

## ON THE COMPUTATION OF THE BEST INTEGER EQUIVARIANT ESTIMATOR

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### ABSTRACT

Carrier phase integer ambiguity resolution is the key to high precision Global Navigation Satellite System (GNSS) positioning and navigation. In this contribution we study some of the computational aspects of best integer equivariant estimation. The best integer equivariant (BIE) estimator is the optimal estimator of the class of integer equivariant estimators, which is one of the three classes of estimators for carrier phase ambiguity resolution. The two other classes are the class of integer estimators and the class of integer aperture estimators. Since the BIE-estimator can not be computed exactly, it is shown how to approximate this estimator while retaining the property of integer equivariance. It is also shown how the decorrelating  $Z$ -transformation and the integer search of the LAMBDA method can be used to speed up the computation of the BIE-estimator.

**Keywords:** GNSS ambiguity resolution, integer least-squares, best integer equivariant estimation

### 1 INTRODUCTION

Global Navigation Satellite System (GNSS) ambiguity resolution is the process of resolving the unknown cycle ambiguities of double difference (DD) carrier phase data. It is the key to fast and high-precision GNSS relative positioning. An overview of GNSS carrier phase models, together with their applications in surveying, navigation, geodesy and geophysics, can be found in textbooks such as [Hofmann-Wellenhof et al., 2001], [Leick, 1995], [Misra and Enge, 2001], [Parkinson and Spilker, 1996], [Strang and Borre, 1997] and [Teunissen and Kleusberg, 1998].

In order to describe the problem of GNSS ambiguity resolution, we take as our point of departure the following system of linear observation equations

$$E\{y\} = Aa + Bb, \quad a \in Z^n, \quad b \in R^p \quad (1)$$

with  $E\{\cdot\}$  the mathematical expectation operator,  $y$  the  $m$ -vector of observables,  $a$  the  $n$ -vector of unknown integer parameters and  $b$  the  $p$ -vector of unknown real-valued parameters. The  $m \times (n + p)$  design matrix  $(A, B)$  is assumed to be of full rank.

All the linear(ized) GNSS models can in principle be cast in the above frame of observation equations. The data vector  $y$  will then usually consist of the 'observed minus computed' single- or dual- frequency double-difference (DD) phase and/or pseudorange (code) observations accumulated over all observation epochs. The entries of vector  $a$  are then the DD carrier phase ambiguities, expressed in units of cycles rather than range, while the entries of the vector  $b$  will consist of the remaining unknown parameters, such as for instance baseline components (coordinates) and possibly atmospheric delay parameters (troposphere, ionosphere).

The procedure which is usually followed for solving the GNSS model can be divided into three steps. In the *first* step one simply discards the integer constraints  $a \in Z^n$  and performs a standard least-squares (LS) adjustment. As a result one obtains the LS-estimators of  $a$  and  $b$  as  $\hat{a}$  and  $\hat{b}$ , respectively. This solution is usually referred to as the 'float' solution. In the *second* step the 'float' solution  $\hat{a}$  is further adjusted so as to take the integerness of the ambiguities into account in some pre-defined way. This gives

$$\hat{a}_S = S(\hat{a}) \quad (2)$$

in which  $S$  is an  $n$ -dimensional mapping that takes the integerness of the ambiguities into account. The estimator  $\hat{a}_S$  is then used in the final and *third* step to adjust the 'float' estimator  $\hat{b}$ . As a result one obtains the so-called 'fixed' estimator of  $b$  as

$$\hat{b}_S = \hat{b} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} (\hat{a} - \hat{a}_S) \quad (3)$$

in which  $Q_{\hat{a}}$  denotes the variance-covariance (vc-) matrix of  $\hat{a}$  and  $Q_{\hat{b}\hat{a}}$  denotes the covariance matrix of  $\hat{b}$  and  $\hat{a}$ .

The above three-step procedure is still ambiguous in the sense that it leaves room for choosing the  $n$ -dimensional map  $S$ . Different choices for  $S$  will lead to different ambiguity estimators and thus also to different baseline estimators  $\hat{b}_S$ . One can therefore now think of constructing a family of maps  $S$  with certain desirable properties. Three such classes of ambiguity estimators are the class of integer estimators, the class of integer aperture estimators, and the class of integer equivariant estimators. These classes were introduced in, respectively, [Teunissen, 1999, 2003, 2002]. These three classes are subsets of one another. The first class, the class of integer estimators, is the most restrictive class. This is due to the fact that the outcomes of any estimator within this class are required to be integer. The integer least-squares (ILS) estimator can be shown to be the optimal estimator within this class, see [Teunissen, 1999]. It is defined as

$$\check{a}_{LS} = \arg \min_{z \in Z^n} \| \hat{a} - z \|_{Q_{\hat{a}}}^2 \quad (4)$$

and it can be shown to have the largest possible probability of correct integer estimation. In contrast to integer rounding and integer bootstrapping, an integer search is needed to compute  $\check{a}_{LS}$ . The ILS-estimator and the integer search are efficiently mechanized in the LAMBDA method [Teunissen, 1993, 1995], which is currently one of the most applied methods for GNSS carrier phase ambiguity resolution. In particular the decorrelating  $Z$ -transformation of the LAMBDA-method is responsible for speeding up the integer search.

Practical results obtained with the LAMBDA method can be found, for example, in [Boon and Ambrosius, 1997], [Boon et al., 1997], [Cox and Brading, 1999], [de Jonge and Tiberius, 1996b], [de Jonge et al., 1996], [Han, 1995], [Peng et al., 1999], [Tiberius and de Jonge, 1995].

The second class, the class of integer aperture estimators, encompasses the class of integer estimators. The integer aperture estimators are of a hybrid nature in the sense that their outcomes are either integer or noninteger. Examples of different integer aperture estimators and their properties can be found in [Teunissen, 2003a, 2004, 2005]. The most relaxed of the three classes is the class of integer equivariant estimators. These estimators are real-valued and they only obey the integer remove-restore principle. The best integer equivariant (BIE) estimator can be shown to be the optimal estimator within this relaxed class, see [Teunissen, 2003b]. Here optimality is measured by minimizing the mean squared error of the estimator. When using the BIE-estimator care should be taken in how it is computed. The purpose of the current contribution is to show how the BIE-estimator should be computed and which pitfall should be avoided.

This contribution is organized as follows. In section 2 we give a brief review of the theory of integer equivariant estimation. It includes the definition of the class of integer equivariant estimators. In section 3 we give the BIE-estimator for an arbitrary probability density function (PDF) of the ambiguity float solution. It follows from minimizing the mean squared error within the class of integer equivariant estimators. Although the BIE-estimator holds true for any probability density function the data might have, we assume in section 4 that the data are normally distributed. For this case it follows that the BIE-estimator of the baseline can be obtained in a way which is very similar to the three-step procedure of current methods of ambiguity resolution. The only difference being that the integer ambiguity estimator needs to be replaced by its BIE-counterpart. Since the BIE-estimator of the ambiguities contains an infinite sum, it can not be evaluated in an exact manner. It is shown how to approximate the BIE-estimator while retaining the property of integer equivariance. It is also shown how the decorrelating  $Z$ -transformation and the integer search of the LAMBDA method can be used to speed up the computation of the BIE-estimator.

## 2 INTEGER EQUIVARIANT ESTIMATION

In order to describe the class of integer equivariant (IE) estimators, we consider estimating an arbitrary linear function of the two type of unknown parameters of the GNSS model (1),

$$\theta = l_a^T a + l_b^T b, \quad l_a \in R^n, \quad l_b \in R^p \quad (5)$$

It seems reasonable that an IE estimator should at least obey the *integer remove-restore* principle, see [Teunissen, 2002]. When estimating ambiguities in case of GNSS for instance, one would like, when adding an arbitrary number of cycles to the carrier phase data, that the solution of the integer ambiguities gets shifted by the same integer amount. For the estimator of  $\theta$  this would mean that adding  $Az$  to  $y$ , with arbitrary  $z \in Z^n$ , must result in a shift of  $l_a^T z$ . Likewise it seems reasonable to require of the estimator that adding  $B\zeta$  to  $y$ , with arbitrary  $\zeta \in R^p$ , results in a shift  $l_b^T \zeta$ . Afterall we would not like the integer part of the estimator to become contaminated by such an addition to  $y$ . Estimators of  $\theta$  that fulfil these two conditions are called *integer equivariant*. Hence, they are defined as follows

**Definition 1** (*IE estimators*)

The estimator  $\hat{\theta}_{IE} = f_{\theta}(y)$ , with  $f_{\theta} : R^m \mapsto R$ , is said to be an *integer equivariant* estimator of  $\theta = l_a^T a + l_b^T b$  if

$$\begin{cases} f_{\theta}(y + Az) = f_{\theta}(y) + l_a^T z, \forall y \in R^m, z \in Z^n \\ f_{\theta}(y + B\zeta) = f_{\theta}(y) + l_b^T \zeta, \forall y \in R^m, \zeta \in R^p \end{cases} \quad (6)$$

It is easy to verify that integer estimators, like integer rounding, integer bootstrapping or integer least-squares, are integer equivariant. Simply check that the above two conditions are indeed fulfilled by integer estimators. The converse, however, is not necessarily true. The class of IE-estimators is therefore a larger class than the class of integer estimators.

The class of IE-estimators is also larger than the class of linear unbiased estimators. Let  $f_{\theta}^T y$ , for some  $f_{\theta} \in R^m$ , be the linear estimator of  $\theta = l_a^T a + l_b^T b$ . For it to be unbiased we need, using  $E\{y\} = Aa + Bb$ , that  $f_{\theta}^T Aa + f_{\theta}^T Bb = l_a^T a + l_b^T b, \forall a \in R^n, b \in R^p$  holds true, or that both  $l_a = A^T f_{\theta}$  and  $l_b = B^T f_{\theta}$  hold true. But this equivalent to stating that

$$\begin{cases} f_{\theta}^T (y + Aa) = f_{\theta}^T y + l_a^T a, \forall y \in R^m, a \in R^n \\ f_{\theta}^T (y + Bb) = f_{\theta}^T y + l_b^T b, \forall y \in R^m, b \in R^p \end{cases} \quad (7)$$

Comparing this result with (6) shows that the condition of linear unbiasedness is more restrictive than the condition of integer equivariance. Hence, the class of linear unbiased estimators is a subset of the class of integer equivariant estimators.

### 3 BEST INTEGER EQUIVARIANT ESTIMATION

Having defined the class of IE-estimators we will now look for an IE-estimator which is 'best' in a certain sense. We will denote our *best integer equivariant* (BIE) estimator of  $\theta$  as  $\hat{\theta}_{BIE}$  and use the mean squared error (MSE) as our criterion of 'best'. The best integer equivariant estimator will therefore be defined as

$$\hat{\theta}_{BIE} = \arg \min_{f_{\theta} \in IE} E\{(f_{\theta}(y) - \theta)^2\} \quad (8)$$

in which *IE* stands for the class of IE-estimators. The minimization is thus taken over all integer equivariant functions that satisfy the conditions of Definition 1.

The reason for choosing the MSE-criterion is twofold. First, it is a well-known probabilistic criterion for measuring the closeness of an estimator to its target value, in our case  $\theta$ . Second, the MSE-criterion is also often used as measure for the quality of the 'float' solution itself. The following theorem gives the solution to the above minimization problem (8).

**Theorem 1** (*BIE estimation*)

Let  $y \in R^m$  have mean  $E\{y\} = Aa + Bb$  and probability density function (PDF)  $p_y(y)$ , and let  $\hat{\theta}_{BIE}$  be the best integer equivariant estimator of  $\theta = l_a^T a + l_b^T b$ . Then

$$\hat{\theta}_{BIE} = \frac{\sum_{z \in Z^n} \int_{R^p} (l_a^T z + l_b^T \beta) p_y(y + A(a - z) + B(b - \beta)) d\beta}{\sum_{z \in Z^n} \int_{R^p} p_y(y + A(a - z) + B(b - \beta)) d\beta} \quad (9)$$

