THE MEAN AND THE VARIANCE MATRIX OF THE 'FIXED' GPS BASELINE

P.J.G. Teunissen
Delft Geodetic Computing Centre (LGR)
Faculty of Geodesy
Delft University of Technology
Thijsseweg 11
2629 JA Delft, The Netherlands
Fax: ++ 31 15 278 3711
e-mail: P.J.G.Teunissen@geo.tudelft.nl

ABSTRACT
In this contribution we determine the first two moments of the 'fixed' GPS baseline. The first two moments of the 'float' solution are well-known. They follow from standard adjustment theory. In order to determine the corresponding moments of the 'fixed' solution, the probabilistic characteristics of the integer least-squares ambiguities need to be taken into account. It is shown that the 'fixed' GPS baseline estimator is unbiased in case the probability density function of the real-valued least-squares ambiguity vector is symmetric about its integer mean. We also determine the variance matrix of the 'fixed' GPS baseline. This matrix differs from the one which is usually used in practice. The difference between the two matrices is made up of the precision contribution of the integer least-squares ambiguities.

Keywords: bias, precision, GPS baseline, ambiguity resolution

1 Introduction
GPS ambiguity resolution is the process of resolving the unknown cycle ambiguities of the double-differenced (DD) carrier phase data as integers [Hofmann-Wellenhof et al., 1997], [Kleusberg and Teunissen, 1996], [Leick, 1995], [Stang and Borre, 1997]. In this process the ambiguities are usually treated as if they were deterministic constants. From a theoretical point of view this is not correct [Teunissen, 1990]. The estimated ambiguities, although integer, are still random variates. They have been computed from the data and since the vector of observables is assumed to be random, also the integer ambiguity estimator is a random vector.

In this contribution we investigate how the random characteristics of the integer least-squares ambiguities contribute to the first two moments of the 'fixed' GPS solution. From standard adjustment theory it is well-known that the 'float' solutions are unbiased. A corresponding result
for the 'fixed' solution did not yet exist. We show, in case the probability density function of the real-valued least-squares ambiguities is symmetric about its integer mean, that a corresponding result holds true as well for both the integer least-squares ambiguities and the 'fixed' GPS baseline.

The precision of the 'fixed' GPS baseline is described by its variance matrix. In order to determine this matrix, the precision characteristics of the integer least-squares ambiguities need to be taken into account as well. The exact expression of the variance matrix of the 'fixed' baseline is determined and it is shown by how much this matrix differs from the one used in practice. It is also shown that the difference between the two matrices reduces the more peaked the probability mass function of the integer least-squares ambiguities gets.

2 The 'fixed' GPS baseline

As our point of the departure we take the following system of linearized observation equations

\[
y = Aa + Bb + e, \quad a \in Z^n, \quad b \in R^q, \quad e \in R^m
\]  

where \(y\) is the given data vector, \(a\) and \(b\) are the unknown parameter vectors and \(e\) is the noise vector. The matrices \(A\) and \(B\) are the corresponding design matrices, where matrix \((A, B)\) is assumed to be of full rank. In principle all the GPS models can be cast in this frame of observation equations. The data vector will then usually consist of the 'observed minus computed' single- or dual-frequency double-differenced (DD) phase and/or pseudo range (code) observations, accumulated over all observation epochs. The entries of the \(n\)-vector \(a\) are the DD carrier phase ambiguities, expressed in units of cycles rather than range. They are known to be integers. The entries of the \(q\)-vector \(b\) consist of the remaining unknown parameters, such as for instance baseline components (coordinates) and possibly atmospheric delay parameters (troposphere, ionosphere).

When using the integer least-squares principle, the above system of observation equations can be solved in three steps [Teunissen, 1993]. In the first step one simply disregards the integer constraints on the ambiguities and performs a standard least-squares adjustment. As a result one obtains the (real-valued) least-squares estimates of \(a\) and \(b\), together with their variance-covariance matrix

\[
\begin{bmatrix}
\hat{a} \\
\hat{b}
\end{bmatrix}, \quad \begin{bmatrix}
Q_{\hat{a}} & Q_{\hat{a}\hat{b}} \\
Q_{\hat{b}\hat{a}} & Q_{\hat{b}}
\end{bmatrix}
\]

This solution is referred to as the 'float' solution. In the second step the 'float' ambiguity estimate \(\hat{a}\) and its variance-covariance matrix are used to compute the corresponding integer least-squares ambiguity estimate. This implies solving the minimization problem

\[
\min_{z \in Z^n} \| \hat{a} - z \|_{Q_{\hat{a}}}^2
\]

with the squared weighted norm \(\| . \|_{Q_{\hat{a}}}^2 = (.)^T Q_{\hat{a}}^{-1}(.)\). Its solution will be denoted as \(\hat{a}\). Finally in the third step, the integer least-squares ambiguities are used to correct the 'float' estimate \(\hat{b}\). As a result one obtains the 'fixed' solution

\[
\hat{b} = \hat{b} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1}(\hat{a} - \hat{a})
\]
Thus the least-squares solution of (1) is given by the integer vector $\hat{a} \in \mathbb{Z}^n$ and the real-vector $\hat{b} \in \mathbb{R}^q$. The entries of vector $\hat{b}$ contain all noninteger parameters of the model. For short baselines these entries contain exclusively the baseline components, but for longer baselines the ionospheric and/or tropospheric delays could be included as well. Although $\hat{b}$ may contain real-valued parameters other than the baseline components, we will - for ease of reference - still refer to $\hat{b}$ as the 'fixed' baseline estimator.

From standard adjustment theory, the mean and the variance matrix of the above 'float' solution are well known. When the noise vector is zero-mean, the means of $\hat{a}$ and $\hat{b}$ follow as
\[ E\{\hat{a}\} = a \quad \text{and} \quad E\{\hat{b}\} = b \quad (5) \]
where $E\{\cdot\}$ denotes the mathematical expectation operator. Similarly, the corresponding variance matrices are known to be given as
\[ Q_{\hat{a}} = (\bar{A}^T Q_y^{-1} \bar{A})^{-1} \quad \text{and} \quad Q_{\hat{b}} = (\bar{B}^T Q_y^{-1} \bar{B})^{-1} \quad (6) \]
where $\bar{A} = P_B \bar{A}$, $\bar{B} = P_A \bar{B}$ with the orthogonal projectors $P_B = I_m - A(A^T Q_y^{-1} A)^{-1} A^T Q_y^{-1}$ and $P_A = I_m - B(B^T Q_y^{-1} B)^{-1} B^T Q_y^{-1}$, and with $Q_y$ being the variance-covariance matrix of $y$. Thus the mean and the variance matrix of the solution of the above first step are known. However, the mean and the variance matrix of the solutions of the following two steps are not yet known. This is quite unsatisfactory. After all, when ambiguity resolution is in place, $\hat{b}$ is the baseline solution provided to the user.

The 'float' solution $\hat{b}$ is known to be unbiased, see (5), but a corresponding result does not yet exist for the 'fixed' solution. Thus we do not even know whether $\hat{b}$ is biased or not. Also a proper precision description of $\hat{b}$ is not yet available. In the practice of GPS, the precision of $\hat{b}$ is thought to be described by the variance matrix that follows from applying the error propagation law to (4), while assuming the integer least-squares ambiguities to be nonrandom. But this is wrong in principle, since $\hat{a}$ is a random vector. The integer least-squares ambiguities follow from a mapping of $\hat{a}$ and since $\hat{a}$ is random, the integer vector $\hat{a}$ is random as well. Thus when applying the error propagation law to (4), the precision of the integer least-squares ambiguities should be taken into account as well.

It is the purpose of this contribution to complement the above results, (5) and (6), by determining the first two moments of the 'fixed' solution. We will first determine the expectations of $\hat{a}$ and $\hat{b}$, and then their variance matrices.

### 3 The mean of the 'fixed' GPS baseline

The mean of the 'fixed' baseline estimator follows from taking the expectation of (4). By making use of the fact that both $\hat{a}$ and $\hat{b}$ are unbiased, we obtain
\[ E\{\hat{b}\} = b - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} (a - E\{\hat{a}\}) \quad (7) \]
This shows that the 'fixed' baseline is unbiased whenever the integer least-squares ambiguities are unbiased. Thus in order to determine the mean of $\hat{b}$, we first need the mean of the integer least-squares ambiguities. Since $\hat{a}$ is an element of the $n$-dimensional space of integers, its distribution
will be of the discrete type. It is a probability mass function, having zero probability at all noninteger points and nonzero probabilities at some or all integer grid points. This probability mass function will be denoted as \( P(\hat{a} = z) \), with \( z \in \mathbb{Z}^n \). The expectation of \( \hat{a} \) follows then by definition as

\[
E\{\hat{a}\} = \sum_{z \in \mathbb{Z}^n} z P(\hat{a} = z)
\]  

(8)

It equals a weighted sum of all integer grid points \( z \in \mathbb{Z}^n \), with the weights given by the corresponding probability masses. We will now prove that the integer least-squares ambiguities are also unbiased. This will be done in a few steps. We start by showing that the property of unbiasedness holds when the probability mass function is symmetric about \( a \). That is, when

\[
P(\hat{a} = a + z) = P(\hat{a} = a - z) \quad \forall z \in \mathbb{Z}^n
\]

(9)

Since the probability masses all add up to one, \( \sum_{z \in \mathbb{Z}^n} P(\hat{a} = z) = 1 \), we may write \( E\{\hat{a}\} = a + \sum_{z \in \mathbb{Z}^n} (z - a)P(\hat{a} = z) \). This leaves us to show that the second term on the right-hand-side equals zero in case of (9). By substituting \( u = z - a \) and recognizing that \( a \) is an integer vector as well, we may write \( \sum_{z \in \mathbb{Z}^n} (z - a)P(\hat{a} = z) = \sum_{u \in \mathbb{Z}^n} uP(\hat{a} = a + u) = \sum_{u \in \mathbb{Z}^n} uP(\hat{a} = a - u) \), where the last equality follows from using (9). Since the same expression, but now with a change of sign, is obtained when we substitute \(-u = z - a\), it follows that \( \sum_{z \in \mathbb{Z}^n} (z - a)P(\hat{a} = z) \) has to be zero. This shows that the integer least-squares ambiguities are unbiased in case (9) holds true.

In order to verify whether (9) holds true or not, we will now have a closer look at the probability mass function itself. To determine this probability mass function, we first need to consider the mapping from \( \hat{a} \) to \( a \). This integer least-squares map is defined as

\[
\hat{a} = \arg \min_{z \in \mathbb{Z}^n} \| \hat{a} - z \|^2_{Q_{\hat{a}}}
\]

(10)

This map from \( \mathbb{R}^n \) to \( \mathbb{Z}^n \) is a many-to-one map. That is, it are subsets of \( \mathbb{R}^n \) which are mapped to the integer grid points. These subsets follow from the integer least-squares principle as

\[
S_z = \{ x \in \mathbb{R}^n \mid \| x - z \|^2_{Q_{\hat{a}}} \leq \| x - u \|^2_{Q_{\hat{a}}} , \forall u \in \mathbb{Z}^n \} , \forall z \in \mathbb{Z}^n
\]

(11)

These subsets act as pull-in-regions for the real-valued least-squares ambiguities: \( \hat{a} \in S_z \iff \hat{a} = z \). Thus \( z \) is the integer least-squares solution whenever \( \hat{a} \) lies in the subset \( S_z \). This implies that the probability that \( \hat{a} \) coincides with \( z \), \( P(\hat{a} = z) \), equals the integral of the probability density function (pdf) of \( \hat{a} \) over the pull-in-region \( S_z \). The probability mass function of the integer least-squares ambiguities reads therefore

\[
P(\hat{a} = z) = \int_{S_z} p_\hat{a}(x)dx
\]

(12)

where \( p_\hat{a}(x) \) denotes the pdf of \( \hat{a} \), which in our case is given by the multivariate normal distribution having \( a \) as its integer mean and \( Q_{\hat{a}} \) as its variance matrix. This probability mass function was introduced in [Temanisken, 1998] and coined the integer normal distribution.

Now in order to prove (9), we need to show that

\[
\int_{S_{a+z}} p_\hat{a}(x)dx = \int_{S_{a-z}} p_\hat{a}(x)dx , \forall z \in \mathbb{Z}^n
\]

(13)
For that purpose we first show that the pull-in-region $S_{a-z}$ can be obtained from $S_{a+z}$ by means of a reflection about $a$. Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ denote the reflection $y = 2a - x$. Then $F(S_{a+z}) = F(\{ x \in \mathbb{R}^n \mid \| x - a - z \|_Q^2 \leq \| x - u \|_Q^2, \forall u \in \mathbb{Z}^n \}) = \{ y \in \mathbb{R}^n \mid \| y - a + z \|_Q^2 \leq \| y - 2a + u \|_Q^2, \forall u \in \mathbb{Z}^n \}$. This last subset equals $S_{a-z}$, since $a$ is an integer vector as well. Thus $F(S_{a+z}) = S_{a-z}$. If we now apply the reflection about $a$ as a change of variable transformation to the integral on the left-hand-side of (13), we obtain

$$
\int_{S_{a+z}} p_\tilde{a}(x) dx = \int_{S_{a-z}} p_\tilde{a}(2a - y) dy, \forall z \in \mathbb{Z}^n
$$

(14)

This shows that (13) holds true whenever the pdf of $\tilde{a}$ itself is symmetric about $a$. That is, when $p_\tilde{a}(y) = p_\tilde{a}(2a - y)$. This property of symmetry certainly holds true for the multivariate normal distribution of $\tilde{a}$.

Summarizing, we may thus conclude that the integer least-squares ambiguities are unbiased, whenever the pdf of the real-valued least-squares ambiguities is symmetric about its integer mean $a$. We are thus in the happy situation that not only the real-valued least-squares ambiguities are unbiased, but their integer least-squares counterparts as well,

$$
E\{ \hat{a} \} = E\{ \tilde{a} \} = a
$$

(15)

From the unbiasedness of the integer least-squares ambiguities follows immediately the unbiasedness of the 'fixed' baseline estimator, see (7). Thus we also have

$$
E\{ \hat{b} \} = E\{ \tilde{b} \} = b
$$

(16)

With this result we thus have proven that the inclusion of the integer constraints does not introduce any biases into the 'fixed' baseline solution. That is, the 'fixed' solutions are unbiased whenever the 'float' solutions are. Biases may still be introduced of course, when the models which are used are misspecified. That is, when model errors such as cycle slips or outliers are present in the data. But these biases are then due to mismodelling and not due to the estimation principle used.

4 The variance matrix of the 'fixed' GPS baseline

In order to determine the variance matrix of $\hat{b}$, we need to apply the error propagation law to

$$
\hat{b} = \hat{b} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1}(\hat{a} - \tilde{a})
$$

(17)

It is the practice of GPS to apply this error propagation under the assumption that the integer ambiguity vector $\hat{a}$ is deterministic. As a result one obtains the variance matrix

$$
Q_{\hat{b}(a)} = Q_{\hat{b}} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} Q_{\hat{a}\hat{b}}
$$

(18)

But this matrix is not the variance matrix of the 'fixed' baseline estimator $\hat{b}$. Instead it is a conditional variance matrix. It is the variance matrix of the conditional estimator

$$
\hat{b}(a) = \hat{b} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1}(\hat{a} - a)
$$

(19)
Note that this estimator follows from replacing the random vector \( \hat{a} \) in (17) by the nonrandom vector \( a \). Since (18) neglects the contribution of the random integer least-squares ambiguities, one can expect this matrix to give a too optimistic precision description of the 'fixed' baseline. Thus

\[ Q_{\hat{b}(a)} \leq Q_{\hat{b}} \]  

(20)

Now in order to determine the correct variance matrix \( Q_{\hat{b}} \) of the 'fixed' baseline, we need to know the contribution of the integer least-squares ambiguities. For the variance matrix of the integer least-squares ambiguities, we have

\[ Q_{\hat{a}} = \sum_{z \in Z^n} (z - a)(z - a)^T P(\hat{a} = z) \]  

(21)

Apart from this variance matrix, it may seem at first instance that we also need the covariances between the integer least-squares ambiguities on the one hand and the 'float' estimators \( \hat{a} \) and \( \hat{b} \) on the other hand. Fortunately this turns out not to be the case. In order to see this, we first partition (17) as

\[ \hat{b} = \hat{b}(a) + Q_{\hat{b}a} Q_a^{-1}(\hat{a} - a) \]  

(22)

From standard adjustment theory it is well known that \( \hat{a} \) and \( \hat{b}(a) \) are distributed as

\[
\begin{bmatrix}
\hat{a} \\
\hat{b}(a)
\end{bmatrix} \sim N(
\begin{bmatrix}
a \\
b
\end{bmatrix};
\begin{bmatrix}
Q_{\hat{a}} & 0 \\
0 & Q_{\hat{b}(a)}
\end{bmatrix})
\]  

(23)

This shows that \( \hat{a} \) and \( \hat{b}(a) \) are independent. But this implies, since \( \hat{a} \) follows from a mapping of \( \hat{a} \), that also \( \hat{a} \) and \( \hat{b}(a) \) are independent. From using this property of independence when applying the error propagation law to (22), the correct variance matrix of the 'fixed' baseline finally follows as

\[ Q_{\hat{b}} = Q_{\hat{b}(a)} + Q_{\hat{b}a} Q_a^{-1} Q_{\hat{a}} Q_a^{-1} Q_{\hat{b}a} \]  

(24)

This shows that the matrix inequality of (20) indeed holds true. The result also shows by how much the variance matrix of the 'fixed' baseline differs from the one used in practice. The latter matrix may be considered an acceptable approximation when the second term on the right-hand-side of (24) is sufficiently small. This will be the case when the integer least-squares ambiguities are determined with sufficient precision. This in turn requires that the probability mass function of the integer least-squares ambiguities is sufficiently peaked [Teunissen, 1997, 1998].

5 Summary

In this contribution we determined the first two moments of the least-squares estimators when ambiguity resolution is in place. Following the three steps of the estimation process of ambiguity resolution, one first determines the 'float' solution, then the integer least-squares ambiguities and finally the 'fixed' GPS baseline. The first moment of the 'float' solution is well-known. It follows from standard adjustment theory that both \( \hat{a} \) and \( \hat{b} \) are unbiased. A similar result was not yet available for the 'fixed' solution. In this contribution it has been shown that also \( \hat{a} \) and \( \hat{b} \) are
unbiased, whenever the probability density function of the 'float' ambiguities is symmetric about the integer mean \( a \). We thus have

\[
\begin{align*}
E\{\hat{a}\} &= E\{\hat{a}\} = a \\
E\{\hat{b}\} &= E\{\hat{b}\} = b
\end{align*}
\] (25)

This is a comforting result since it shows that the inclusion of the integer ambiguity constraints does not introduce any biases into the 'fixed' GPS baseline.

The variance matrix, the (central) second moment, of the 'float' solution is also well-known. This is not the case however for the variance matrix of the 'fixed' solution. It was shown that the matrix which is usually thought to be the variance matrix of the 'fixed' baseline is not correct. This matrix would only give a correct precision description of the 'fixed' baseline in case the integer least-squares ambiguities are nonrandom. But this is not the case, since the integer least-squares ambiguities are functionally related to the random real-valued least-squares ambiguities. In this contribution it has been shown that the correct variance matrix of the 'fixed' GPS baseline is given as

\[
Q_{\hat{b}} = Q_{b(a)} + Q_{ba}Q_a^{-1}Q_\hat{a}Q_\hat{a}^{-1}Q_{\hat{ab}}
\] (26)

with

\[
Q_\hat{a} = \sum_{z \in \mathbb{Z}^n} (z - a)(z - a)^T P(\hat{a} = z)
\]

the variance matrix of the integer least-squares ambiguities. The first term on the right-hand-side of (26) equals the matrix which is usually thought to be the variance matrix of the 'fixed' baseline, while the second term summarizes the contribution of the integer least-squares ambiguities. This result shows that matrix \( Q_{b(a)} \) may only be considered an acceptable approximation in case the integer least-squares ambiguities are determined with sufficient precision. But this requires checking whether the probability mass function of the ambiguities is sufficiently peaked.

To conclude, we point out that our results not only apply to the actual baseline components, but also, when present in the model, to the ionospheric and/or tropospheric delay parameters. The parameter vector \( b \) was namely defined to contain all real-valued parameters of the GPS model.

6 References


