THE DISTRIBUTION OF THE GPS BASELINE IN CASE OF INTEGER LEAST-SQUARES AMBIGUITY ESTIMATION

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Abstract

This contribution presents the probability distribution of the 'fixed' GPS baseline. This is the baseline which is used in fast and high precision GPS kinematic positioning. It follows from an ambiguity resolution process in which the carrier phase ambiguities are estimated as integers. For the estimation of the carrier phase ambiguities the principle of integer least-squares is used. By means of the 'fixed' baseline distribution it becomes possible to infer the quality of the positioning results. In particular their dependence on the quality of GPS ambiguity resolution is made clear. The mean and variance matrix of the 'fixed' baseline estimator are also determined. It shows that the 'fixed' baseline estimator is unbiased and that the difference of its precision with that of its conditional counterpart is governed by the precision of the integer least-squares ambiguities.

Keywords: Integer least-squares, probability, ambiguity resolution, GPS baseline.

1 Introduction

The purpose of GPS ambiguity resolution is to improve the precision of the baseline estimator. As a consequence shorter observation time spans can be used, than would have been necessary otherwise to obtain a comparable precision. As a measure for the precision of the 'fixed' baseline one usually takes the variance matrix that follows from assuming the ambiguities to be deterministic and known. From a theoretical point of view this is not correct, since the estimated integer ambiguities are not deterministic but random variates. Hence in order to describe the precision of the 'fixed' baseline, the random characteristics of the estimated integer ambiguities will have to be taken into account as well.

It is the purpose of this contribution to present the probability distribution of the 'fixed' baseline. This will be done for the case the ambiguities are estimated using the integer least-squares principle. Once this distribution is known, one is in the position to describe, in a qualitative as well as quantitative way, the quality that can be attached to the 'fixed' baseline estimator. One will then also be able to determine its first two moments, that is, its mean and variance matrix.

This contribution is organized as follows. In Sect. 2 we formulate our assumptions and briefly present the basic steps for solving an integer least-squares problem. In Sect. 3 we discuss the scalar case and present the corresponding ambiguity distribution as well as baseline distribution. The results of this section are generalized to the more realistic vectorial case in Sect. 4. In this section we first present the distribution of the integer least-squares ambiguities. It is a probability mass function, which is referred to as the integer normal distribution. Using the probability mass function of the ambiguities, the distribution of the 'fixed' baseline estimator is presented next. It equals a weighted sum of conditional distributions, with the weights given by the probability masses of the ambiguity distribution. From this distribution of the baseline estimator we then finally determine the first two moments.

2 Integer least-squares estimation

In principle all the GPS models can be cast in the following conceptual frame of linear(ized) observation equations

$$y = Aa + Bb + e \tag{1}$$

where y is the given data vector, a and b are the unknown parameter vectors of order n and q respectively, and where e is the noise vector of order m. The matrices A and B are the corresponding design matrices of order $m \times n$ and $m \times q$ respectively. The matrix (A, B) is assumed to be of full rank. In case of GPS the data vector will usually consist of the 'observed minus computed' single- or dual-frequency double-differenced (DD) phase and/or pseudo range (code) observations, accumulated over all observation epochs. The entries of vector a are the DD carrier phase ambiguities, expressed in units of cycles rather than range. They are known to be integers. The entries of vector b consist of the remaining unknown parameters, such as for instance baseline components (coordinates) and possibly atmospheric delay parameters (troposphere, ionosphere).

When using the least-squares principle, the above system of observation equations can be solved by means of the minimization problem

$$\min_{a,b}(y - Aa - Bb)^T Q_y^{-1}(y - Aa - Bb) \ \ , \ \ a \in Z^n \ , \ b \in R^q \tag{2}$$

with Q_y the variance-covariance matrix of the observables and where Z^n and R^q denote the n-dimensional space of integers and the q-dimensional space of real numbers respectively. This is a nonstandard least-squares problem, due to the integer constraints on the ambiguities. This type of least-squares problem was first introduced in [*Teunissen*, 1993] and has been coined with the term '*integer least-squares*'.

Conceptually one can divide the computation of (2) into three different steps. In the first step one simply disregards the integer constraints on the ambiguities and performs a standard least-squares adjustment. As a result one obtains the (real-valued) least-squares estimates of a and b, together with their variance-covariance matrix

$$\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix} , \begin{bmatrix} Q_{\hat{a}} & Q_{\hat{a}\hat{b}} \\ Q_{\hat{b}\hat{a}} & Q_{\hat{b}} \end{bmatrix} \tag{3}$$

This solution is often referred to as the 'float' solution. In the second step the 'float' ambiguity estimate $\hat{a}$ and its variance- covariance matrix are used to compute the corresponding integer ambiguity estimate. This implies that one has to solve the minimization problem

$$\min_{a \in Z^n} (\hat{a}-a)^T Q_{\hat{a}}^{-1} (\hat{a}-a) \tag{4}$$

Its solution will be denoted as $\check{a}$. Once this integer solution is computed, it is finally used in the third step to correct the 'float' estimate of b. As a result one obtains the 'fixed' solution

$$\check{b} = \hat{b} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} (\hat{a} - \check{a}) \tag{5}$$

From a computational point of view, the most difficult part in the above three steps is the computation of the solution of (4). It requires the minimization of a quadratic form over the whole n- dimensional space of integers. In [*Teunissen*, 1993] the least- squares ambiguity decorrelation adjustment (LAMBDA) was introduced as a method for computing the integer least-squares ambiguities in a rigorous and efficient way, see also [*Teunissen*, 1995], [*de Jonge and Tiberius*, 1996] and the textbooks [*Kleusberg and Teunissen*, 1996], [*Hofmann-Wellenhof et al.*, 1997] and [*Strang and Borre*, 1997].

In this contribution we will not discuss the computational intricacies of the above procedure. Instead, we will present the statistical properties of the 'fixed' estimator $\check{b}$. For that purpose we first have to state our assumptions concerning the vector of observables y. It will be assumed that y is normally distributed as

$$y \sim N(Aa + Bb, Q_y) \tag{6}$$

This implies that the least-squares principle (2) corresponds to finding the maximum likelihood solution. It is well-known from adjustment theory, that if y is distributed as (6), then $\hat{a}$ and $\hat{b}$ are normally distributed too as well as unbiased. Their (marginal) probability density distributions are given as

$$p_{\hat{a}}(\xi) = \frac{1}{\sqrt{\det(Q_{\hat{a}})}(2\pi)^{\frac{1}{2}n}} \exp\{-\frac{1}{2}(\xi - a)^T Q_{\hat{a}}^{-1} (\xi - a)\} \tag{7}$$

and

$$p_{\hat{b}}(\zeta) = \frac{1}{\sqrt{\det(Q_{\hat{b}})}(2\pi)^{\frac{1}{2}q}} \exp\{-\frac{1}{2}(\zeta - b)^T Q_{\hat{b}}^{-1} (\zeta - b)\} \tag{8}$$

It is the purpose of this distribution to find the corresponding distributions for the 'fixed' solution. To this end we will consider the scalar case, $n = 1$ and $q = 1$, first.

3 The distribution in the scalar case

In this section we will assume that both $\check{a}$ and $\check{b}$ are scalars. Although not realistic within the context of GPS ambiguity resolution, the scalar case has the advantage of being relatively straightforward, while it still retains most of the characteristics of the vectorial case. Treatment of the scalar case therefore prepares us for our discussion of the vectorial case. We will first present the distribution of $\check{a}$ and then the distribution of $\check{b}$.

3.1 The distribution of $\check{a}$

In the scalar case we have

$$\hat{a} \sim N(a, \sigma_{\hat{a}}^2) \ , \ a \in Z \tag{9}$$

Hence, $\hat{a}$ is normally distributed with integer mean a and variance $\sigma_{\hat{a}}^2$. Its probability density function is given as

$$p_{\hat{a}}(\xi) = \frac{1}{\sigma_{\hat{a}}\sqrt{2\pi}} \exp\{-\frac{1}{2}(\frac{\xi - a}{\sigma_{\hat{a}}})^2\} \tag{10}$$

In the scalar case the minimization problem (4) reduces simply to a rounding of the 'float' ambiguity $\hat{a}$ to its nearest integer. If we denote 'rounding to the nearest integer' by '[.]', the integer least-squares ambiguity reads

$$\check{a} = [\hat{a}] \tag{11}$$

In order to determine the distribution of the integer least-squares estimator, we need the probability that $\check{a} = i$, for $i \in Z$. This probability equals the area under the normal distribution of $\hat{a}$ over the interval $(i - \frac{1}{2}, i + \frac{1}{2})$. Hence, the distribution of $\check{a}$ reads

$$P(\check{a} = i) = P(\mid \hat{a} - i \mid \leq \frac{1}{2}) = \int_{i-\frac{1}{2}}^{i+\frac{1}{2}} p_{\hat{a}}(\xi) d\xi \tag{12}$$

Note that we use capital 'P' for probability and small 'p' for the density. The result (12) shows that the distribution of $\check{a}$ is of the discrete type. It is a probability *mass* function. Note that it is symmetric about the integer mean a and that it reaches its maximum for $i = a$. Both these properties are of importance. The property of symmetry about a implies that the integer estimator $\check{a} = [\hat{a}]$ is unbiased. Thus $E\{\check{a}\} = a$. Also the second property, $\max_i P(\check{a} = i) = P(\check{a} = a)$, is a comforting one. It states that of all integers, the largest probability mass is located at the integer a.

For GPS ambiguity resolution the probability $P(\check{a} = a)$ is particularly of relevance. It is the probability of correct integer estimation. In the scalar case, this probability is relatively easy to evaluate. To see this, we first use the integral of the *standard* normal distribution

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}\xi^2\} d\xi$$

to write the above probability mass function as

$$P(\check{a} = i) = \Phi(\frac{i - a}{\sigma_{\hat{a}}} + \frac{1}{2\sigma_{\hat{a}}}) - \Phi(\frac{i - a}{\sigma_{\hat{a}}} - \frac{1}{2\sigma_{\hat{a}}}) \tag{13}$$

The probability of correct integer estimation follows then as

$$P(\check{a}=a) = 2\Phi(\frac{1}{2\sigma_{\hat{a}}}) - 1 \tag{14}$$

Note, as expected, that the probability of rounding to the correct integer value increases as the standard deviation of $\hat{a}$ gets smaller.

3.2 The distribution of $\check{b}$

Since the distribution of the 'float' solution is known, it is not too difficult to make qualitative statements about its probabilistic characteristics. For instance, let $R_{i,\zeta_0}(\xi,\zeta)$ be a rectangle in R^2 which is centred at (i,ζ_0) and which has side lengths of 1 and 2β respectively. Thus

$$R_{i,\zeta_0}(\xi,\zeta) = \{(\xi,\zeta)\in R^2 \mid \mid \xi - i \mid \leq \frac{1}{2}, \mid \zeta - \zeta_0 \mid \leq \beta\} \tag{15}$$

The probability that the float solution lies in this rectangle reads then

$$P((\hat{a},\hat{b})\in R_{i,\zeta_0}(\xi,\zeta)) = \int\int_{R_{i,\zeta_0}(\xi,\zeta)} p_{\hat{a}\hat{b}}(\xi,\zeta)d\xi d\zeta \tag{16}$$

with $p_{\hat{a}\hat{b}}(\xi,\zeta)$ the joint density of $\hat{a}$ and $\hat{b}$. To obtain the (marginal) probability for $\hat{b}$, we first note that the intervals $\mid \xi - i \mid \leq \frac{1}{2}$, $i \in Z$, divide the real-axis R in an almost mutually exclusive way. The only overlap these intervals have, occurs at their boundaries. This implies that we can sum (16) over all integers $i \in Z$ to obtain the probability

$$\begin{aligned} P(\mid \hat{b} - \zeta_0 \mid \leq \beta) &= \textstyle\sum_{i\in Z}\int\int_{R_{i,\zeta_0}(\xi,\zeta)} p_{\hat{a}\hat{b}}(\xi,\zeta)d\xi d\zeta \\ &= \textstyle\int_{\mid\zeta-\zeta_0\mid\leq\beta} p_{\hat{b}}(\zeta)d\zeta \\ &= \textstyle\int_{\mid\zeta-\zeta_0\mid\leq\beta} \frac{1}{\sigma_{\hat{b}}\sqrt{2\pi}}\exp\{-\frac{1}{2}(\frac{\zeta-b}{\sigma_{\hat{b}}})^2\}d\zeta \end{aligned} \tag{17}$$

This result holds for the 'float' solution $\hat{b}$, but not for the 'fixed' solution $\check{b}$. In order to obtain a corresponding result for the 'fixed' solution, we first need to find the region of integration that would correspond with (15). For that purpose consider the estimation rule of integer least-squares. It reads

$$\hat{a}\rightarrow\check{a} \implies \hat{b}\rightarrow\check{b} = \hat{b} - \sigma_{\hat{b}\hat{a}}\sigma_{\hat{a}}^{-2}(\hat{a}-\check{a}) \tag{18}$$

Thus when $\hat{a}$ gets mapped to $\check{a}$, then $\hat{b}$ gets mapped to $\check{b}$. Geometrically, this mapping can be described as follows. Since the region covered by the confidence ellipse of the 'float' solution is given as

$$C(\xi,\zeta) = \{(\xi,\zeta)\in R^2 \mid \begin{bmatrix}\xi-a\\ \zeta-b\end{bmatrix}^T \begin{bmatrix}\sigma_{\hat{a}}^2 & \sigma_{\hat{a}\hat{b}}\\ \sigma_{\hat{b}\hat{a}} & \sigma_{\hat{b}}^2\end{bmatrix}^{-1} \begin{bmatrix}\xi-a\\ \zeta-b\end{bmatrix} \leq \chi^2\} \tag{19}$$

the line through its centre (a,b) intersecting the ellipse at the two points where it has a vertical tangent, reads $\zeta = b - \sigma_{\hat{b}\hat{a}}\sigma_{\hat{a}}^{-2}(a-\xi)$. Parallel to this line we have the line $\zeta = \hat{b} - \sigma_{\hat{b}\hat{a}}\sigma_{\hat{a}}^{-2}(\hat{a} -$

ξ). It passes through the 'float' solution $(\hat{a}, \hat{b})$, as well as through the 'fixed' solution $(\check{a}, \check{b})$. The conclusion reads therefore that *every* potential 'float' solution (ξ, ζ) for which $\mid \xi - i \mid \leq \frac{1}{2}$ and $\zeta = \hat{b} - \sigma_{\hat{b}\hat{a}}\sigma_{\hat{a}}^{-2}(\hat{a} - \xi)$ holds true, gets mapped to *the same* integer least-squares solution $\check{a} = i$ and $\check{b} = \hat{b} - \sigma_{\hat{b}\hat{a}}\sigma_{\hat{a}}^{-2}(\hat{a} - i)$. This implies that it is the region

$$S_{i,\zeta_0}(\xi,\zeta) = \{(\xi,\zeta) \in R^2 \mid \; \mid \xi - i \mid \leq \frac{1}{2}, \mid (\zeta - \zeta_0) - \sigma_{\hat{b}\hat{a}}\sigma_{\hat{a}}^{-2}(\xi - i) \mid \leq \beta\} \tag{20}$$

which gets mapped by the integer least-squares rule to the rectangle (15). Note that also $S_{i,\zeta_0}(\xi,\zeta)$ is centred at (i, ζ_0). It is not a rectangle however, but a *parallelogram* which has the same area as the rectangle.

We are now in the position to determine the probability for the 'fixed' solution $\check{b}$. From the above relation between (15) and (20), it follows that

$$P((\hat{a}, \check{b}) \in R_{i,\zeta_0}(\xi,\zeta)) = P((\hat{a}, \hat{b}) \in S_{i,\zeta_0}(\xi,\zeta)) \tag{21}$$

This shows that the probability for the 'fixed' solution can be computed from the joint distribution of the 'float' solution, using the appropriate region of integration. For the 'fixed' solution $\check{b}$ we therefore have instead of (17),

$$\begin{aligned}
P(\mid \check{b} - \zeta_0 \mid \leq \beta) &= \textstyle\sum_{i \in Z} \int\int_{S_{i,\zeta_0}(\xi,\zeta)} p_{\hat{a}\hat{b}}(\xi,\zeta) d\xi d\zeta \\
&= \textstyle\sum_{i \in Z} \int\int_{S_{i,\zeta_0}(\xi,\zeta)} p_{\hat{b}|\hat{a}}(\zeta \mid \xi) p_{\hat{a}}(\xi) d\zeta d\xi \\
&= \textstyle\sum_{i \in Z} \int\int_{S_{i,\zeta_0}(\xi,\zeta)} \frac{1}{\sigma_{\hat{b}|\hat{a}}\sqrt{2\pi}} \exp\{-\frac{1}{2}(\frac{\zeta - b(\xi)}{\sigma_{\hat{b}|\hat{a}}})^2\} \frac{1}{\sigma_{\hat{a}}\sqrt{2\pi}} \exp\{-\frac{1}{2}(\frac{\xi - a}{\sigma_{\hat{a}}})^2\} d\zeta d\xi \\
&= \textstyle\sum_{i \in Z} [\int_{|\zeta - \zeta_0| \leq \beta} \frac{1}{\sigma_{\hat{b}|\hat{a}}\sqrt{2\pi}} \exp\{-\frac{1}{2}(\frac{\zeta - b(i)}{\sigma_{\hat{b}|\hat{a}}})^2\} d\zeta] P(\mid \hat{a} - i \mid \leq \frac{1}{2})
\end{aligned} \tag{22}$$

with

$$b(i) = b - \sigma_{\hat{b}\hat{a}}\sigma_{\hat{a}}^{-2}(a - i) \;\text{ and }\; \sigma^2_{\hat{b}|\hat{a}} = \sigma^2_{\hat{b}} - \sigma^2_{\hat{a}\hat{b}}/\sigma^2_{\hat{a}}$$

being the conditional mean and conditional variance respectively.

The last equation of (22) is our sought for expression for the probability of the 'fixed' solution $\check{b}$. It equals a weighted sum of conditional probabilities. The weights are given by the probability mass function of $\check{a}$ and the conditional probabilities are those of the distribution $p_{\hat{b}|\hat{a}}(\zeta \mid \xi = i)$.

As with (13), also the above expression can be formulated in terms of the function $\Phi(x)$. It reads

$$P(\mid \check{b} - \zeta_0 \mid \leq \beta) = \sum_{i \in Z}[\Phi(\frac{\zeta_0 - b(i)}{\sigma_{\hat{b}|\hat{a}}} + \frac{\beta}{2\sigma_{\hat{b}|\hat{a}}}) - \Phi(\frac{\zeta_0 - b(i)}{\sigma_{\hat{b}|\hat{a}}} - \frac{\beta}{2\sigma_{\hat{b}|\hat{a}}})][\Phi(\frac{i-a}{\sigma_{\hat{a}}} + \frac{1}{2\sigma_{\hat{a}}}) - \Phi(\frac{i-a}{\sigma_{\hat{a}}} - \frac{1}{2\sigma_{\hat{a}}})] \tag{23}$$

4 The distribution in the vectorial case

In this section we will generalize the results of the previous section to the vectorial case. Hence, $\check{a}$ and $\check{b}$ will be treated as vectors of dimension $n \geq 1$ and $q \geq 1$ respectively. Within the context of GPS, the vector $\check{a}$ refers then to the integer least-squares solution of the carrier phase ambiguities and the vector $\check{b}$ to the corresponding solution of the noninteger parameters in the model of observation equations. Hence for a single-baseline model of DD observation equations for which the atmospheric delays are assumed absent, the vector $\check{b}$ contains the solution of the baseline coordinates and is therefore of dimension $q = 3$. The dimension of this vector will be higher than 3 though, when the model includes parameters for the atmospheric delays as well. The same holds true when a multi-baseline model is used, that is, when more than two receivers are used. Thus although $\check{b}$ is referred to as the baseline estimator, it is in fact the 'fixed' estimator of all noninteger parameters. For positioning purposes though, the baseline components of the estimator will be the most relevant ones.

4.1 The distribution of $\check{a}$

In the scalar case, rounding the 'float' ambiguity to its nearest integer is equivalent to solving the minimization problem (4). In general this fails to be true for the vectorial case. It would only be true when the ambiguity variance-covariance matrix is a diagonal matrix, which is not the case with GPS. Hence, for the vectorial case we can not simply take the cube $\cap_{j=1}^{n}\{\mid \hat{a}_j - i_j \mid \leq \frac{1}{2}\}$ as a generalization of the interval $\mid \hat{a} - i \mid \leq \frac{1}{2}$. In order to find the proper generalization of the one-dimensional interval, we need to consider the minimization problem (4) again. Note that $z \in Z^n$ would solve the minimization problem (4) if and only if the 'float' solution $\hat{a}$ satisfies

$$(\hat{a} - z)^T Q_{\hat{a}}^{-1} (\hat{a} - z) \leq (\hat{a} - \zeta)^T Q_{\hat{a}}^{-1} (\hat{a} - \zeta) \ , \ \forall \zeta \in Z^n \tag{24}$$

This inequality of the two quadratic forms can also be written as an inequality which is linear in $\hat{a}$,

$$(\zeta - z)^T Q_{\hat{a}}^{-1} (\hat{a} - z) \leq \frac{1}{2} (\zeta - z)^T Q_{\hat{a}}^{-1} (\zeta - z) \ , \ \forall \zeta \in Z^n \tag{25}$$

Since both z and ζ are integer, and since the inequality should hold for all $\zeta \in Z^n$, we may replace $(\zeta - z)$ by $c \in Z^n$ and write (25) as

$$\mid w(\hat{a}, c, z) \mid \leq \frac{1}{2} \sqrt{c^T Q_{\hat{a}}^{-1} c} \ , \ \forall c \in Z^n \tag{26}$$

with

$$w(\hat{a}, c, z) = \frac{c^T Q_{\hat{a}}^{-1} (\hat{a} - z)}{\sqrt{c^T Q_{\hat{a}}^{-1} c}}$$

Note that $w(\hat{a}, c, z)$ is the well-known w-test statistic for testing one-dimensional alternative hypotheses [*Baarda*, 1968], [*Teunissen*, 1985]. It is the test statistic used for testing H_o : $E\{\hat{a}\} = z$ against H_a : $E\{\hat{a}\} = z + c\nabla$, with $\frac{1}{2}\sqrt{c^T Q_{\hat{a}}^{-1} c}$ as 'critical value'. Geometrically, the $w(\hat{a}, c, z)$ can be interpreted as orthogonal projectors that project $(\hat{a} - z)$ onto the direction vectors $c \in Z^n$. Hence for a single vector $c \in Z^n$, the inequality of (26) describes the region between two parallel hyperplanes which are a distance $\sqrt{c^T Q_{\hat{a}}^{-1} c}$ apart, centred at z and which have the vector c as their normal. Since the inequality must hold for all integer vectors c, the multivariate generalization of the one- dimensional interval $\mid \xi - i \mid \leq \frac{1}{2}$ equals the intersection of all subsets bounded by these parallel hyperplanes. It reads

$$S_z(\xi) = \cap_{\forall c \in Z^n} \{ \xi \in R^n \mid \ \mid w(\xi, c, z) \mid \leq \frac{1}{2} \sqrt{c^T Q_{\hat{a}}^{-1} c} \} \tag{27}$$

This region contains all values of $\hat{a} = \xi$ which will be mapped to the single integer grid point $z \in Z^n$ when solving the integer least- squares problem (4). Thus the integer least-squares solution equals z when $\hat{a}$ lies in $S_z(\xi)$ and vice versa. Hence

$$\hat{a} \in S_z(\xi) \Longleftrightarrow \check{a} = z \tag{28}$$

Note that each integer grid point $z \in Z^n$ has such a subset assigned to it and that $R^n = \cup_{\forall z \in Z^n} S_z(\xi)$. These subsets act as *pull-in-regions* for $\hat{a} \in R^n$. That is, whenever $\hat{a}$ lies in such a subset $S_z(\xi)$, it is pulled to z, being the centre grid point of the set. Also note that $S_z(\xi)$ can be seen as a generalization of the interval $\mid \xi - i \mid \leq \frac{1}{2}$ or of the cube $\cap_{j=1}^{n} \{ \mid \xi_j - i_j \mid \leq \frac{1}{2} \}$. It reduces to the interval when $n = 1$ and it reduces the the cube when the matrix $Q_{\hat{a}}$ is diagonal. We are therefore now in the position to generalize (12) to the vectorial case. The probability mass function of $\check{a}$ reads

$$P(\check{a} = z) = P(\hat{a} \in S_z(\xi)) = \int_{S_z(\xi)} \frac{1}{\sqrt{\det(Q_{\hat{a}})} (2\pi)^{\frac{1}{2}n}} \exp\{ -\frac{1}{2} (\xi - a)^T Q_{\hat{a}}^{-1} (\xi - a) \} d\xi \tag{29}$$

The discrete distribution of the integer least-squares ambiguities follows thus from mapping the volume of the multivariate normal distribution over the subsets $S_z(\xi)$ to their respective centre grid points $z \in Z^n$. This distribution was introduced in [*Teunissen*, 1998] where it was called the *integer normal distribution*.

4.2 The distribution of $\check{b}$

In order to determine the distribution of the 'fixed' solution $\check{b}$, we proceed in a way which is similar to the approach used for the scalar case. We therefore first generalize (21). It reads

$$P((\hat{a}, \check{b}) \in R_{z,\zeta_0}(\xi, \zeta)) = P((\hat{a}, \hat{b}) \in S_{z,\zeta_0}(\xi, \zeta)) \tag{30}$$

with

$$\begin{array}{lcl} R_{z,\zeta_0}(\xi, \zeta) & = & S_z(\xi) \times R_{\zeta_0}(\zeta) \\ S_{z,\zeta_0}(\xi, \zeta) & = & S_z(\xi) \times T_{z,\zeta_0}(\xi, \zeta) \\ R_{\zeta_0}(\zeta) & = & \{ \zeta \in R^p \mid \ \parallel \zeta - \zeta_0 \parallel \leq \beta \} \\ T_{z,\zeta_0}(\xi, \zeta) & = & \{ \zeta \in R^p \mid \ \parallel (\zeta - \zeta_0) - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} (\xi - z) \parallel \leq \beta \} \end{array}$$

This shows that the probability for the 'fixed' solution can be computed from the joint distribution of the 'float' solution, using the appropriate region of integration. In this region of integration we recognize the Cartesian product of the pull-in-regions $S_z(\xi)$ with the $T_{z,\zeta_0}(\xi, \zeta)$. For the 'fixed' solution $\check{b}$ we therefore have,

$$\begin{array}{lcl} P(\parallel \check{b} - \zeta_0 \parallel \leq \beta) & = & \sum_{z \in Z^n} \int \int_{S_{z,\zeta_0}(\xi,\zeta)} p_{\hat{a}\hat{b}}(\xi, \zeta) d\xi d\zeta \\ & = & \sum_{z \in Z^n} \int \int_{S_{z,\zeta_0}(\xi,\zeta)} p_{\hat{b}|\hat{a}}(\zeta \mid \xi) p_{\hat{a}}(\xi) d\zeta d\xi \\ & = & \sum_{z \in Z^n} [\int_{R_{\zeta_0}(\zeta')} p_{\hat{b}|\hat{a}}(\zeta' \mid \xi = z) d\zeta'] [\int_{S_z(\xi)} p_{\hat{a}}(\xi) d\xi] \end{array} \tag{31}$$

for which the change-of-variable transformation

$$\begin{bmatrix} \xi \\ \zeta' \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ -Q_{\hat{b}\hat{a}}Q_{\hat{a}}^{-1} & I_p \end{bmatrix} \begin{bmatrix} \xi - z \\ \zeta \end{bmatrix} + \begin{bmatrix} z \\ 0 \end{bmatrix}$$

was used to obtain the last equality. In this last equation we recognize the conditional distribution of $\hat{b}$, $p_{\hat{b}|\hat{a}}(\zeta \mid \xi = z)$, and the probability mass function of $\check{a}$. Hence, we may also write

$$\boxed{\begin{array}{c} P(\| \check{b} - \zeta_0 \| \leq \beta) = \\ = \sum_{z\in Z^n}[\int_{\|\zeta-\zeta_0\| \leq \beta} \frac{1}{\sqrt{\det(Q_{\hat{b}|\hat{a}})(2\pi)^{\frac{1}{2}q}}} \exp\{-\frac{1}{2}(\zeta - b(z))^T Q_{\hat{b}|\hat{a}}^{-1}(\zeta - b(z))\}d\zeta]P(\check{a} = z) \end{array}} \tag{32}$$

with

$$b(z) = b - Q_{\hat{b}\hat{a}}Q_{\hat{a}}^{-1}(a - z) \text{ and } Q_{\hat{b}|\hat{a}} = Q_{\hat{b}} - Q_{\hat{a}\hat{b}}Q_{\hat{a}}^{-1}Q_{\hat{a}\hat{b}}$$

being the conditional mean and conditional variance matrix respectively. This result is the sought for multivariate generalization of (22). Note that we did not yet specify the norm taken in (32). If we choose the norm to satisfy $\| \,.\, \|^2_{Q_{\hat{b}|\hat{a}}} = (.)^T Q_{\hat{b}|\hat{a}}^{-1}(.)$, the above probability can be expressed in terms of noncentral Chi-square distributions. We have

$$P(\| \check{b} - \zeta_0 \|_{Q_{\hat{b}|\hat{a}}} \leq \beta) = \sum_{z\in Z^n} P(\chi^2(q, \lambda_z) \leq \beta^2)P(\check{a} = z) \tag{33}$$

with $\chi^2(q, \lambda_z)$ the noncentral Chi-square distribution with q degrees of freedom and noncentrality parameter $\lambda_z = \| b(z) - \zeta_0 \|^2_{Q_{\hat{b}|\hat{a}}}$. This result follows when using the property that if a random q-vector x is normally distributed as $x \sim N(\mu_x, Q_x)$, then $x^T Q_x^{-1} x \sim \chi^2(q, \mu_x^T Q_x^{-1}\mu_x)$.

In (33) one is still free in choosing ζ_0. In case one wants to know by how much the 'fixed' baseline estimator deviates from its mean, the choice $\zeta_0 = b$ should be taken. The expression for the noncentrality parameter simplifies then to

$$\lambda_z = (a - z)^T (Q_{\hat{a}|\hat{b}}^{-1} - Q_{\hat{a}}^{-1})(a - z) \tag{34}$$

4.3 The mean and variance matrix of $\check{b}$

Having obtained the distribution of $\check{b}$, we will now determine the expectation and dispersion of the 'fixed' baseline estimator, that is, its mean and variance matrix. They are defined as

$$\begin{array}{rcccl} E\{\check{b}\} & = & \mu_{\check{b}} & = & \int \zeta p_{\check{b}}(\zeta) d\zeta \\ D\{\check{b}\} & = & Q_{\check{b}} & = & \int (\zeta - \mu_{\check{b}})(\zeta - \mu_{\check{b}})^T p_{\check{b}}(\zeta) d\zeta \end{array} \tag{35}$$

Using the results of the previous subsection, we obtain for the mean

$$\begin{array}{rcl} E\{\check{b}\} & = & \sum_{z\in Z^n}[\int \zeta p_{\hat{b}|\hat{a}}(\zeta \mid \xi = z) d\zeta] P(\check{a} = z) \\ & = & \sum_{z\in Z^n} b(z) P(\check{a} = z) \\ & = & b - Q_{\hat{b}\hat{a}}Q_{\hat{a}}^{-1}(a - \sum_{z\in Z^n} z P(\check{a} = z)) \end{array} \tag{36}$$

In this last expression we recognize the mean of $\check{a}$. It equals a since the probability mass function $P(\check{a}=z)$ is symmetric about a. From this it follows that

$$\boxed{E\{\check{b}\}=E\{\hat{b}\}=b} \tag{37}$$

Hence not only the 'float' solution is unbiased, but the 'fixed' solution as well. This is a comforting result. It shows that the integer least-squares ambiguity estimation does not introduce any biases into the 'fixed' baseline estimator. Of course, biases may still be present in the 'fixed' baseline estimator due to e.g. cycle slips or outliers. But this is then due to the fact that a misspecified model is used. It is not a consequence of having used the principle of integer least-squares.

Again using the results of the previous subsection, we obtain for the variance matrix of the 'fixed' baseline estimator

$$\begin{aligned} Q_{\check{b}} &= \textstyle\sum_{z\in Z^n}[\int(\zeta-b)(\zeta-b)^T p_{\hat{b}|\hat{a}}(\zeta\mid\xi=z)d\zeta]P(\check{a}=z) \\ &= \textstyle\sum_{z\in Z^n}[\int(\zeta-b(z))(\zeta-b(z))^T p_{\hat{b}|\hat{a}}(\zeta\mid\xi=z)+(b-b(z))(b-b(z))^T]P(\check{a}=z) \\ &= Q_{\hat{b}|\hat{a}}+Q_{\hat{b}\hat{a}}Q_{\hat{a}}^{-1}[\textstyle\sum_{z\in Z^n}(z-a)(z-a)^T P(\check{a}=z))]Q_{\hat{a}}^{-1}Q_{\hat{a}|\hat{b}} \end{aligned} \tag{38}$$

In this last expression we recognize the variance matrix of $\check{a}$. The variance matrix of the 'fixed' baseline estimator follows therefore as

$$\boxed{Q_{\check{b}}=Q_{\hat{b}|\hat{a}}+Q_{\hat{b}\hat{a}}Q_{\hat{a}}^{-1}Q_{\check{a}}Q_{\hat{a}}^{-1}Q_{\hat{a}|\hat{b}}} \tag{39}$$

This shows that the precision of the 'fixed' baseline estimator $\check{b}$ is always poorer than that of the conditional baseline estimator $\hat{b}\mid\hat{a}$. The difference of the two is governed by the precision of the vector of integer least-squares ambiguities $\check{a}$, which on its turn is governed by the probability mass function $P(\check{a}=z)$. The two variance matrices will only coincide when $P(\check{a}=a)=1$, that is, when the probability of correct integer estimation equals one.

5 Summary

In this contribution the probability distribution of the 'fixed' GPS baseline estimator was presented. By means of this distribution it becomes possible to diagnose the quality of the 'fixed' GPS baseline and to infer the contribution of GPS ambiguity resolution. For the estimation of the carrier phase ambiguities the integer least-squares principle was used.

It was shown that the distribution of the 'fixed' baseline equals a weighted sum of conditional distributions, with the weights being determined by the probability mass function of the integer least-squares ambiguities. The conditional distributions in this sum differ only in their means.

Using the distribution it was also shown that the 'fixed' baseline estimator is an unbiased estimator. Hence, the inclusion of the integer ambiguity constraints and the estimation of the carrier phase ambiguities by means of the integer least-squares principle does not introduce any biases into the baseline solution. Finally, the variance matrix of the 'fixed' baseline estimator was determined. The expression so obtained clearly shows by how much the precision of the 'fixed' baseline estimator differs from its conditional counterpart, that is, from the estimator that follows from assuming the ambiguities to be deterministic and known.

Finally we note, although the term 'baseline estimator' was used, that the results presented hold for all noninteger parameters in the GPS model of observation equations, thus also, if applicable, for the atmospheric delay parameters such as used for the ionosphere and troposphere. For positioning purposes though, the baseline components of the estimator will then be the most relevant ones.

6 References

[1] Baarda, W. (1968): *A testing procedure for use in geodetic networks*, Netherlands Geodetic Commission, Publications on Geodesy, New Series, Vol. 2, No. 5.

[2] de Jonge, P.J., C.C.J.M. Tiberius (1996): The LAMBDA method for integer ambiguity estimation: implementation aspects. Publications of the Delft Geodetic Computing Centre, *LGR Series*, No. 12. 49 p.

[3] Hofmann-Wellenhof, B., H. Lichtenegger, J. Collins (1997): *Global Positioning System: Theory and Practice*. 4th edition. Springer Verlag.

[4] Kleusberg, A., P.J.G. Teunissen (eds) (1996): *GPS for Geodesy*, Lecture Notes in Earth Sciences, Vol. 60, Springer Verlag.

[5] Strang, G. and K. Borre (1997): *Linear Algebra, Geodesy, and GPS*, Wellesley-Cambridge Press.

[6] Teunissen, P.J.G. (1985): Quality control in geodetic networks. Chapter 17 in *Optimization and Design of Geodetic Networks*, E. Grafarend and F. Sanso (eds), Springer Verlag.

[7] Teunissen, P.J.G. (1993): Least-squares estimation of the integer GPS ambiguities. Invited Lecture, Section IV Theory and Methodology, IAG General Meeting, Beijing, China, August 1993. Also in: *LGR Series*, No. 6, Delft Geodetic Computing Centre.

[8] Teunissen, P.J.G. (1995): The least-squares ambiguity decorrelation adjustment: a method for fast GPS integer ambiguity estimation. *Journal of Geodesy*, Vol. 70, No. 1-2, pp. 65-82.

[9] Teunissen, P.J.G. (1998): On the integer normal distribution of the GPS ambiguities. Submitted to *Artificial Satellites.*