, the search space is usually extremely elongated, due to the high correlations between the ambiguities. Since this extreme elongation usually hinders the computational efficiency of the search, the search space is first transformed to a more spherical shape,

$$\Omega_{\hat{z}} = \{z \in Z^n \mid \; ||\hat{z} - z||^2_{Q_{\hat{z}}} \; \leq \; \chi^2\} \tag{10}$$

using an admissible ambiguity transformation: $z = Ta$, $\hat{z} = T\hat{a}$, $Q_{\hat{z}} = TQ_{\hat{a}}T^T$. Ambiguity transformations T are said to be admissible when both T and its inverse T^{-1} have integer entries. Such matrices preserve the integer nature of the ambiguities. In order for the transformed search space to become more spherical, the volume-preserving T-transformation is constructed as a transformation that decorrelates the ambiguities as much as possible. Using the triangular decomposition of $Q_{\hat{z}}$, the left-hand side of the quadratic inequality in (10) is then written as a sum-of-squares:

$$\sum_{i=1}^{n} \frac{(\hat{z}_{i|I} - z_i)^2}{\sigma^2_{i|I}} \; \leq \; \chi^2 \tag{11}$$

On the left-hand side one recognizes the conditional least-squares estimate $\hat{z}_{i|I}$, which follows when the conditioning takes place on the integers $z_1, z_2, \ldots, z_{i-1}$. Using the sum-of-squares structure, one can finally set up the n intervals which are used for the search. These n sequential intervals are given as

$$(\hat{z}_1 - z_1)^2 \leq \sigma_1^2 \chi^2 \ , \ \ldots \ , \ (\hat{z}_{n|N} - z_n)^2 \leq \sigma_{n|N}^2 \left(\chi^2 - \sum_{i=1}^{n-1} \frac{(\hat{z}_{i|I} - z_i)^2}{\sigma_{i|I}^2} \right) \tag{12}$$

For more information on the LAMBDA method, we refer to e.g. (*Teunissen*, 1993), (*Teunissen*, 1995) and (*de Jonge and Tiberius*, 1996a) or to the textbooks (*Hofmann-Wellenhof*, 1997), (*Strang and Borre*, 1997), (*Teunissen and Kleusberg*, 1998), (*Misra and Enge*, 2006). Examples of applications can be found in e.g. (*Boon and Ambrosius*, 1997), (*Cox and Brading*, 1999), (*de Jonge and Tiberius*, 1996b), or (*de Jonge et al.*, 1996).

3 THE GNSS COMPASS MODEL

3.1 THE GNSS COMPASS MODEL: CONSTRAINED VERSION

For the GNSS compass model, vector b is taken as a 3×1 vector that consists of the three baseline components. Thus the atmospheric delays are assumed absent. Although this assumption is not really necessary for the methods described in this and the following sections, the assumption is made for reasons of simplicity. Moreover, in most applications of the GNSS compass model, the length of the baseline is such that atmospheric delays can indeed be neglected.

An essential assumption for the GNSS compass model is that the length of the baseline is assumed known. Thus $||b||_{I_3} = l$, with l known. Hence, the GNSS compass model follows from the standard GNSS model (1) by adding the length-constraint of the baseline,

$$E(y) = Aa + Bb \ , \ ||b||_{I_3} = l \ , \ a \in Z^n, b \in R^3 \tag{13}$$

This formulation of the GNSS compass model will be referred to as the constrained version. It is parametrized in the three baseline components of b and it shows the baseline constraint explicitly by means of the equation $||b||_{I_3} = l$.

3.2 THE GNSS COMPASS MODEL: UNCONSTRAINED VERSION

It is also possible to formulate an unconstrained version of the GNSS compass model. This is done by reparametrizing b such that the constraint is automatically fulfilled. This can be done by reparametrizing b in spherical coordinates,

$$b(\gamma) = l \begin{bmatrix} \cos\alpha \cos\beta \\ \cos\alpha \sin\beta \\ \sin\alpha \end{bmatrix} \tag{14}$$

with $\gamma = [\alpha, \beta]^T$. Substitution into (13) gives

$$E(y) = Aa + Bb(\gamma) \ , \ a \in Z^n, \gamma \in R^2 \tag{15}$$

The two model formulations, (13) and (15), are equivalent. In (13), the unknown parameters are a and b, whereas in (15), they are a and γ. Thus the reparametrization turns the originally constrained model (13) into a nonlinear, but *unconstrained* model (15). The three unknown components of b have been reduced to the two unknown angles α and β.

In Sections 4 and 5, we will consider the constrained version for ambiguity resolution. First, however, we consider the unconstrained version.

3.3 INTEGER ESTIMATION BASED ON LINEARIZATION

The unconstrained model (15) is nonlinear in γ. It can be brought into the standard form (1) by means of a linearization. Let the approximate values for the angles be provided by $\gamma_0 = [\alpha_0, \beta_0]^T$. Linearization of the baseline vector gives then

$$b(\gamma) = b(\gamma_0) + C(\gamma_0)\Delta\gamma \tag{16}$$

with

$$C(\gamma_0) = \begin{bmatrix} -\sin\alpha_0 \cos\beta_0 & -\cos\alpha_0 \sin\beta_0 \\ -\sin\alpha_0 \sin\beta_0 & \cos\alpha_0 \cos\beta_0 \\ \cos\alpha_0 & 0 \end{bmatrix} \tag{17}$$

and $\Delta\gamma = \gamma - \gamma_0$. Substitution into the observation equations of (15), gives the unconstrained linearized GNSS model as

$$E(\Delta y) = Aa + BC(\gamma_0)\Delta\gamma \;,\; a \in R^n, \Delta\gamma \in R^2 \tag{18}$$

with $\Delta y = y - Bb(\gamma_0)$. Since this unconstrained GNSS compass model is of the type (1), the same steps can be used as described in Section 2 for ambiguity resolution. Matrix $BC(\gamma_0)$ and vector $\Delta\gamma$ in (18) play the role of matrix B and vector b in (1).

The above approach is based on a linearization and thus requires approximate values. These approximate values need to be close enough to the sought for minimizers in order for the linearization to be valid. Such approximate values could possibly be obtained from sensors that externally aid the attitude determination process (e.g. IMU) or from the GNSS float solution itself. In the latter case, however, the requirements on the float solution become more stringent the shorter the baseline is. That is, for long baselines, the float solution is often good enough for it to be used as a way of computing approximate values for the angles. For short baselines, however, this may not be the case.

Another aspect that one has to keep in mind, is that the above linearized approach (possibly with iterations included) does not guarantee that one will obtain a global minimum of the original constrained integer least-squares problem. We will now present a solution that does guarantee a global minimum.

4 BASELINE CONSTRAINED INTEGER AMBIGUITY RESOLUTION (I)

4.1 INTEGER LEAST-SQUARES SOLUTION

From now on we will work with the constrained model formulation (13). The ILS-problem to be solved is then

$$\min_{a \in Z^n, b \in R^3, ||b|| = l} \parallel y - Aa - Bb \parallel^2_{Q_y} \tag{19}$$

With the use of the orthogonal decomposition (3), we can write the objective function of (19) as a sum of squares. This gives, instead of (7), the minimization problem

$$\begin{aligned} &\min_{a\in Z^n, b\in R^3, \|b\|=l} \| y - Aa - Bb \|^2_{Q_y} = \\ &= \| \hat{e} \|^2_{Q_y} + \min_{a\in Z^n, b\in R^3, \|b\|=l} \left(\| \hat{a} - a \|^2_{Q_{\hat{a}}} + \| \hat{b}(a) - b \|^2_{Q_{\hat{b}(a)}} \right) \\ &= \| \hat{e} \|^2_{Q_y} + \min_{a\in Z^n} \left(\| \hat{a} - a \|^2_{Q_{\hat{a}}} + \min_{b\in R^3, \|b\|=l} \| \hat{b}(a) - b \|^2_{Q_{\hat{b}(a)}} \right) \end{aligned} \tag{20}$$

Note that, due to the constraint, the third term in the last equation can now not be made equal to zero. If we define

$$\check{b}(a) = \arg \min_{b\in R^3, \|b\|=l} \| \hat{b}(a) - b \|^2_{Q_{\hat{b}(a)}} \tag{21}$$

then the ILS-solution is given as

$$\begin{aligned} \check{a} &= \arg\min_{a\in Z^n} \left(\| \hat{a} - a \|^2_{Q_{\hat{a}}} + \| \hat{b}(a) - \check{b}(a) \|^2_{Q_{\hat{b}(a)}} \right) \\ \check{b} &= \check{b}(\check{a}) \end{aligned} \tag{22}$$

From the baseline solution $\check{b}$, the necessary compass information of heading and elevation (or yaw and pitch) can be recovered. Compare (22) with (8) and note that the computation of $\check{b}(a)$ requires the solution of a quadratically constrained least-squares problem.

4.2 QUADRATICALLY CONSTRAINED LEAST-SQUARES

There are different ways of tackling the quadratic least-squares problem (21). Let us first consider the problem from a geometric point of view. The problem reads

$$\min_{b\in R^3} ||\hat{b}(a) - b||^2_{Q_{\hat{b}(a)}} \quad \text{subject to} \quad ||b||^2_{I_3} = l^2 \tag{23}$$

Note that $||\hat{b}(a) - b||^2_{Q_{\hat{b}(a)}} = c^2$ is the equation of an ellipsoid centred at $\hat{b}(a)$ and that $||b||^2_{I_3} = l^2$ is the equation of a sphere, with radius l, centred at the origin. Thus the problem amounts to finding the smallest ellipsoid that just touches the sphere. At this point of contact the ellipsoid and the sphere will have the same tangent plane and the same normal vector (note: in exceptional cases, which we disregard, there may be more than one point of contact; the solution to (23) is then nonunique). The outer normal vector (i.e. direction of steepest ascent) of the ellipsoid is given by the gradient of $||\hat{b}(a) - b||^2_{Q_{\hat{b}(a)}}$ and the outer normal vector of the sphere is given by the gradient of $||b||^2_{I_3}$. Hence, at the point where the two surfaces touch, the normal vector of the ellipsoid is a scaled version of the normal vector of the sphere. Denoting the scalar by λ and equating the two normal vectors results then in the normal equations

$$(Q^{-1}_{\hat{b}(a)} - \lambda I_3)b = Q^{-1}_{\hat{b}(a)} \hat{b}(a) \tag{24}$$

(Note: this is an eigenvalue problem if $\hat{b}(a) = 0$. For $\hat{b}(a) \neq 0$, the normal equations can be rewritten, with the aid of the baseline constraint, as a quadratic eigenvalue problem. We will, however, not pursue this approach further in this contribution). The scalar λ (the Lagrange multiplier) can be positive, zero or negative. It is zero when $\hat{b}(a)$ already lies on the sphere. The scalar λ will be positive when $\hat{b}(a)$ lies inside the sphere, since then the two outer normal vectors of ellipsoid and sphere will point in the same direction. The scalar λ will be negative when $\hat{b}(a)$ lies outside the sphere, since then the two outer normal vectors will point in opposite direction. Another way of understanding the sign of λ is to note that the length of b must be smaller than that of $\hat{b}(a)$ if $||\hat{b}(a)|| > l$, i.e. if the vector lies outside the sphere. This is accomplished by having $-\lambda > 0$ in the normal equations (24).

The above normal equations will have different solutions b_λ for different values of λ. To determine λ such that b_λ is the solution of (23), we set $b_\lambda^T b_\lambda = l^2$ and find the optimal value for λ as the smallest root of this nonlinear equation, whereby the singular value decomposition (SVD) can be used to bring the above normal equations in canonical form, thereby facilitating the formulation of the nonlinear equation, see e.g. (*Gander*, 1981), (*Bjork*, 1996). Once the optimal value for λ has been found, say $\hat{\lambda}$, the sought for solution follows as $\check{b}(a) = b_{\hat{\lambda}}$.

Note that the quadratically constrained least-squares problem becomes trivial in case of a scaled unit matrix, $Q_{\hat{b}(a)} = \mu^{-1} I_3$. In that case we have

$$\check{b}(a) = l\hat{b}(a)/||\hat{b}(a)||_{I_3} \ , \ \hat{\lambda} = \mu(1 - ||\hat{b}(a)||_{I_3}/l) \tag{25}$$

and

$$\min_{b \in R^3, ||b||=l} ||\hat{b}(a) - b||^2_{Q_{\hat{b}(a)}} = \mu(||\hat{b}(a)||_{I_3} - l)^2 \tag{26}$$

Intermezzo We will now show, for later reference, how $\check{b}(a)$ can be viewed as a solution to an *unconstrained* least-squares problem. For $\hat{\lambda} < 0$, the normal equations (24) can be written as

$$(Q^{-1}_{\hat{b}(a)} + |\hat{\lambda}| I_3)\check{b}(a) = Q^{-1}_{\hat{b}(a)} \hat{b}(a) \tag{27}$$

and for $\hat{\lambda} > 0$, as

$$(Q^{-1}_{\hat{b}(a)} + \hat{\lambda} I_3)\check{b}(a) = Q^{-1}_{\hat{b}(a)} \left(2\check{b}(a) - \hat{b}(a)\right) \tag{28}$$

Both normal equations have a positive definite normal matrix with the same structure. Their right hand sides, however, differ. Note that the point $2\check{b}(a) - \hat{b}(a)$ is the reflection of $\hat{b}(a)$ about $\check{b}(a)$. Thus if $\hat{b}(a)$ lies inside the sphere, then $2\check{b}(a) - \hat{b}(a)$ lies outside the sphere. Thus we achieved the same structure for the normal matrix, by having right hand sides that in both cases consists of a point outside the sphere.

From the structure of the above normal equations it follows that $\check{b}(a)$ can be seen to be the solution of the unconstrained least-squares problem,

$$\min_{b \in R^3} || \begin{bmatrix} \beta \\ 0 \end{bmatrix} - \begin{bmatrix} I_3 \\ I_3 \end{bmatrix} b ||^2_{Q_\beta} \ \text{ with } \ Q_\beta = \begin{bmatrix} Q_{\hat{b}(a)} & 0 \\ 0 & |\hat{\lambda}|^{-1} I_3 \end{bmatrix} \tag{29}$$

and where $\beta = \hat{b}(a)$ if $\hat{\lambda} < 0$ and $\beta = 2\check{b}(a) - \hat{b}(a)$ if $\hat{\lambda} > 0$.

An alternative approach for solving the quadratically constrained least-squares problem, and one which avoids the use of the SVD and the root-finding of $b_\lambda^T b_\lambda = l^2$, is based on an iteration of orthogonal projections onto an ellipsoid. Note that problem (23) can be reformulated as the problem of finding the closest point to an ellipsoid. This problem is similar to the problem of computing geodetic coordinates from Cartesian coordinates.

Since this latter approach turned out to be somewhat more efficient than the one based on the SVD (albeit with no guaranteed convergence to a global minimum if a poor initial is used), it was used in (*Park and Teunissen*, 2003).

4.3 INTEGER SEARCH (I)

One will need an integer search for computing the integer least-squares ambiguity vector $\check{a}$, cf. (22). We define the integer search space as

$$\Psi(\chi^2) = \{a \in Z^n | \parallel \hat{a} - a \parallel^2_{Q_{\hat{a}}} + \parallel \hat{b}(a) - \check{b}(a) \parallel^2_{Q_{\hat{b}(a)}} \leq \chi^2\} \tag{30}$$

This search space is not ellipsoidal anymore (compare with (9)), due to the presence of the residual baseline term. The idea behind the search for $\check{a}$ is as follows. Assuming that $\Psi(\chi^2)$ is not empty, one first collects all integer vectors inside $\Psi(\chi^2)$ and then one selects the one which returns the smallest value for the objective function of (22). To set up the integer search, we introduce the auxiliary ellipsoidal search space

$$\Psi_0(\chi^2) = \{a \in Z^n | ||\hat{a} - a||^2_{Q_{\hat{a}}} \leq \chi^2\} \tag{31}$$

Note that $\Psi(\chi^2) \subset \Psi_0(\chi^2)$. Thus $\Psi_0(\chi^2)$ contains all integer vectors of $\Psi(\chi^2)$ and thus also the sought for solution $\check{a}$. The search starts with collecting all integer vectors inside $\Psi_0(\chi^2)$, which can be done efficiently with the LAMBDA method as described in Section 2. From this set, we then retain only those that satisfy the inequality $\parallel \hat{b}(a) - \check{b}(a) \parallel^2_{Q_{\hat{b}(a)}} \leq \chi^2 - \parallel \hat{a} - a \parallel^2_{Q_{\hat{a}}}$. This resulting set is $\Psi(\chi^2)$.

The positive scalar χ^2 sets the size of the search space. In order to avoid an abundance of integer vectors inside the search space, one would prefer a small value for χ^2. However, in order to guarantee that the search space is not empty, χ^2 should not be chosen too small. It will be clear that the search space is not empty, if χ^2 is chosen as

$$\chi^2(a) = \parallel \hat{a} - a \parallel^2_{Q_{\hat{a}}} + \parallel \hat{b}(a) - \check{b}(a) \parallel^2_{Q_{\hat{b}(a)}} \tag{32}$$

for some $a \in Z^n$. To obtain a small enough value, one would prefer to choose a close to $\check{a}$ (note: $\check{a}$ itself is, of course, no option, since this is the solution we are looking for). For GNSS baseline models that have enough strength, a good approach is to use either the bootstrapped solution $\check{a}_B \in Z^n$, based on $\hat{a}$ and $Q_{\hat{a}}$ (of course after the decorrelation step), or the integer least-squares solution $\check{a}_{ILS} = \arg\min_{a \in Z^n} ||\hat{a} - a||^2_{Q_{\hat{a}}}$. For such models it can be shown that the probability of correct integer estimation (the success rate) of $\check{a}_B$ or $\check{a}_{ILS}$ is already very close to one, thus indicating that they are good candidates for setting the size of the search space $\Psi(\chi^2)$. GNSS models that have enough strength are, for instance, short baselines models using multiple frequencies. Hence, if these models are used for the GNSS compass, the above approach may be used for setting the size of the search space.

4.4 INTEGER SEARCH (II)

A much more challenging situation occurs if one considers the GNSS compass model based on *single epoch, single frequency* data. In this case the success rate of $\check{a}_B$ or $\check{a}_{ILS}$ is too low, as a consequence of which $\chi^2(\check{a}_B)$ or $\chi^2(\check{a}_{ILS})$ will often be too large. This can be explained as follows. In the single epoch, single frequency (say L_1) phase-code case, the design matrices A and B are structured as $A = [\lambda_1 I_m, 0]^T$ ($m+1$ is the number of satellites tracked) and $B = [G^T, G^T]^T$, with λ_1 the L_1 wavelength and G the geometry matrix, which contains the unit direction vectors to the satellites. The variance matrices used in (32) are then given as

$$Q_{\hat{a}} = \frac{\sigma_\phi^2}{\lambda_1^2}\left(Q + \frac{\sigma_p^2}{\sigma_\phi^2} G(G^T Q^{-1} G)^{-1} G^T\right) \ , \ Q_{\hat{b}(a)} = \frac{\sigma_\phi^2}{1+\sigma_\phi^2/\sigma_p^2}(G^T Q^{-1} G)^{-1} \qquad (33)$$

with Q the cofactor matrix due to the double differencing and σ_ϕ^2, σ_p^2 the variances of the phase and code data, respectively. This shows, since $\sigma_\phi^2 << \sigma_p^2$, that the precision of the float solution $\hat{a}$ is dominated by the relatively imprecise code data, whereas the precision of the conditional baseline solution $\hat{b}(a)$ is governed by the very precise phase data. Hence, for most a the second term on the right hand side of (32) will be much larger than the first term. As a consequence, χ^2 will be large too, which implies that $\Psi_0(\chi^2)$ will contain many integer vectors, most of which will be rejected again by the inequality check $\| \hat{b}(a) - \check{b}(a) \|^2_{Q_{\hat{b}(a)}} \leq \chi^2 - \| \hat{a} - a \|^2_{Q_{\hat{a}}}$. Thus in this case many of the collected integer vectors will be computed with no avail (search halting) and, moreover, for the many integer vectors inside $\Psi_0(\chi^2)$, one will have to compute $\check{b}(a)$, which may considerably slow down the estimation process. The conclusion is therefore, that for the single epoch, single frequency case, an alternative approach is needed for selecting χ^2. Moreover, it would be helpful, if, in the evaluation of the integer candidates, one can avoid the necessity of having to compute $\| \hat{b}(a) - \check{b}(a) \|^2_{Q_{\hat{b}(a)}}$ too often. We will first address this latter problem.

4.4.1 Bounding the search space

The computation of $\check{b}(a)$ is easy if the matrix $Q_{\hat{b}(a)}$ is a scaled unit matrix. In our applications, however, this is not the case. The computation of $\check{b}(a)$ may then become a computational burden if it needs to be done for many integer candidates a. We will now show how this can be avoided, at the expense, however, of a change in the search space.

Let λ_{min} and λ_{max} be the smallest and largest eigenvalue of $Q^{-1}_{\hat{b}(a)}$. Then

$$\lambda_{min} \min_{||b||=l} ||\hat{b}(a) - b||^2_{I_3} \leq \min_{||b||=l} ||\hat{b}(a) - b||^2_{Q_{\hat{b}(a)}} \leq \lambda_{max} \min_{||b||=l} ||\hat{b}(a) - b||^2_{I_3} \qquad (34)$$

Since $\min_{||b||=l} ||\hat{b}(a) - b||^2_{I_3}$ is the problem of finding the closest vector on a sphere of radius l, we have $\min_{||b||=l} ||\hat{b}(a) - b||^2_{I_3} = (||\hat{b}(a)||_{I_3} - l)^2$. Hence, (34) can be written as

$$\lambda_{min}(||\hat{b}(a)||_{I_3} - l)^2 \leq \min_{||b||=l} ||\hat{b}(a) - b||^2_{Q_{\hat{b}(a)}} \leq \lambda_{max}(||\hat{b}(a)||_{I_3} - l)^2 \qquad (35)$$

By noting that $\min_{||b||=l} ||\hat{b}(a) - b||^2_{Q_{\hat{b}(a)}} \leq ||\hat{b}(a) - b||^2_{Q_{\hat{b}(a)}}$ for $b = l\hat{b}/||\hat{b}||_{I_3}$, a somewhat sharper upper bound can be obtained that avoids the computation of λ_{max}. Hence,

$$\lambda_{min}(||\hat{b}(a)||_{I_3} - l)^2 \leq \min_{||b||=l} ||\hat{b}(a) - b||^2_{Q_{\hat{b}(a)}} \leq ||\hat{b}||^2_{Q_{\hat{b}(a)}}(||\hat{b}(a)||_{I_3} - l)^2/||\hat{b}||^2_{I_3} \qquad (36)$$

We now define the functions

$$\begin{array}{rcl} F_1(a) & = & ||\hat{a} - a||^2_{Q_{\hat{a}}} + \lambda_{min}(||\hat{b}(a)||_{I_3} - l)^2 \\ F(a) & = & ||\hat{a} - a||^2_{Q_{\hat{a}}} + ||\hat{b}(a) - \check{b}(a)||^2_{Q_{\hat{b}(a)}} \\ F_2(a) & = & ||\hat{a} - a||^2_{Q_{\hat{a}}} + ||\hat{b}||^2_{Q_{\hat{b}(a)}}(||\hat{b}(a)||_{I_3} - l)^2/||\hat{b}||^2_{I_3} \end{array} \qquad (37)$$

and the integer sets

$$\begin{array}{lcl} \Psi_1(\chi^2) & = & \{a \in Z^n | F_1(a) \leq \chi^2\} \\ \Psi(\chi^2) & = & \{a \in Z^n | F(a) \leq \chi^2\} \\ \Psi_2(\chi^2) & = & \{a \in Z^n | F_2(a) \leq \chi^2\} \end{array} \tag{38}$$

Then

$$F_1(a) \leq F(a) \leq F_2(a) \text{ and } \Psi_2(\chi^2) \leq \Psi(\chi^2) \leq \Psi_1(\chi^2) \tag{39}$$

Hence, we have created, by means of the two inequalities of (36), two new integer sets, $\Psi_1(\chi^2)$ and $\Psi_2(\chi^2)$, for which it will be easier to collect all integer candidates.

4.4.2 Using the bounding sets for the search

So far we did not describe how (38) can be used to set up the search. Different approaches are possible. Here we will describe the one which has been successfully used in (*Park and Teunissen*, 2003). It uses $\Psi_1(\chi^2)$ and a small value for χ^2, which, if necessary, is incremented to the point that $\Psi(\chi^2)$ is nonempty. The search goes as follows. We start with an initial value χ_0^2 and then collect all integer vectors inside $\Psi_0(\chi_0^2)$. Of all integer vectors inside $\Psi_0(\chi_0^2)$, one then collects those that are in $\Psi_1(\chi_0^2)$. The following two situations may now occur: $\Psi_1(\chi_0^2)$ can either be empty or not empty. If $\Psi_1(\chi_0^2)$ is empty, then so will $\Psi(\chi_0^2)$ be. The value χ_0^2 is then increased, say to χ_1^2, so that $\Psi_1(\chi_1^2)$ is not empty. Of those (few) candidates inside $\Psi_1(\chi_1^2)$, one then checks whether they are in $\Psi(\chi_1^2)$ as well. If not, one increases χ_1^2 again and repeats the process, otherwise one selects the candidate in $\Psi(\chi_1^2)$ which returns the smallest value for $F(a)$. This latter candidate will then be the solution sought.

There are different ways for choosing a starting value for χ^2. One can use $\chi_0^2 = ||\hat{a} - \check{a}_B||^2_{Q_{\hat{a}}}$. One can also use a probabilistic approach by making use of the chi-squared distribution, i.e. determine the value of χ_0^2 that corresponds to a selected probability that the true value a is contained in the search space. Or, noting that the volume of $\Psi_0(\chi^2)$ is easy to compute, one can make use of the fact that the volume of a region is an estimate for the number of integer vectors contained in it. Or one can make use of a quadratic approximation of the function F and use its integer minimizer as an approximation of $\check{a}$. In our experience, the first choice already works fine (*Park and Teunissen*, 2003).

5 BASELINE CONSTRAINED INTEGER AMBIGUITY RESOLUTION (II)

5.1 THE CONSTRAINED FLOAT SOLUTION

Note that $\hat{a}$ in (22), the least-squares estimator of a, is not based on the baseline constraint $||b||_{I_3} = l$. Hence an improved estimator of a can be obtained by incorporating this constraint. We therefore consider the baseline constrained least-squares problem

$$\min_{a \in R^n, b \in R^3, ||b||_{I_3} = l} ||y - Aa - Bb||^2_{Q_y} \tag{40}$$

Note that this minimization problem differs from (19), since it does not include the integer constraints on a. It does, however, include the baseline constraint. The solution to (40), which we denote as $\hat{a}_l$ and $\hat{b}_l$, will therefore be referred to as the constrained float solution. To determine this solution, we make use of the orthogonal decomposition

$$\| y - Aa - Bb \|^2_{Q_y} = \| \hat{e} \|^2_{Q_y} + \| \hat{b} - b \|^2_{Q_{\hat{b}}} + \| \hat{a}(b) - a \|^2_{Q_{\hat{a}(b)}} \tag{41}$$

which follows from (3) by interchanging the role of a and b. Substitution of (41) into (40), gives

$$\begin{aligned} &\min_{a \in R^n, b \in R^3, ||b||=l} \| y - Aa - Bb \|^2_{Q_y} = \\ &= \| \hat{e} \|^2_{Q_y} + \min_{a \in R^n, ||b||=l} \left(\| \hat{b} - b \|^2_{Q_{\hat{b}}} + \| \hat{a}(b) - a \|^2_{Q_{\hat{a}(b)}} \right) \\ &= \| \hat{e} \|^2_{Q_y} + \min_{||b||=l} \left(\| \hat{b} - b \|^2_{Q_{\hat{b}}} + \min_{a \in R^n} \| \hat{a}(b) - a \|^2_{Q_{\hat{a}(b)}} \right) \end{aligned} \tag{42}$$

Note that the third term in the last equation can be made zero for any b. Hence, the constrained float solution is given as

$$\begin{aligned} \hat{b}_l &= \arg\min_{b \in R^3, ||b||=l} ||\hat{b} - b||^{-1}_{Q_{\hat{b}}} \\ \hat{a}_l &= \hat{a}(\hat{b}_l) = \hat{a} - Q_{\hat{a}\hat{b}} Q^{-1}_{\hat{b}} (\hat{b} - \hat{b}_l) \end{aligned} \tag{43}$$

Thus $\hat{b}_l$ (having the property that $||\hat{b}_l|| = l$) follows from a quadratically constrained least-squares problem having $\hat{b}$ as input and $\hat{a}_l$ follows from adjusting $\hat{a}$ on the basis of the residual baseline vector $\hat{b} - \hat{b}_l$.

Since both $\hat{a}_l$ and $\hat{b}_l$ can expected to be improvements over $\hat{a}$ and $\hat{b}$, respectively, the question arises whether they can be used in the integer search in the same way as the unconstrained float solution was used. The answer is in the affirmative, as the next sections will show.

5.2 AN ALTERNATIVE ORTHOGONAL DECOMPOSITION

In this section we will formulate an orthogonal decomposition similar to the one of (3), but with $\hat{a}_l$ and $\hat{b}_l$ now playing the role of $\hat{a}$ and $\hat{b}$, respectively. This alternative decomposition is made possible by formulating the quadratically constrained least-squares problem as an unconstrained least-squares problem. We have, similar to (24), that the constrained float solution $\hat{a}_l$, $\hat{b}_l$ satisfies the normal equations

$$\begin{bmatrix} A^T Q_y^{-1} A & A^T Q_y^{-1} B \\ B^T Q_y^{-1} A & B^T Q_y^{-1} B + \hat{\lambda} I_3 \end{bmatrix} \begin{bmatrix} \hat{a}_l \\ \hat{b}_l \end{bmatrix} = \begin{bmatrix} A^T Q_y^{-1} y \\ B^T Q_y^{-1} y \end{bmatrix} \tag{44}$$

in which $-\hat{\lambda} > -\lambda_1$ is the resolved Lagrange multiplier and λ_1 the smallest eigenvalue of the reduced normal matrix $\bar{B}^T Q_y^{-1} \bar{B}$. Hence, $\hat{a}_l$ and $\hat{b}_l$ are also the solution of the *unconstrained* least-squares problem

$$\min_{a \in R^n, b \in R^3} ||y_l - A_l a - B_l b||^2_{Q_{y_l}} \tag{45}$$

with

$$y_l = \begin{bmatrix} y \\ 0 \end{bmatrix}, \; A_l = \begin{bmatrix} A \\ 0 \end{bmatrix}, \; B_l = \begin{bmatrix} B \\ I_3 \end{bmatrix}, \; Q_{y_l} = \begin{bmatrix} Q_y & 0 \\ 0 & \hat{\lambda}^{-1} I_3 \end{bmatrix}$$

We can therefore, similarly to (3), decompose the objective function of (45) as

$$\| y_l - A_l a - B_l b \|^2_{Q_{y_l}} = \| \hat{e}_l \|^2_{Q_{y_l}} + \| \hat{a}_l - a \|^2_{Q_{\hat{a}_l}} + \| \hat{b}_l(a) - b \|^2_{Q_{\hat{b}_l(a)}} \tag{46}$$

with

$$\begin{array}{lcl} \hat{e}_l & = & y_l - A_l\hat{a}_l - B_l\hat{b}_l \\ \hat{a}_l & = & (\bar{A}_l^T Q_{y_l}^{-1}\bar{A}_l)^{-1}\bar{A}_l^T Q_{y_l}^{-1} y_l \\ \hat{b}_l & = & (\bar{B}_l^T Q_{y_l}^{-1}\bar{B}_l)^{-1}\bar{B}_l^T Q_{y_l}^{-1} y_l \\ \hat{b}_l(a) & = & (B_l^T Q_{y_l}^{-1}B_l)^{-1}B_l^T Q_{y_l}^{-1}(y_l - A_l a) \end{array} \tag{47}$$

and where $Q_{\hat{a}_l} = (\bar{A}_l^T Q_{y_l}^{-1}\bar{A}_l)^{-1}$ and $Q_{\hat{b}_l(a)} = (B_l^T Q_{y_l}^{-1}B_l)^{-1}$. Furthermore, we have the equality

$$\| y - Aa - Bb \|_{Q_y}^2 = \| y_l - A_l a - B_l b \|_{Q_{y_l}}^2 - \hat{\lambda}||b||_{I_3}^2 \tag{48}$$

and therefore the decomposition

$$\| y - Aa - Bb \|_{Q_y}^2 = \| \hat{e}_l \|_{Q_{y_l}}^2 - \hat{\lambda}||b||_{I_3}^2 + \| \hat{a}_l - a \|_{Q_{\hat{a}_l}}^2 + \| \hat{b}_l(a) - b \|_{Q_{\hat{b}_l(a)}}^2 \tag{49}$$

The first term on the right hand side is independent of a and b, and therefore constant. This also holds true for the second term on the right hand side, since $\hat{\lambda}$ is independent of a and b, and the length of b is known to be equal to the constant l. Hence, for the baseline constrained integer minimization, we only need to consider the last two terms.

5.3 INTEGER LEAST-SQUARES SOLUTION

With the use of decomposition (49), we have

$$\begin{array}{l} \min_{a\in Z^n, b\in R^3, ||b||_{I_3}=l} \| y - Aa - Bb \|_{Q_y}^2 = \\ = \| \hat{e}_l \|_{Q_{y_l}}^2 + \min_{a\in Z^n, b\in R^3, ||b||_{I_3}=l}\left(\| \hat{a}_l - a \|_{Q_{\hat{a}_l}}^2 + \| \hat{b}_l(a) - b \|_{Q_{\hat{b}_l(a)}}^2 - \hat{\lambda}||b||_{I_3}^2\right) \\ = \| \hat{e}_l \|_{Q_{y_l}}^2 + \min_{a\in Z^n}\left(\| \hat{a}_l - a \|_{Q_{\hat{a}_l}}^2 + \min_{b\in R^3, ||b||_{I_3}=l}\left(\| \hat{b}_l(a) - b \|_{Q_{\hat{b}_l(a)}}^2 - \hat{\lambda}||b||_{I_3}^2\right)\right) \\ = \| \hat{e}_l \|_{Q_{y_l}}^2 - \hat{\lambda}l^2 + \min_{a\in Z^n}\left(\| \hat{a}_l - a \|_{Q_{\hat{a}_l}}^2 + \min_{b\in R^3, ||b||_{I_3}=l} \| \hat{b}_l(a) - b \|_{Q_{\hat{b}_l(a)}}^2\right) \end{array} \tag{50}$$

Thus if we define

$$\check{b}_l(a) = \arg\min_{b\in R^3, ||b||=l} \| \hat{b}_l(a) - b \|_{Q_{\hat{b}_l(a)}}^2 \tag{51}$$

then the integer least-squares solution is given as

$$\begin{array}{lcl} \check{a} & = & \arg\min_{a\in Z^n}\left(\| \hat{a}_l - a \|_{Q_{\hat{a}_l}}^2 + \| \hat{b}_l(a) - \check{b}_l(a) \|_{Q_{\hat{b}_l(a)}}^2\right) \\ \check{b} & = & \check{b}_l(\check{a}) \end{array} \tag{52}$$

Compare this formulation with that of (22). Both have the same structure and can therefore be tackled in the same way.

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