Some Remarks on GPS Ambiguity Resolution

P.J.G. Teunissen

Delft Geodetic Computing Centre (LGR)
Faculty of Geodesy
Delft University of Technology
The Netherlands
E-mail: P.J.G.Teunissen@geo.tudelft.nl

ABSTRACT
In this contribution we will discuss some issues of GPS ambiguity resolution. The issues are related to inconsistencies and misconceptions which are unfortunately still present in some of the practical methods used for ambiguity resolution. In this contribution four such pitfalls will be identified and discussed. The first two pitfalls are concerned with the way the integer ambiguities are sometimes computed. They show what might happen with the solution when the neccessary precautions are not taken. The last two pitfalls are related to the statistical properties which are sometimes thought to be valid for the estimated ambiguities.

1. INTRODUCTION

GPS ambiguity resolution is the process of resolving the unknown cycle ambiguities of the double-difference (DD) carrier phase data as integers. It is the key to high precision relative GPS positioning, when only short observation time spans are used. Once the integer ambiguities are resolved, the carrier phase measurements will start to act if they were high-precision pseudo range measurements, thereby allowing the remaining parameters, such as the baseline coordinates, to be estimated with a comparable high precision.
The topic of resolving the integer values of the DD ambiguities has been a rich source of GPS research in the last decade or so. Examples are: [*Counselman and Gourevitch*, 1981], [*Hatch*, 1982], [*Remondi*, 1986], [*Blewitt*, 1989], [*Wubbena*, 1989], [*Frei*, 1991], [*Teunissen*, 1993]. It resulted in a variety of different methods and proposals for efficiently estimating the integer ambiguities. In [*ibid*] the sequence of computational steps for efficiently solving the carrier phase ambiguities by means of an integer least-squares problem was introduced. However, despite the progress made, there are unfortunately still a number of misconceptions that persist in some of the earlier methods in use for ambiguity resolution. It is the purpose of this contribution to identify four of them and to explain the associated pitfalls involved. Some of the pitfalls are related to the mathematical inconsistencies which are unfortunately still present in some of the methods documented. As a result, no guarantee can be given that these methods indeed provide the integer least-squares solution. At best, an approximation to it is obtained. The other pitfalls are related to the statistical properties which are sometimes claimed of the 'ambiguity-fixed' solutions. This contribution was presented at the symposium of the 'Geodätische Woche', 6-11 Oct. 97, Berlin, Germany.
In Sect. 2 we first give a brief overview of the steps which are involved in ambiguity resolution. In this section it is also stressed that the methodology of ambiguity resolution should be, and in fact can be formulated independently of the particular GPS application at hand. Hence, there is no need to design or formulate 'new' methods of ambiguity resolution for each new application. In Sect. 3, the first pitfall is identified. It shows the risk of using the so-called 'bootstrapping' technique as a method for integer ambiguity estimation. Although this technique may, from time to time, provide the integer least-squares ambiguities as its solution, there is no guarantee that it will do so all the time. The second pitfall, which is addressed in Sect. 4, is related to the so-called ambiguity search space. For computing the integer least-squares solution, it is stressed that the search space should be of ellipsoidal shape and constructed on the basis of the complete variance-covariance matrix of the ambiguities. Some methods however, still make use of 'boxes' or 'truncated boxes' as their search space. As a consequence, integer ambiguities are obtained, which again cannot be guaranteed to be the most likely integer ambiguities. In Sect. 5 the third pitfall is identified. For the purpose of validation, so-called 'discrimination tests' are performed between the most likely and the second most likely integer ambiguities. Although many of these tests seem to work quite satisfactorily in practice, the associated probability statements made about these tests are quite often in error. This is due to the erroneous claims which are sometimes made about the probability distributions involved. In Sect. 6, we finally consider the fourth and last pitfall. It concerns the fact that one often still considers the solved for integer ambiguities as being deterministic. It is emphasized however, that the

integer least-squares ambiguities are random variates, having their own discrete probability distribution. It is therefore this probability distribution that needs to form the basis for inferring whether or not the estimated integer ambiguities may be considered sufficiently 'nonrandom' or not.

2. THE GPS MODEL OF OBSERVATION EQUATIONS

There exists a whole suite of different GPS models. All these models however, can be cast in the following frame of linear(ized) observation equations

$$y = Aa + Bb + e \tag{2.1}$$

where *y* is the given data vector, *a* and *b* are the unknown parameter vectors and *e* is the noise vector. The data vector *y* will usually consist of the 'observed minus computed' single- or dual frequency DD phase and/or pseudo range (code) observations, accumulated over all observation epochs. The entries of vector *a* are then the DD carrier phase ambiguities, expressed in units of cycles rather than range. They are known to be integer valued. The entries of vector *b* will then consist of the remaining unknown parameters, such as for instance baseline components (coordinates) and possibly atmospheric delay parameters (troposphere, ionosphere). Note that we have followed the customary practice of using the DD version of the code and carrier phase observation equations. This is however not strictly needed. As an alternative one can work with undifferenced or single differenced observations as well. In that case the non-integer ambiguities will have to be reparametrized so as to obtain integer ambiguities again.

As pointed out, there exists a whole suite of GPS models one can think of. A GPS model for relative positioning may be based on the simultaneous use of two receivers (single baseline) or more than two receivers (multi-baseline). It may have the relative receiver-satellite geometry included (geometry-based) or exluded (geometry-free). The geometry is included through the unit direction vectors in the design matrix. When it is excluded, not the baseline components are involved as unknowns in the model, but instead, the receiver-satellite ranges themselves. GPS models may also be discriminated as to whether the slave receiver(s) are in motion (non-stationary) or not (stationary). When in motion, one solves for one or more trajectories, since with the receiver-satellite geometry included, one will have new coordinate unknowns for each new epoch. One may also discriminate as to whether the differential ionospheric delays are included as unknowns or not. Continuing along this line, one can thus identify a whole suite of different GPS models.

For each of the above models, there already exists a vast literature on GPS ambiguity resolution. Some typical examples of the above applications are: [*Hwang*, 1991], [*Brown*, 1992], [*Dedes and Goad*, 1994], [*Mervart*, 1995], [*Tiberius and de Jonge*, 1995]. Other examples can be found in the textbooks [*Borre*, 1995], [*Kleusberg and Teunissen*, 1996], [*Leick*, 1995], [*Parkinson et al.*, 1996], [*Hofmann-Wellenhof et al.*, 1997]. Although in all these examples, the models themselves differ greatly in complexity and diversity, they all have in common that they can be formulated as Eq.(2.1), irrespective of whether single or dual frequency data, and phase only or phase and code data are used. When using the least-squares principle, this therefore implies that all these different models can be solved by means of the minimization problem

$$\min_{a,b} \|y - Aa - Bb\|^2 \quad , \quad a \ \text{ integer} \ , \quad b \ \text{ real} \tag{2.2}$$

This is a nonstandard least-squares problem, due to the integer constraints on the ambiguities. This type of least-squares problem was first introduced in [*Teunissen, 1993*] and has been coined with the term '*integer least-squares'*. Conceptually one can divide the computation of Eq. (2.2) into three different steps. In the first step one simply disregards the integer constraints on the ambiguities and performs a standard least-squares adjustment. As a result one obtains the (real-valued) least-squares estimates of *a* and *b*, together with their variance-covariance matrix

$$\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix} \quad ; \quad \begin{bmatrix} Q_{\hat{a}} & Q_{\hat{a}\hat{b}} \\ Q_{\hat{b}\hat{a}} & Q_{\hat{b}} \end{bmatrix} \tag{2.3}$$

This solution is often referred to as the 'float' solution. In the second step the 'float' ambiguity estimate $\hat{a}$ and its variance matrix are used to compute the corresponding integer ambiguity estimate. This implies that one has to solve the minimization problem

$$\min_{a} (\hat{a} - a)^T Q_{\hat{a}}^{-1} (\hat{a} - a) \quad , \quad a \ \text{ integer} \tag{2.4}$$

Its solution will be denoted as $\breve{a}$. This estimate is then finally used in the third step to correct the 'float' estimate of *b*. As a result one obtains the 'fixed' estimate

$$\breve{b} = \hat{b} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} (\hat{a} - \breve{a}) \tag{2.5}$$

These three steps can be recognized in most of the approaches that have been suggested in the literature for ambiguity resolution. Difficulties and misconceptions remain however with the second step, the step that needs to provide the integer ambiguity estimate from its floated counterpart. In the sections following we will identify four such pitfalls.

3. THE FIRST PITFALL

After the first step we have the float solution $\hat{a}$ and its variance matrix $Q_{\hat{a}}$. In the second step we have to compute the corresponding integer solution $\breve{a}$. In general this is a far from trivial problem [*Teunissen, 1993*]. There are only two cases for which this problem becomes trivial. The first case occurs when only one single ambiguity is involved, that is, when $\hat{a}$ is a scalar instead of a vector. The second case occurs when all ambiguities are completely decorrelated, that is, when the variance matrix $Q_{\hat{a}}$ is diagonal.

If $\hat{a}$ is a scalar, the integer least-squares solution of the ambiguity is simply given by the integer nearest to $\hat{a}$. Hence,

$$\breve{a} = [\hat{a}] \tag{3.1}$$

where '[.]' stands for 'rounding to the nearest integer'. This simple 'rounding to the nearest integer' may also be used in the multivariate case, provided the variance matrix is diagonal. Diagonality of the variance matrix implies namely a complete decorrelation of the ambiguities, which on its turn implies that the multivariate problem becomes decoupled into a multitude of scalar integer estimation problems. Thus when $Q_{\hat{a}}$ is diagonal, we have

$$\breve{a}_1 = [\hat{a}_1] \quad , \quad \breve{a}_2 = [\hat{a}_2] \quad , \quad \cdots \quad , \quad \breve{a}_n = [\hat{a}_n] \tag{3.2}$$

It will be clear that these two trivial cases are not met in actual practice. First, one is almost always confronted with more than one ambiguity, and second, the variance matrix of the ambiguities will not be diagonal. In fact, the least-squares DD ambiguities can be shown to be far from decorrelated, in particular when short time spans of data are used.

Since the integer ambiguity estimation problem becomes trivial in the scalar case, it seems intuitively appealing to try to formulate a similar 'scalar approach' for the multivariate case as well. However, since the ambiguity variance matrix is nondiagonal, such an approach should be able to take the nonzero correlations between the ambiguities into account as well. These considerations have led some researchers to propose a procedure where the integer ambiguities are estimated sequentially, that is, one ambiguity is estimated at a time. This approach is sometimes referred to as the 'bootstrapping' technique. Examples are [*Blewitt*, 1989], [*Dong and Bock*, 1989]. The 'bootstrapping' technique goes as follows. If n ambiguities are available, one starts with the first ambiguity and computes its most likely integer value using the simple 'rounding to the nearest integer' approach of above. Having obtained the integer value of this first ambiguity, the (real-valued) estimates of all remaining ambiguities are then corrected by virtue of their correlation with the first ambiguity. Then the second, but now corrected, ambiguity estimate is used to compute its corresponding most likely integer value, again using the scalar approach of above. Having obtained the integer value of the second ambiguity, the (real-valued) estimates of all remaining *n-2* ambiguities are then again corrected, but now by virtue of their correlation with the second ambiguity. This process is then continued until all ambiguities are taken care of.

In essence this 'bootstrapping' technique boils down to the use of a *sequential conditional least-squares adjustment* [*Teunissen*, 1993, 1996], with a conditioning on the integer ambiguity values obtained in the previous steps. Thus if $\hat{a}_{i/I}$ denotes the (real-valued) least-squares estimate of the *i*th ambiguity, conditioned on a fixing of the previous *(i-1)* ambiguities at their sequentially integer rounded values $\breve{a}_1, \breve{a}_2, \ldots, \breve{a}_{i-1}$, the integer ambiguity vector obtained by the 'bootstrapping' technique equals

$$\breve{a}_1 = [\hat{a}_1] \quad , \quad \breve{a}_2 = [\hat{a}_{2|1}] \quad , \quad \cdots \quad , \quad \breve{a}_n = [\hat{a}_{n|1,\ldots,(n-1)}] \tag{3.3}$$

Note that this approach has some appeal in the sense that it seems to be a straightforward generalization of Eq.(3.1) and Eq.(3.2), in particular since the variance matrix of the (real-valued) sequential conditional least-squares solution is diagonal. Nevertheless, this solution should not be used in general as the solution for the integer ambiguities. The pitfall involved is namely that one cannot guarantee beforehand that solution Eq.(3.3) will indeed equal the sought for *n*-dimensional integer least-squares solution. This becomes clear for instance, once one recognizes that solution Eq.(3.3) is not invariant against a reordering of the ambiguities. Different solutions can be obtained when the order of the sequence in which the conditioning is done, is changed. For the cases where the 'bootstrapping' technique is used in actual practice, one tries to circumvent this by starting the sequence with the most precise ambiguities. In large networks for instance, this implies that first the short baselines are tackled, then the medium baselines and finally the long baselines. Although this ordering on the basis of the precision of the ambiguities may help considerably, in the sense that it increases the chance of having solution Eq.(3.3) coincide with the integer least-squares solution, the equality of the two solutions is still not guaranteed. The conclusion reads therefore that 'integer rounding' will generally fail to produce the sought for integer least-squares solution. This will particularly be true when the ambiguities are of poor precision and highly correlated.

Although it cannot be guaranteed beforehand that solution Eq. (3.3) will indeed equal the sought for integer least-squares solution, it is possible, once the 'bootstrapped' solution has been computed, to check whether it coincides with the integer least-squares solution. In fact this is possible for any integer solution. Let a^0 be such an integer solution, for instance Eq. (3.2) or Eq. (3.3) or any other integer estimate, and consider the cosine-rule-based decomposition $||\hat{a} - a^0 + \nabla||^2 = ||\hat{a} - a^0||^2 + ||\nabla||^2 + 2||\hat{a} - a^0||\,||\nabla||\cos\alpha$, with α the enclosed angle. Then a^0 will coincide with the integer least-squares solution when the sum of the last two terms is larger than or equal to zero for any nonzero integer vector ∇. This certainly will be the case when $||\nabla|| \geq 2||\hat{a} - a^0||$ and thus also when

$$1/\sqrt{\lambda_{\max}} \geq 2\left\|\hat{a} - a^0\right\| \tag{3.4}$$

where λ_{max} equals the largest eigenvalue of the ambiguity variance matrix. Hence, a check to see whether a^0 equals the integer least-squares solution is to check whether the reciprocal of the square root of the largest eigenvalue is larger than or equal to twice the distance between $\hat{a}$ and a^0. One may replace the largest eigenvalue also by the largest variance of $\hat{a}_i$ in Eq. (3.2) or by the largest (conditional) variance of $\hat{a}_{i|I}$ in Eq. (3.3), to get looser upperbounds. This follows from the sum of squares property of the sequential conditional least-squares adjustment. In both cases though, to get even looser upperbounds, it is preferred to base the computations on the variance matrix of the decorrelated ambiguities.

4. THE SECOND PITFALL

Since there is no simple method available that will produce the integer least-squares solution directly from its floated counterpart, one will have to employ a search for the integer least-squares solution. Solving Eq.(2.4) by means of a search, implies in all its simplicity, that one decides on a set of integer vectors, draws members from it and then computes the corresponding value of the objective function of Eq.(2.4). The integer vector that returns the smallest value of the objective function is then voted to be the solution sought.

First we need to decide upon a set of integer vectors, the so-called ambiguity search space, from which the integer vectors (grid points) are drawn. And here again, one should be aware of some pitfalls. The search space should be chosen such that one can guarantee that it indeed contains the solution sought. Every (nonempty) set will always contain an integer vector that minimizes the objective function of Eq.(2.4) within this set. This does not guarantee however, that it also minimizes the objective function over the whole space of integers. But this is what is required in order to solve Eq.(2.4), that is, in order to find the integer least-squares solution. Methods have been proposed in the literature, that unfortunately failed to take this into account.

We will discuss one example of a 'search space' which, by the way it is constructed, does not guarantee that it contains the sought for integer least-squares solution. The 'search space' is in this case a rectangular box, centred at the (real-valued) least squares solution $\hat{a}$, having side lengths that are based on choosing an interval per ambiguity [*Landau and Euler*, 1992]. The length of the interval can be either fixed, in which case the 'search space' becomes a box with all sides of equal length, or the interval length can be chosen to depend on the standard deviation of the ambiguity, in which case the 'search space' reads

$$(\hat{a}_1 - a_1)^2 \leq \sigma_1^2\chi^2 \quad , \quad (\hat{a}_2 - a_2)^2 \leq \sigma_2^2\chi^2 \quad , \quad \cdots \quad , \quad (\hat{a}_n - a_n)^2 \leq \sigma_n^2\chi^2 \tag{4.1}$$

where χ^2 is a positive constant. In both cases though, one cannot guarantee that the box will contain the solution sought. This is perhaps best explained if one considers what happens when χ^2 is varied. Provided the box is nonempty, every value of χ^2 will result in a grid point that minimizes the objective function of Eq.(2.4) for the box. These grid points however, will generally not be the same. Thus different solutions are obtained when varying χ^2 and none of these grid points can be guaranteed to coincide with the integer least-squares solution. Such a grid point could only be guaranteed to coincide with the integer least-squares solution, if one knew beforehand that the integer least-squares solution is indeed located inside the box. But such a mechanism is usually not built in. Checking whether the minimizer of the box happens to coincide with the integer least-squares solution is of course still possible by means of Eq. (3.4). The conclusion reads therefore, when the search is performed using the box as search space, that one always will find a grid point that minimizes the objective function of Eq.(2.4) for the box, but that this 'solution' may not be the grid point that minimizes the objective function for the whole space of integers. A similar situation happens in the FARA-approach as described in [*Frei*, 1991]. In this approach the concept of the box is refined. Instead of using only the diagonal entries of the ambiguity variance matrix, it uses the off-diagonal entries as well. In this way a 'search space' is constructed that more closely resembles the shape of the ellipsoid as it is defined by the variance matrix $Q_{\hat{a}}$. When compared with the box, this has the advantage that the 'search space' admits less unnecessary grid points. But also with this 'truncated' box, it is still not guaranteed that it contains the solution sought.

In order to ensure that one is computing the solution sought, it is preferable to use instead of the above search spaces, the search space that follows from using the objective function of Eq. (2.4) directly. It reads

$$(\hat{a}-a)^T Q_{\hat{a}}^{-1}(\hat{a}-a)=\sum_{i=1}^{n}(\hat{a}_{i|1,\cdots,(i-1)}-a_i)^2 / \sigma^2_{i|1,\cdots,(i-1)} \leq \chi^2 \tag{4.2}$$

where the decomposition of the quadratic form follows from using the sequential conditional least-squares adjustment. In order to make sure that this search space does contain the solution sought, one needs to choose an appropriate value for χ^2. Here we can make use of the integer estimates Eq. (3.2) or Eq. (3.3). By substituting either one of these integer vectors for a in the quadratic form of Eq. (4.2), one obtains a value for χ^2 that is large enough to ensure that the search space contains the integer least-squares solution. Of course, in order to avoid an abundance of grid points in the search space, χ^2 should not be too large. One should therefore work with the decorrelated ambiguities instead of with the original DD ambiguities. Experience has shown that Eq. (3.2) works quite well for single baseline models and Eq. (3.3) for network models.

Once the search space has been appropriately scaled, one can think of formulating the search bounds. Instead of Eq. (4.1), they now follow from decomposition Eq. (4.2) as

$$\begin{array}{ccc} (\hat{a}_1-a_1)^2 & \leq & \sigma_1^2\chi^2 \\ (\hat{a}_{2|1}-a_2)^2 & \leq & \sigma^2_{2|1}(\chi^2-(\hat{a}_1-a_1)^2/\sigma^2_{2|1}) \\ & \vdots & \\ (\hat{a}_{n|1,\cdots,(n-1)}-a_n)^2 & \leq & \sigma^2_{n|1,\cdots,(n-1)}(\chi^2-\sum_{j=1}^{n-1}(\hat{a}_{j|1,\cdots,(j-1)}-a_j)^2/\sigma^2_{j|1,\cdots,(j-1)}) \end{array} \tag{4.3}$$

All integer vectors that satisfy these n bounds also satisfy Eq. (4.2) and vice versa. These bounds are then sequentially searched for candidate grid points. Each time an integer vector is found which does not coincide with the integer least-squares solution, the search space can be rescaled to find an integer vector which reduces the objective function still further. More details on the search and its variations can be found in e.g. [*Teunissen, 1993*], [*de Jonge and Tiberius, 1996*].

5. THE THIRD PITFALL

In the actual practice of GPS ambiguity resolution, one not only estimates the integer ambiguities, but also validates this solution. The validation part is of importance in its own right and quite distinct from the estimation part. One will namely always be able to compute an integer least-squares solution, whether it is of poor quality or not. The question addressed by the validation part is therefore, whether the quality of the computed integer least-squares solution is such that one is also willing to accept this solution.

In many of the current approaches to ambiguity resolution, a ‘discrimination test’ is conducted to infer whether the likelihood of the integer least-squares solution - which by definition is the most likely solution - differs sufficiently from the likelihood of the second most likely integer ambiguity solution. The rationale behind this test is, if the difference between the two likelihoods is small, then insufficient ‘evidence’ is considered available to believe that the integer least-squares ambiguities may be used to compute the ‘fixed’ estimate of b from its floated counterpart.

In the literature, different formulations of the ‘discrimination test’ can be found. Most of them make use of the integer least-squares solution $\breve{a}$ and the second most likely integer solution, which we denote as $\breve{a}'$. Although most of the approaches currently in use seem to work satisfactorily in actual practice, there are unfortunately still some pitfalls involved. These pitfalls concern claims which are sometimes made about the probability distributions of the test statistics used for ‘discrimination’. As a consequence, the probability statements, for instance such as those which are related to the errors of the first and second kind, will be wrong. We will consider two examples.

One test statistic which is often used for ‘discrimination’ reads [*Rothacher and Mervart*, 1996]

$$D_1=\frac{\breve{\sigma}'^2}{\breve{\sigma}^2}=\frac{\|\hat{e}\|^2+\|\hat{a}-\breve{a}'\|^2}{\|\hat{e}\|^2+\|\hat{a}-\breve{a}\|^2} \tag{5.1}$$

It is the ratio of two a posteriori variance factors. One corresponds with the second most likely integer ambiguity vector and the other with the most likely integer ambiguity vector. Geometrically, D_1 equals the ratio of two squared distances. The denominator equals the squared distance from the sample point y to the integer least-squares solution, whereas the numerator equals the squared distance from the sample point to the second most likely solution. The norms in the ratio are taken with respect to the appropriate variance matrices and $\hat{e}$ is the least-squares residual vector that corresponds with the ‘float’ solution.

Sufficient discrimination between the two integer solutions is said to exist when the value of D_1 is large enough. Although this ‘discrimination test’ seems to work quite well in actual practice, some authors claim that the test statistic D_1 has an F-distribution. This is not true however. There are two ways of seeing this. For D_1 to have an F-distribution, the numerator and denominator need to be independent, but they are not. Another way of seeing that D_1 cannot have an

F-distribution, is to note that D_1 is always larger than one. This is simply a consequence of the way the test statistic is constructed, using the most likely and second most likely integer solutions.
A second test statistic which is sometimes used for 'discrimination', reads

$$D_2 = \frac{(\breve{a}'-\breve{a})^T Q_{\hat{a}}^{-1}(\hat{a}-\breve{a})}{\sqrt{(\breve{a}'-\breve{a})^T Q_{\hat{a}}^{-1}(\breve{a}'-\breve{a})}} \tag{5.2}$$

Geometrically, this test statistic equals the length of the orthogonal projection of $(\hat{a}-\breve{a})$ onto $(\breve{a}'-\breve{a})$. Sufficient discrimination between the two integer solutions is said to exist when the absolute value of D_2 is small enough. Again this 'discrimination test' seems to work quite well in practice. Some authors claim however that D_2 has a standard normal distribution [*Wang et al.*, 1997]. But this is not true. It is perhaps tempting to assume that D_2 has such a distribution. D_2 is namely linear in $\hat{a}$ and when $\hat{a}$ is normally distributed with mean $\breve{a}$ and variance matrix $Q_{\hat{a}}$, a normalization as given in Eq.(5.2) usually results in a variate which has a standard normal distribution. Not so however in the present case. This is again a consequence of the way the test statistic is constructed, using the most likely and second most likely integer solutions. As a result the absolute value of D_2 will always be smaller or at the most equally small as half times the distance between $\breve{a}'$ and $\breve{a}$. The test statistic will thus fail to have 'infinite tails'.

6. THE FOURTH PITFALL

The fourth and last pitfall has to do with $\breve{a}$ itself. It seems that one often still considers $\breve{a}$ to be a deterministic quantity. Evidence of this can be found for example, in the way one applies the 'classical' theory of hypothesis testing. Using this framework, one tests the significance of the constraint $a=\breve{a}$ by 'classical' means, despite the fact that the data used for the testing are the same data that have been used in computing $\breve{a}$. The assumption that $\breve{a}$ is deterministic also underlies the computation of the precision of the 'fixed' solution. The variance matrix of $\breve{b}$ is usually computed as

$$Q_{\breve{b}} = Q_{\hat{b}} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} Q_{\hat{a}\hat{b}} \tag{6.1}$$

This result follows from applying the error propagation law to Eq.(2.5), when one assumes $\breve{a}$ to be nonstochastic. However, the integer least-squares ambiguities are not nonstochastic. They have been computed from the data and since the vector of observables is assumed to be a random vector, usually normally distributed, also the integer least-squares ambiguity estimator is a random vector. We have

$$\breve{a} = F(y) \tag{6.2}$$

where $F(.)$ denotes the mapping from the continuous vector of observables to the integer vector of ambiguities. Thus when y is random, $\breve{a}$ is random as well. In fact, the integer least-squares estimator has a probability density (mass) function which is of the discrete type [*Teunissen*, 1990]. Nonzero probability masses are located at the grid points and zero probability masses everywhere else.
From Eq.(6.2) and the probability distribution of y, one can compute the probability density function of $\breve{a}$. This is a non trivial problem due to the complexity of the mapping $F(.)$. It is possible however to compute this distribution by means of a simulation. One first draws members from the multivariate normal distribution $\hat{a} \sim N(a, Q_{\hat{a}})$, where one may assume the mean to be zero. To get a single sample of $\hat{a}$, one uses a random generator to generate n independent samples from the standard normal distribution and then transforms this vector by means of the Cholesky factor of $Q_{\hat{a}}$. With each sample of $\hat{a}$ so obtained one then solves the integer least-squares problem Eq. (2.4) to get the corresponding integer least-squares sample. With this generated set of integer least-squares samples the discrete distribution is constructed. To perform these computations efficiently though, it is preferable to make use of the far more precise ambiguities that follow from the decorrelation process [*Teunissen, 1993*]. Once the discrete distribution is known, one knows how the probability masses are distributed over the grid points. Ideally one would like to have for the probability

$$p(\breve{a}=a)=1 \tag{6.3}$$

Only then will the variance matrix of Eq.(6.1) give an exact description of the precision of the 'fixed' solution. In practice though, condition Eq.(6.3) will never be reached. One will then have to be content with a somewhat smaller value of this probability. The point is however, that one should compute the probability $p(\breve{a}=a)$ and check whether it is sufficiently close to one. Only then will one have enough confidence that the integer least-squares solution coincides with the integer mean of $\hat{a}$ and that 'discrimination' is sufficient since the complement of this probability will then be sufficiently close to zero. An alternative to computing the probability $p(\breve{a}=a)$ exactly, is to make use of approximations, see e.g. [*Teunissen et al. 1996*], [*Teunissen, 1997*]. Different approaches are possible, depending on how the various characteristics of the ambiguity variance matrix are used. For instance, one may bound the ambiguity variance matrix itself or the ambiguity region for which all 'float' solutions are mapped to the same grid point. By bounding the ambiguity variance matrix from above using the largest eigenvalue, one gets

$$[2\Phi(1/2\sqrt{\lambda_{\max}}) - 1]^n \leq p(\breve{a} = a) \tag{6.4}$$

where $\Phi(x)$ equals the integral of the standard normal distribution from minus infinity to x. In a similar way one can use the smallest eigenvalue. Since the resulting interval gets smaller the smaller the elongation of the ambiguity search space is, one should base the eigenvalues on the variance matrix of the decorrelated ambiguities. Instead of the probability of the 'box' in Eq. (6.5), one can also use the fact that a^0 will coincide with the integer least-squares solution when $\|\nabla\| \geq 2\|\hat{a} - a^0\|$, and therefore take the probability over the ambiguity search space with χ^2 equal to the square of half the minimum distance between two grid points. Another approach is to approximate the probabilities that occur in the chain rule of conditional probabilities. This chain rule reads as

$$p(\breve{a} = a) = \prod_{i=1}^{n} p(\breve{a}_i = a_i | \breve{a}_1 = a_1, \cdots, \breve{a}_{i-1} = a_{i-1}) \tag{6.5}$$

For the integer least-squares solution these conditional probabilities are still difficult to compute analytically. For the 'bootstrapped' solution however, this is possible due to the diagonality of its variance matrix. By computing the probability for the 'bootstrapped' solution, using the conditional variances, an approximated from below is obtained for the sought for probability. At the expense of a poorer approximation one may also use the ambiguity variances themselves. In both cases though one should again use the decorrelated ambiguities, since otherwise the increasing number of products in Eq. (6.4) soon will deteriorate the approximation due to the generally poor precision of the DD ambiguities.

7. CONCLUSIONS

In this contribution we tried to point out some pitfalls in the process of GPS ambiguity resolution. Four pitfalls were addressed. First it was explained why (sequential) integer rounding should not be used for integer ambiguity estimation. Although the method has some appeal, it will generally not produce the sought for integer least-squares solution. This is of course not to say, that the (sequential) integer rounding has no place in the whole framework of ambiguity resolution. In fact, it plays an important role in helping to solve the integer least-squares problem efficiently [*Teunissen et al.*, 1996], [*de Jonge and Tiberius*, 1996]. Applied to the decorrelated ambiguities, it can be used to set the size of the search space and to obtain approximations of the sought for probabilities.

The second pitfall was concerned with the use of 'search spaces' for which it is unknown a priori whether they actually contain the solution sought. Unfortunately, methods have been proposed that contain this deficiency. The consequence is that one computes an integer solution, believes it to be the integer least-squares solution, while no mechanism is built in that guarantees it or checks it.

The third pitfall was related to the probability distributions of some of the test statistics that are used for ambiguity discrimination. Most of the test statistics which are used for 'discrimination' seem to work quite well in practice. The problem is however, that some authors claim that these test statistics have well known distributions like the *F*-distribution or the normal distribution. This is unfortunately not true, which implies that the corresponding qualitative statements about the associated probabilities involved are not correct.

Finally the last pitfall was related to the statistical properties of the ambiguities themselves. It was emphasized, although the ambiguities are not continuous but discrete-like, that the integer least-squares ambiguities are still random variates. It is therefore their probability mass function, approximated or obtained by simulation, which should form the basis for inferences on the precision of the estimated integer ambiguities.

8. REFERENCES

Blewitt, G. (1989): Carrier-phase ambiguity resolution for the Global Positioning System applied to geodetic baselines up to 2000 km. *Journal of Geophysical Research*, Vol. 94, No. B8, pp. 10.187-10.302.

Borre, K. (1995): *GPS i landmaelingen*, Aalborg.

Brown, R.A. (1992): Instantaneous GPS attitude determination. *IEEE Aerospace and Electronic Systems Magazine*, pp. 3-8.

Counselman, C.C., S.A. Gourevitch (1981): Miniature interferometer terminals for earth surveying: ambiguity and multipath with Global Positioning System. *IEEE Transactions on Geoscience and Remote Sensing*, Vol. GE-19, No. 4, pp. 244-252.

Dedes, G., C. Goad (1994): Real-time cm-level GPS positioning of cutting blade and earth movement equipment. In*: Proc. 1994 Nat. Tech. Meeting ION*, San Diego, California, pp. 587-594.

Dong, D., Y. Bock (1989): Global Positioning System network analysis with phase ambiguity resolution applied to crustal deformation studies in California. *J. Geophys. Res.*, 94, pp. 3949-3966.

Frei, E. (1991): *Rapid differential positioning with the Global Positioning System*. In: Geodetic and Geophysical Studies in Switzerland, Vol. 44.

Hwang, P.Y.C. (1991): Kinematic GPS for differential positioning: resolving integer ambiguities on the fly. In: *Navigation*, Vol. 38, No. 1, pp. 1-15.

Hatch, R. (1982): The synergism of GPS code and carrier measurements. In: *Proc. 3rd Int. Geod. Symp. Satellite Positioning*. Las Vegas, New Mexico, 8-12 February, Vol. 2, pp. 1213-1231.

Hofmann-Wellenhof, B., H. Lichtenegger, J. Collins (1997): *Global Positioning System: Theory and Practice*. 4th edition. Springer Verlag.

Jonge de P.J., C.C.J.M. Tiberius (1996): The LAMBDA method for integer ambiguity estimation: implementation aspects. Delft Geodetic Computing Centre *LGR Series* No. 12, Delft University of Technology.

Kleusberg, A., P.J.G. Teunissen (eds) (1996): *GPS for Geodesy*, Lecture Notes in Earth Sciences, vol. 60. Springer Heidelberg New York.

Landau, H., H-J. Euler (1992): On-the-fly ambiguity resolution for precise differential positioning. In: *Proceedings of ION GPS-92*, pp. 607-613.

Leick (1995): *GPS Satellite Surveying*, 2nd edition. John Wiley, New York.

Mervart, L. (1995): *Ambiguity resolution techniques in geodetic and geodynamic applications of the Global Positioning System*. PhD thesis, Astronomical Institute, University of Bern, Switzerland.

Parkinson, B., J.J. Spilker (eds) (1996): *GPS: Theory and Applications*, vols 1 and 2. AIAA, Washington DC.

Remondi, B.W. (1986): Performing cm-level surveys in seconds with GPS carrier phase: initial results. *Journal of Navigation*, Vol. III, The Institute of Navigation.

Rothacher, M., L. Mervart (Eds.) (1996): *Bernese GPS software version 4.0*. Astronomical Institute, University of Berne.

Teunissen, P.J.G. (1990): GPS op afstand bekeken. In: *Een halve eeuw in de goede richting*, Lustrumboek Snellius pp. 215-233.

Teunissen, P.J.G. (1993): Least-squares estimation of the integer GPS ambiguities. Invited Lecture, Section IV Theory and Methodology, IAG General Meeting, Beijing, China. Also in: Delft Geodetic Computing Centre *LGR Series* No. 6, Delft University of Technology.

Teunissen, P.J.G. (1996): GPS carrier phase ambiguity fixing concepts. Chapter 8 in: *GPS for Geodesy*. A. Kleusberg and P.J.G. Teunissen (eds). Lecture Notes in Earth Sciences, Vol. 60, Springer Verlag, pp. 263-335.

Teunissen, P.J.G., P.J. de Jonge, C.C.J.M. Tiberius (1994): On the spectrum of the GPS DD-ambiguities. In: *Proceedings of ION GPS-94*, pp. 115-124.

Teunissen, P.J.G., P.J. de Jonge, C.C.J.M. Tiberius (1996): The volume of the GPS ambiguity search space and its relevance for integer ambiguity resolution. In: *Proc. ION GPS-96*. Kansas City, USA, Sept. 17-20, pp. 889-898.

Teunissen, P.J.G. (1997a): A canonical theory for short GPS baselines. *Journal of Geodesy*, Vol. 71. Part I: the baseline precision, No. 6, pp 320-336. Part II: the ambiguity precision and correlation, No. 7, pp 389-401. Part III: the geometry of the ambiguity search space, No. 8, in press. Part IV: precision versus reliability, No. 9, in press.

Teunissen, P.J.G. (1997b): Closed form expressions for the volume of the GPS ambiguity search spaces. *Artificial Satellites*, Vol. 32, No. 1, pp 5-20.

Tiberius, C.C.J.M., P.J. de Jonge (1995): Fast positioning using the LAMBDA method. In*: Proc. DSNS-95*, Bergen, Norway, April 24-28, Paper No. 30.

Wang, J., M. Stewart, M. Tsakiri (1997): A new validation test procedure for ambiguity resolution on-the-fly. *Proceedings KIS97*, pp. 279-285.

Wubbena, G. (1989): The GPS adjustment software package - GEONAP - concepts and models. *Proceedings 5th Geodetic Symp. on Satellite Positioning*, pp. 452-461.