

INTEGER APERTURE LEAST-SQUARES ESTIMATION

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ABSTRACT

GNSS carrier phase ambiguity resolution is the key to fast and high-precision satellite positioning and navigation. It applies to a great variety of current and future models of GPS, modernized GPS and Galileo. In [Teunissen, 2003] we described the general principle of integer aperture (IA) ambiguity estimation. In the present contribution we introduce one particular IA estimator, the integer aperture least-squares (IALS) estimator. The motivation for studying this estimator stems from the known optimality of the integer least-squares estimator. It is shown how the IALS estimator extends the integer least-squares estimator and how its performance can be measured by means of its fail-rate and success-rate.

1 INTRODUCTION

Global Navigation Satellite System (GNSS) ambiguity resolution is the process of resolving the unknown cycle ambiguities of double difference (DD) carrier phase data. Its practical importance becomes clear when one realizes the great variety of current and future GNSS models to which it applies. An overview of GNSS models, together with their applications in surveying, navigation, geodesy and geophysics, can be found in textbooks such as [Hofmann-Wellenhof et al., 2001], [Leick, 1995], [Misra and Enge, 2001], [Parkinson and Spilker, 1996], [Strang and Borre, 1997] and [Teunissen and Kleusberg, 1998].

In [Teunissen, 2003] we introduced the class of integer aperture (IA) estimators for carrier phase ambiguity resolution. This class allows one to design ambiguity estimators such that the ambiguity resolution process will have a user-defined *fixed fail-rate*. In this contribution we will introduce the integer aperture least-squares (IALS) estimator as an extension of the well-known integer least-squares estimator. We start with a brief review of integer estimation and of integer least-squares estimation in particular. Then we describe the general principle of integer aperture estimation and introduce the integer aperture least-squares estimator. It is shown how the framework of integer aperture estimation incorporates the important problem of ambiguity discernibility. By setting the size and shape of the integer aperture pull-in region, the user has control over the fail-rate of the

integer aperture estimator and thus also over the amount of discernibility. In case of the IALS estimator the aperture pull-in region is chosen as a down-sized version of the integer least-squares pull-in region. It is shown how the aperture of the pull-in region governs the fail-rate and the success-rate of the IALS estimator and how lower bounds and upper bounds of these probabilities can be computed.

2 INTEGER LEAST-SQUARES ESTIMATION

2.1 THE GNSS MODEL

As our point of departure we take the following system of linear observation equations

$$E\{y\} = Aa + Bb, \quad a \in Z^n, \quad b \in R^p \quad (1)$$

with $E\{\cdot\}$ the mathematical expectation operator, y the m -vector of observables, a the n -vector of unknown integer parameters and b the p -vector of unknown real-valued parameters. The $m \times (n + p)$ design matrix (A, B) is assumed to be of full rank.

All the linear(ized) GNSS models can in principle be cast in the above frame of observation equations. The data vector y will then usually consist of the 'observed minus computed' single- or dual- frequency double-difference (DD) phase and/or pseudorange (code) observations accumulated over all observation epochs. The entries of vector a are then the DD carrier phase ambiguities, expressed in units of cycles rather than range, while the entries of the vector b will consist of the remaining unknown parameters, such as for instance baseline components (coordinates) and possibly atmospheric delay parameters (troposphere, ionosphere).

The procedure which is usually followed for solving the GNSS model can be divided into three steps. In the *first* step one simply discards the integer constraints $a \in Z^n$ and performs a standard least-squares (LS) adjustment. As a result one obtains the LS-estimators of a and b as

$$\begin{cases} \hat{a} = (\bar{A}^T Q_y^{-1} \bar{A})^{-1} \bar{A}^T Q_y^{-1} y \\ \hat{b} = (\bar{B}^T Q_y^{-1} \bar{B})^{-1} \bar{B}^T Q_y^{-1} y \end{cases} \quad (2)$$

with Q_y the vc-matrix of the observables, $\bar{A} = P_B^\perp A$, $\bar{B} = P_A^\perp B$, and the two orthogonal projectors $P_B^\perp = I_m - B(B^T Q_y^{-1} B)^{-1} B^T Q_y^{-1}$ and $P_A^\perp = I_m - A(A^T Q_y^{-1} A)^{-1} A^T Q_y^{-1}$. This solution is usually referred to as the 'float' solution.

In the *second* step the 'float' estimator \hat{a} is used to compute the corresponding integer estimator $\check{a} \in Z^n$. This implies that a mapping S from the n -dimensional space of reals to the n -dimensional space of integers is introduced such that

$$\check{a} = S(\hat{a}), \quad S: R^n \mapsto Z^n \quad (3)$$

This integer estimator is then used in the final and *third* step to adjust the 'float' estimator \hat{b} . As a result one obtains the so-called 'fixed' estimator of b as

$$\check{b} = \hat{b} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} (\hat{a} - \check{a}) \quad (4)$$

in which $Q_{\hat{a}}$ denotes the vc-matrix of \hat{a} and $Q_{\hat{b}\hat{a}}$ denotes the covariance matrix of \hat{b} and \hat{a} . This 'fixed' estimator can alternatively be expressed as $\check{b} = (B^T Q_y^{-1} B)^{-1} B^T Q_y^{-1} (y - A\check{a})$.

Note that only two of the three steps are needed in case one only would be interested in obtaining an integer solution for a . In the case of GNSS, however, one is particularly interested in the solution of the third step as it contains the solution for the baseline coordinates. All three steps are therefore needed in case of GNSS.

2.2 INTEGER AMBIGUITY ESTIMATION

The above three-step procedure is still ambiguous in the sense that it leaves room for choosing the integer map S . It will be clear that the map S will not be one-to-one due to the discrete nature of Z^n . Instead it will be a many-to-one map. This implies that different real-valued vectors will be mapped to one and the same integer vector. One can therefore assign a subset $S_z \subset R^n$ to each integer vector $z \in Z^n$:

$$S_z = \{x \in R^n \mid z = S(x)\}, \quad z \in Z^n \quad (5)$$

The subset S_z contains all real-valued vectors that will be mapped by S to the same integer vector $z \in Z^n$. This subset is referred to as the *pull-in region* of z . It is the region in which all vectors are pulled to the same integer vector z .

Since the pull-in regions define the integer estimator completely, one can define classes of integer estimators by imposing various conditions on the pull-in regions. One such class was introduced by Teunissen (1999a) as follows.

Definition 1 (*Integer estimators*)

The mapping $\tilde{a} = S(\hat{a})$ is said to be an integer estimator if its pull-in regions satisfy

- (i) $\bigcup_{z \in Z^n} S_z = R^n$
- (ii) $Int(S_{z_1}) \cap Int(S_{z_2}) = \emptyset, \quad \forall z_1, z_2 \in Z^n, z_1 \neq z_2$
- (iii) $S_z = z + S_0, \quad \forall z \in Z^n$

This definition is motivated as follows. Each one of the above three conditions describe a property of which it seems reasonable that it is possessed by an arbitrary integer estimator. The first condition states that the pull-in regions should not leave any gaps and the second that they should not overlap. The absence of gaps is needed in order to be able to map any float solution $\hat{a} \in R^n$ to Z^n , while the absence of overlaps is needed to guarantee that the float solution is mapped to just one integer vector. Note that we allow the pull-in regions to have common boundaries. This is permitted if we assume to have zero probability that \hat{a} lies on one of the boundaries. This will be the case when the probability density function (PDF) of \hat{a} is continuous.

The third and last condition of the definition follows from the requirement that $S(x+z) = S(x) + z, \forall x \in R^n, z \in Z^n$. Also this condition is a reasonable one to ask for. It states that when the float solution \hat{a} is perturbed by $z \in Z^n$, the corresponding integer solution is perturbed by the same amount. This property allows one to apply the *integer remove-restore* technique: $S(\hat{a} - z) + z = S(\hat{a})$. It therefore allows one to work with the fractional parts of the entries of \hat{a} , instead of with its complete entries.

Using the pull-in regions, one can give an explicit expression for the corresponding integer estimator \tilde{a} . It reads

$$\tilde{a} = \sum_{z \in Z^n} z s_z(\hat{a}) \quad \text{with} \quad s_z(\hat{a}) = \begin{cases} 1 & \text{if } \hat{a} \in S_z \\ 0 & \text{if } \hat{a} \notin S_z \end{cases} \quad (6)$$

Note that the $s_z(\hat{a})$ can be interpreted as weights, since $\sum_{z \in Z^n} s_z(\hat{a}) = 1$. The integer estimator \check{a} is therefore equal to a weighted sum of integer vectors with binary weights.

2.3 INTEGER LEAST-SQUARES AMBIGUITY ESTIMATION

Different choices for S will lead to different integer estimators \check{a} and thus also to different baseline estimators \check{b} . One can therefore now think of constructing integer maps which possess certain desirable properties. Examples are integer rounding, integer bootstrapping and integer least-squares. In this contribution we will make use of the integer least-squares (ILS) estimator. It is defined as

$$\check{a}_{LS} = \arg \min_{z \in Z^n} \|\hat{a} - z\|_{Q_{\hat{a}}}^2 \quad (7)$$

In contrast to integer rounding and integer bootstrapping, an integer search is needed to compute \check{a}_{LS} . The ILS-estimator was introduced in [Teunissen, 1993], see also [Teunissen, 1995]. The ILS procedure is mechanized in the LAMBDA method, which is currently one of the most applied methods for GNSS carrier phase ambiguity resolution. Practical results obtained with it can be found, for example, in [Boon and Ambrosius, 1997], [Boon et al., 1997], [Cox and Brading, 1999], [de Jonge and Tiberius, 1996b], [de Jonge et al., 1996], [Han, 1995], [Jonkman, 1998], [Peng et al., 1999], [Tiberius and de Jonge, 1995], [Tiberius et al., 1997].

To determine the ILS pull-in regions we need to know the set of float solutions $\hat{a} \in R^n$ that are mapped to the same integer vector $z \in Z^n$. This set is described by all $x \in R^n$ that satisfy $z = \arg \min_{u \in Z^n} \|x - u\|_{Q_{\hat{a}}}^2$. The ILS pull-in-region that belongs to the integer vector z follows therefore as

$$S_{LS,z} = \{x \in R^n \mid \|x - z\|_{Q_{\hat{a}}}^2 \leq \|x - u\|_{Q_{\hat{a}}}^2, \forall u \in Z^n\} \quad (8)$$

It consists of all those points which are closer to z than to any other integer point in R^n . The metric used for measuring these distances is determined by the vc-matrix $Q_{\hat{a}}$. An alternative representation of the ILS pull-in regions is

$$S_{LS,z} = \bigcap_{c_i \in Z^n} \{x \in R^n \mid |c_i^T Q_{\hat{a}}^{-1}(x - z)| \leq \frac{1}{2} \|c_i\|_{Q_{\hat{a}}}^2\}, \forall z \in Z^n \quad (9)$$

This shows that the ILS pull-in regions are constructed from intersecting half-spaces. One can show that at most $2^n - 1$ pairs of such half spaces are needed for constructing the pull-in region. The ILS pull-in regions are convex, symmetric sets of volume 1, which satisfy the conditions of Definition 1. They are hexagons in the two-dimensional case. Two-dimensional examples of the pull-in regions of integer least-squares are given in Figure 1.

2.4 PROBABILITY OF CORRECT INTEGER ESTIMATION: THE AMBIGUITY SUCCESS-RATE

For the evaluation of the fixed ambiguities one needs the distribution of the integer estimator \check{a} . This distribution is of the discrete type and it will be denoted as $P(\check{a} = z)$. It is a probability mass function, having zero masses at nongrid points and nonzero masses

