CLOSED FORM EXPRESSIONS FOR THE VOLUME OF THE GPS AMBIGUITY SEARCH SPACES

P.J.G. Teunissen
Delft Geodetic Computing Centre (LGR)
Faculty of Geodesy
Delft University of Technology
Thijseweg 11
2629 JA Delft, The Netherlands

Abstract

In this contribution we present closed form expressions for the volumes of the GPS ambiguity search spaces. This is done for the geometry-free model, the time-averaged model and the geometry-based model. We also discriminate between single-frequency data and dual-frequency data, and between phase-only cases and phase and code cases. Our results provide an intrinsic precision description of the least- squares ambiguities and explicit show the various factors that influence it.

1 Introduction

Integer ambiguity estimation is a prerequisite for fast, high precision GPS relative positioning. The various approaches in use for integer ambiguity estimation, can be distinguished in the way they make use of the available relative receiver-satellite geometry. The simplest approach possible, is one where one opts for dispensing with the receiver-satellite geometry. The code data are then almost directly used to determine the integer ambiguities of the observed phase data. The GPS model on which this approach is based is referred to as the *geometry- free* model. Examples can be found in [Hatch, 1982], [Euler and Goad, 1990], [Dedes and Goad, 1994], [Euler and Hatch, 1994] and [Teunissen, 1996a].

Approaches that do make use of the receiver-satellite geometry are based on the *time-averaged* model or the *geometry-based* model. The latter model is the one which is most commonly used in GPS surveying. It allows for instantaneous relative positioning, depending on whether both code and phase data or only phase data are used. Examples can be found in [Blewitt, 1989], [Frei and Beutler, 1990], [Hatch, 1991], [Teunissen, 1993] and [Tiberius and de Jonge, 1995]. The time-averaged model on the other hand, needs both phase and code data. This model, which was introduced in [Teunissen, 1996b], is based on the time average of the receiver-satellite geometry. It can be seen to be situated in between the geometry-free model and the geometry-based model. In the absence of satellite redundancy, its ambiguity results reduce to that of the geometry-free model and in the absence of a change in the receiver- satellite geometry, they reduce to that of the geometry-based model.

In this contribution we will develop, for the above mentioned three models, closed form expressions for the volumes of the ambiguity search spaces, which themselves are scaled versions of the ambiguity confidence ellipsoids. As experience has shown, the volume of the ambiguity search space is a good indicator for the number of grid points that are located within the search space. Hence, the closed form formulae give an easy way of downsizing the ambiguity search space, such that an abundance of unnecessary grid points is avoided.

Since the volume is defined through the determinant of the ambiguity variance matrix, our results also provide a precision measure of the least-squares ambiguities. In fact, since the determinant is invariant for all admissible ambiguity transformations possible, our results give a truely intrinsic precision description of the least-squares ambiguities. The closed formulae derived, clearly show the impact of the various factors that contribute to the ambiguity precision, such as: use of phase only or of phase and code; use of single-frequency data or of dual-frequency data; change in receiver-satellite geometry; number of satellites tracked and the number of observation epochs used.

This contribution is organized as follows. In section 2 we present the volume of the ambiguity search space and briefly discuss the various steps involved in integer least-squares estimation. In section 3, we start with the simplest single baseline model possible, the geometry- free model, and derive for it the determinant of the ambiguity variance matrix. This is first done for the single-frequency case and then for the dual-frequency case. In section 4 it shown how the determinant of the ambiguity variance matrix changes, when instead of the geometry-free model, the time-averaged model is considered. Finally, the geometry- based model is considered in section 5. Both for the phase-only case and for the phase and code case, it is shown how the determinant of the ambiguity variance matrix is affected by the change in the receiver-satellite geometry. The main results are summarized in section 6.

2 Ambiguity search space and its volume

2.1 Integer least-squares estimation

The linear(ized) GPS model of observation equations on which the estimation of the integer ambiguities is based, is generally of the form

$$y = Aa + Bb + e \tag{1}$$

where y is a vector of 'observed minus computed' double-differenced (DD) GPS observables, a is the vector of unknown integer ambiguities, b is a vector that includes the unknown baseline components and e is the vector that takes care of the measurement noise and remaining unmodelled effects. The matrices A and B are the appropriate design matrices.

In order to solve for the above system of equations, the least-squares principle is applied. Since the ambiguities are known to be integer, we are dealing with an *integer* least-squares problem instead of with a standard least-squares problem. The integer least-squares problem can be solved in three steps. First an ordinary least-squares solution is computed. Hence, in this step the integer constraints on the ambiguities are discarded. As a result, one obtains the *real-valued* least- squares solution and corresponding variance matrices

$$\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix} \;,\; \begin{bmatrix} Q_{\hat{a}} & Q_{\hat{a}\hat{b}} \\ Q_{\hat{b}\hat{a}} & Q_{\hat{b}} \end{bmatrix} \tag{2}$$

This solution is often referred to as the *float* solution.

In the second step, the results $\hat{a}$ and $Q_{\hat{a}}$ of the first step are used to compute the integer least-squares estimates of the ambiguities. The integer least-squares estimate of a is denoted as $\check{a}$, and it is the solution of

$$\min_{a} \, (\hat{a}-a)^T Q_{\hat{a}}^{-1}(\hat{a}-a) \;,\;\; a \text{ integer} \tag{3}$$

Once the minimizer $\check{a}$ has been found, the residual $(\hat{a}-\check{a})$ is used to adjust the float solution $\hat{b}$. This is done in the third step. As a result the *fixed* solution $\check{b}$ and its variance matrix are obtained as

$$\check{b} = \hat{b} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1}(\hat{a}-\check{a}) \;\;,\;\; Q_{\check{b}} = Q_{\hat{b}} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} Q_{\hat{a}\hat{b}} \tag{4}$$

The computations needed for the first and third step are rather straightforward and can be based on standard techniques. Not so however for the second step. Due to the integer constraints on the ambiguities, the solution of (3) must be obtained by means of a search. An efficient method for computing the integer least-squares ambiguities has been introduced in [Teunissen, 1993]. It is the Least-squares AMBiguity Decorrelation Adjustment (LAMBDA). A review of the method is given in the book [Kleusberg and Teunissen (Eds.), 1996] and implementation aspects of the method can be found in [de Jonge and Tiberius, 1996]. The method makes use of the *ambiguity search space*, which when formulated in the space of the original DD ambiguities, is defined as

$$(\hat{a}-a)^T Q_{\hat{a}}^{-1}(\hat{a}-a) \leq \chi^2 \tag{5}$$

It is centred at $\hat{a}$, its shape and orientation are governed by $Q_{\hat{a}}$ and it can be scaled by χ^2.

2.2 The volume of the ambiguity search space

The volume of the ambiguity search space is given as

$$V_n = \chi^n U_n \sqrt{\mid Q_{\hat{a}} \mid} \tag{6}$$

where n is the order of the ambiguity variance matrix $Q_{\hat{a}}$, $\mid Q_{\hat{a}} \mid$ is its determinant and U_n is the volume of the unit sphere in R^n. The volume of the unit sphere is given as $U_n = \pi^{n/2}/\Gamma(n/2+1)$, where $\Gamma(x)$ is the gamma-function. The required function values of the gamma-function can be computed using the recurrence relation $\Gamma(x+1) = x\Gamma(x)$ and the initial values $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(1) = 1$.

It is intuitively clear that the volume of the ambiguity search space and the number of grid points (points with integer coordinates) contained in it, must be related. The smaller (larger) the volume becomes, the less (more) grid points it generally will admit. It is however far less obvious that the volume itself indicates how many grid points are located within the search space. Of course, this can only be an approximation. The volume is namely independent of the location and orientation of the search space. Numerical experiments, as reported in [Teunissen et al., 1996], have shown however, that the volume is indeed an excellent indicator of the number of grid points. This implies that the volume can be used to downsize the ambiguity search space, such that an abundance of unnecessary grid points in it is avoided. This is very beneficial for the efficiency of the search.

The volume depends on the determinant of the ambiguity variance matrix. This determinant, when taken to the power of $\frac{1}{n}$, is often also referred to as the *generalized variance*. It also equals the *geometric mean* of the eigenvalues of the ambiguity variance matrix.

In the sections following we will derive closed form formulae for this determinant. An important property of the determinant is that it is invariant for any admissible ambiguity transformation. As an example consider the DD ambiguities. Since the DD ambiguities are defined with respect to a choice of reference satellite, their variance matrix is dependent on this choice as well. This makes it difficult to evaluate the precision of the ambiguities by means of their variances and covariances. They will namely change, when the choice of reference satellite is changed. The same problem appears if one would use the trace of the variance matrix as precision measure. Not so however, if one uses the determinant. The determinant is invariant for the choice of reference satellite. In fact it is invariant for any admissible ambiguity transformation possible. This is due to the fact that these transformations are volume preserving. The determinant or generalized variance provides therefore a truely intrinsic measure of the precision of the ambiguities.

3 The geometry-free model

The geometry-free model is the simplest single GPS baseline model one can think of and one, that at the same time, still allows one to determine the integer ambiguities. The model is referred to as geometry-free, since it dispenses with the receiver-satellite geometry. In this section, we will first consider the single-frequency case and then the dual-frequency case.

3.1 Single frequency case

First we consider the scalar case. In case the receiver-satellite geometry is dispensed with, the observation equations for a single DD phase observation and a single DD code observation at epoch i, read

$$\begin{array}{rcll} \phi(i) & = & \rho(i) + \lambda N & \\ p(i) & = & \rho(i) & \quad i = 1, \ldots, k. \end{array} \tag{7}$$

where $\phi(i)$ and $p(i)$ are the DD phase and code observations at epoch i, on either the L_1 or the L_2 frequency; $\rho(i)$ is the DD form of the unknown range from receiver to satellite; λ is the known wave length of the carrier and N is the unkown integer ambiguity. It will be assumed that N stays constant in time. The variance of the DD phase observable is given as $2\sigma_\phi^2$, where σ_ϕ^2 is the variance of the single-differenced phase observable. Similarly, the variance of the DD code observable is given as $2\sigma_p^2$.

Would the DD ranges $\rho(i)$ be known, then no code observations would be necessary and the variance of the least-squares ambiguity, based on k epochs of data, would read

$$\sigma_{\hat{N}}^2 = 2\frac{\sigma_\phi^2}{k\lambda^2} \tag{8}$$

Hence, due to the very high precision of the phase observations (e.g. $\sigma_\phi \approx 3\sqrt{2}\ mm$), one would then have no difficulties at all to obtain a precise and reliable integer ambiguity after only a few observation epochs. Unfortunately, the ranges are unknown. The code data are therefore needed per se. For a single epoch one would then get, $\lambda N(i) = \phi(i) - p(i)$. Taking the average over k epochs and applying the error propagation law, gives the ambiguity variance matrix of the least-squares solution as

$$\sigma^2_{\hat{N}} = 2\frac{\sigma^2_\phi}{k\lambda^2}\,(1 + \frac{\sigma^2_p}{\sigma^2_\phi}) \tag{9}$$

This variance is considerably larger in value than the previous one. This is due to the poor precision of the code observables (e.g. $\sigma_p \approx 30\sqrt{2}\ cm$). The situation improves somewhat, if we consider the phase and code data observed to different satellites, simultaneously. Instead of having two DD obervation equations per epoch as above, one would then have two times $(m-1)$ DD observation equations per epoch, with m being the number of satellites tracked. The $(m-1)$ DD phase observables are correlated, due to their DD nature, and the same holds true for the DD code observables. Collecting the $(m-1)$ DD ambiguities in the vector $a = (N_1, \ldots, N_{m-1})^T$, the variance matrix of the least-squares ambiguities reads then $Q_{\hat{a}} = \frac{1}{2}\sigma^2_{\hat{N}} D^T D$, where D^T is the DD matrix operator of order $(m-1)\times m$. The matrix $D^T D$ has two's on its diagonal and one's as off diagonal entries. Since the determinant of $D^T D$ equals m, it follows that

$$\boxed{\mid Q_{\hat{a}} \mid = m\ (\frac{\sigma^2_\phi}{k\lambda^2})^{m-1}\ (1 + \frac{\sigma^2_p}{\sigma^2_\phi})^{m-1} \qquad (\text{cycles}^{2(m-1)})} \tag{10}$$

This result reduces to that of (9), when $m = 2$. In the expression for the determinant we recognize $m(.)^{m-1}$ as a derivative. It is due to the differencing of the DD operator. When we rais the determinant to the power $\frac{1}{m-1}$ and recognize that $\sigma_\phi \ll \sigma_p$, we obtain

$$\mid Q_{\hat{a}} \mid^{\frac{1}{m-1}} \approx\ m^{\frac{1}{m-1}}\ \frac{\sigma^2_p}{k\lambda^2} \qquad (\text{cycles}^2) \tag{11}$$

This shows that the generalized ambiguity variance is governed by the poor precision of the code data and that the correlation due to DD operator helped somewhat ($m^{\frac{1}{m-1}} \leq 2$), but that one still would need quite some observation epochs (k large), to bring the variance down to an acceptable value.

3.2 Dual frequency case

In the dual-frequency case, we have four observation equations available per epoch. Two for the phase data and two for the code data. They read

$$\begin{array}{rcl} \phi_1(i) & = & \rho(i) + \lambda_1 N_1 \\ \phi_2(i) & = & \rho(i) + \lambda_2 N_2 \\ p_1(i) & = & \rho(i) \\ p_2(i) & = & \rho(i) \end{array} \tag{12}$$

The subindex indicates the frequency of the observable, L_1 or L_2. Again no code data would be necessary in case the ranges $\rho(i)$ would be known. In that case the two ambiguities can be determined independently from one another using the phase data. The variance matrix of the two least-squares ambiguities would then be diagonal. Denoting the ambiguity vector as $N = (N_1, N_2)^T$, the determinant of the ambiguity variance matrix reads then

$$\mid Q_{\hat{N}} \mid = 2^2\ (\frac{\sigma_{\phi_1}\sigma_{\phi_2}}{k\lambda_1\lambda_2})^2 \tag{13}$$

Note that we have allowed the precision of the phase data on the two frequencies to differ.

In practice the ranges $\rho(i)$ are unknown and we need the code data in addition to the phase data. For a single epoch, we then have

$$\begin{array}{rcl} \lambda_1 N_1(i) & = & \phi_1(i) - p_w(i) \\ \lambda_2 N_2(i) & = & \phi_2(i) - p_w(i) \end{array} \tag{14}$$

where $p_w(i)$ is the weighted average of $p_1(i)$ and $p_2(i)$,

$$p_w(i) = \frac{\sigma^2_{p_1} p_2(i) + \sigma^2_{p_2} p_1(i)}{\sigma^2_{p_1} + \sigma^2_{p_2}} \quad , \quad \sigma^2_{p_w} = \frac{\sigma^2_{p_1}\sigma^2_{p_2}}{\sigma^2_{p_1} + \sigma^2_{p_2}} \tag{15}$$

The least-squares solution for the ambiguities follows then from taking the time-average of (14). The ambiguity variance matrix reads then

$$Q_{\hat{N}} = \frac{2}{k} \begin{bmatrix} (\sigma^2_{\phi_1} + \sigma^2_{p_w})/\lambda_1^2 & \sigma^2_{p_w}/\lambda_1\lambda_2 \\ \sigma^2_{p_w}/\lambda_1\lambda_2 & (\sigma^2_{\phi_2} + \sigma^2_{p_w})/\lambda_2^2 \end{bmatrix} \tag{16}$$

The determinant of this matrix reads

$$\mid Q_{\hat{N}} \mid = 2^2 \, (\frac{\sigma_{\phi_1}\sigma_{\phi_2}}{k\lambda_1\lambda_2})^2 (1 + \frac{\sigma^2_{p_w}}{\sigma^2_{\phi_w}}) \tag{17}$$

A comparision with (13) shows that the determinant blows up considerably due to the fact that the ranges $\rho(i)$ are unknown. The code-variance ratio $\frac{\sigma^2_{p_w}}{\sigma^2_{\phi_w}}$ is large (e.g. 10^4), due to the poor precision of the code data when compared to the precision of the phase data.

The four equations (12) are based on the tracking of two satellites only. In case m satellites are tracked, an $(m-1)$ number of four such equations are available. The variance matrix of the least-squares ambiguities follows then as

$$Q_{\hat{a}} = \frac{1}{2} Q_{\hat{N}} \otimes D^T D \tag{18}$$

where $a = (\ldots, N_{1\alpha}, \ldots, N_{2\alpha}, \ldots)^T$, $(\alpha = 1, \ldots, (m-1))$ and '$\otimes$' denotes the Kronecker product. Since the determinant of the Kronecker product of two square matrices P and Q of order $p \times p$ and $q \times q$ respectively, is given as $\mid P \otimes Q \mid = \mid P \mid^q \mid Q \mid^p$, the determinant of (18) can be written as $\mid Q_{\hat{a}} \mid = \mid \frac{1}{2} Q_{\hat{N}} \mid^{m-1} \mid D^T D \mid^2$. Together with (17) this gives

$$\boxed{\mid Q_{\hat{a}} \mid = [m \, (\frac{\sigma_{\phi_1}\sigma_{\phi_2}}{k\lambda_1\lambda_2})^{m-1}]^2 \, (1 + \frac{\sigma^2_{p_w}}{\sigma^2_{\phi_w}})^{m-1} \qquad (\text{cycles}^{4(m-1)})} \tag{19}$$

Compare this with the single-frequency result (10). In order to obtain the generalized ambiguity variance, the determinant has to be raised to the power $\frac{1}{2(m-1)}$. If we assume the phase data to be equally precise, the code data to be equally precise and recognize that $\sigma_\phi \ll \sigma_p$, then the generalized variance can be approximated as

$$\mid Q_{\hat{a}} \mid^{\frac{1}{2(m-1)}} \approx m^{\frac{1}{m-1}} \, \frac{\sigma_\phi \sigma_p}{k\lambda_1\lambda_2} \qquad (\text{cycles}^2) \tag{20}$$

If we compare this to (11), we immediately see the important role which is played by having data on a second frequency available. Instead of the proportionality to the code-variance σ^2_p, which we had in the single-frequency case, we now have a proportionality to the product $\sigma_\phi \sigma_p$, which of course is much smaller. The impact of the number of satellites ($m^{\frac{1}{m-1}}$) and the number of observation epochs used (k), has remained the same.

4 The time-averaged model

In the geometry-free model, the observation equations are parametrized in terms of the DD ranges from receiver to satellite. Since these equations are linear from the outset, no linearization needed to take place. The observation equations however, that form the basis of this and the next section, will be parametrized in terms of the three components of the stationary baseline. These equations are nonlinear and therefore need to be linearized first. For a single epoch i, the linearized DD observation equations for phase and code read

$$\begin{array}{rcll} \phi(i) & = & A(i)b + \lambda a & \\ p(i) & = & A(i)b & \quad i = 1, \ldots, k \end{array} \tag{21}$$

Note that $\phi(i)$ and $p(i)$ are now vectors that contain, in case m satellites are tracked, respectively the $m-1$ 'observed minus computed' DD phases and the $m-1$ 'observed minus computed' DD pseudoranges. Their variance matrices are given respectively as $\sigma^2_\phi D^T D$ and $\sigma^2_p D^T D$. The design matrix $A(i)$ is of order $(m-1) \times 3$ and it captures the receiver-satellite geometry. The unknown baseline components are the entries of the vector b and the unknown ambiguities are the entries of the vector a.

In this section we will study the time-averaged version of the above model. First we consider the single-frequency case and then the dual- frequency case.

4.1 Single frequency case

The time-averaged model follows from taking the time average of the vectorial observation equations of (21). It reads

$$\begin{array}{rcl} \bar{\phi} & = & \bar{A}b + \lambda a \\ \bar{p} & = & \bar{A}b \end{array} \tag{22}$$

where $\bar{A}$ is the time average of $A(i)$, $\bar{A} = \sum_{i=1}^{k} A(i)$. The variance matrices of the time averaged observables $\bar{\phi}$ and $\bar{p}$ are given as $\frac{1}{k}\sigma_\phi^2 D^T D$ and $\frac{1}{k}\sigma_p^2 D^T D$.

The reason for studying the time-averaged model is two fold. First, the model is simpler to work with than (21). Secondly, since the GPS receiver-satellite geometry is known to change slowly with time, the results that hold true for the time-averaged model should not differ too much from the result that hold true for the model (21), at least when short observation time spans are involved, such as it is the case with fast ambiguity resolution applications.

The least-squares solution for the ambiguity vector follows from the above model as

$$\hat{a} = \frac{1}{\lambda}[\bar{\phi} - \bar{A}(\bar{A}^T(D^TD)^{-1}\bar{A})^{-1}\bar{A}^T(D^TD)^{-1}\bar{p}] \tag{23}$$

Application of the error propagation law, gives the ambiguity variance matrix as

$$Q_{\hat{a}} = \frac{\sigma_\phi^2}{k\lambda^2} D^T D[I_{m-1} + \frac{\sigma_p^2}{\sigma_\phi^2}(D^TD)^{-1}\bar{A}(\bar{A}^T(D^TD)^{-1}\bar{A})^{-1}\bar{A}^T] \tag{24}$$

Taking the determinant, gives

$$\begin{array}{rcl} \mid Q_{\hat{a}} \mid & = & \mid \frac{\sigma_\phi^2}{k\lambda^2} D^T D \mid \mid I_{m-1} + \frac{\sigma_p^2}{\sigma_\phi^2}(D^TD)^{-1}\bar{A}(\bar{A}^T(D^TD)^{-1}\bar{A})^{-1}\bar{A}^T \mid \\ & = & \mid \frac{\sigma_\phi^2}{k\lambda^2} D^T D \mid \mid I_3 + \frac{\sigma_p^2}{\sigma_\phi^2} I_3 \mid \end{array}$$

The first equality follows, since the determinant of the product of two square and full rank matrices equals the product of their determinants. For the second equality, we have made use of the fact that $\mid I + PQ \mid = \mid I + QP \mid$ for any two matrices P and Q of appropriate order. As a result we get for the determinant of the ambiguity variance matrix

$$\boxed{\mid Q_{\hat{a}} \mid = m \, (\frac{\sigma_\phi^2}{k\lambda^2})^{m-1} \, (1 + \frac{\sigma_p^2}{\sigma_\phi^2})^3 \qquad (\text{cycles}^{2(m-1)})} \tag{25}$$

Compare this result with that of (10) and note that the only difference between the two results is the power to which the large term $(1 + \frac{\sigma_p^2}{\sigma_\phi^2})$ is taken. In case of the geometry-free model the power equals $(m-1)$, whereas in case of the time-averaged model, it equals only 3. This is an important difference, since it implies for the geometry-free model that the large term gets larger when more satellites are tracked, whereas it stays constant for the time-averaged model. The difference in power between the two expressions, is due to the fact that the time-averaged model is based on a three dimensional baseline which stays stationary during the observation time span, whereas the receiver-satellite ranges $\rho(i)$ of the geometry-free model are not linked in time and thus change from epoch to epoch. The two results become identical when $m = 4$, that is, when there is no satellite redundancy available.

If we rais (25) to the power of $\frac{1}{m-1}$ and recognize that $\sigma_\phi \ll \sigma_p$, the generalized ambiguity variance can be approximated as

$$\mid Q_{\hat{a}} \mid^{\frac{1}{m-1}} \approx m^{\frac{1}{m-1}} \, \frac{\sigma_p^2}{k\lambda^2} \, (\frac{\sigma_\phi^2}{\sigma_p^2})^{\frac{m-4}{m-1}} \qquad (\text{cycles}^2) \tag{26}$$

Compare this result with that of (11). We now clearly see the benefit of having more than four satellites available in case of the time-averaged model. The term $m^{\frac{1}{m-1}} \, (\sigma_\phi^2/\sigma_p^2)^{\frac{m-4}{m-1}}$ decreases much faster when m increases $(m > 4)$, than the term $m^{\frac{1}{m-1}}$, which we had for the geometry-free model.

4.2 Dual frequency case

When the data are available on both of the frequencies, the time- averaged model reads

$$\begin{array}{rcl} \bar{\phi}_1 &=& \bar{A}b + \lambda_1 a_1 \\ \bar{\phi}_2 &=& \bar{A}b + \lambda_2 a_2 \\ \bar{p}_1 &=& \bar{A}b \\ \bar{p}_2 &=& \bar{A}b \end{array} \tag{27}$$

The least-squares solution for the ambiguities reads then

$$\begin{array}{rcl} \hat{a}_1 &=& \frac{1}{\lambda_1}[\bar{\phi}_1 - \bar{A}(\bar{A}^T(D^TD)^{-1}\bar{A})^{-1}\bar{A}^T(D^TD)^{-1}\bar{p}_w] \\ \hat{a}_2 &=& \frac{1}{\lambda_2}[\bar{\phi}_2 - \bar{A}(\bar{A}^T(D^TD)^{-1}\bar{A})^{-1}\bar{A}^T(D^TD)^{-1}\bar{p}_w] \end{array} \tag{28}$$

where $\bar{p}_w$ is the time-average of the weighted pseudoranges $p_w(i)$. Application of the error propagation law gives

$$Q_{\hat{a}} = Q_1 \otimes D^TD + Q_2 \otimes \bar{A}(\bar{A}^T(D^TD)^{-1}\bar{A})^{-1}\bar{A}^T \tag{29}$$

with

$$Q_1 = \frac{1}{k}\,\mathrm{diag}(\frac{\sigma^2_{\phi_1}}{\lambda_1^2}, \frac{\sigma^2_{\phi_2}}{\lambda_2^2}) \ , \ Q_2 = \frac{1}{k}\sigma^2_{p_w}\begin{bmatrix} 1/\lambda_1 \\ 1/\lambda_2 \end{bmatrix}\begin{bmatrix} 1/\lambda_1 \\ 1/\lambda_2 \end{bmatrix}^T$$

For the determinant of the ambiguity variance matrix, we get

$$\begin{array}{rcl} \mid Q_{\hat{a}} \mid &=& \mid Q_1 \otimes D^TD \mid \mid I_2 \otimes I_{m-1} + [Q_1^{-1}Q_2 \otimes (D^TD)^{-1}\bar{A}(\bar{A}^T(D^TD)^{-1}\bar{A})^{-1}][I_2 \otimes \bar{A}^T] \mid \\ &=& \mid Q_1 \otimes D^TD \mid \mid I_2 \otimes I_3 + Q_1^{-1}Q_2 \otimes I_3 \mid \\ &=& \mid Q_1 \mid^{m-1} \mid D^TD \mid^2 \mid I_2 + Q_1^{-1}Q_2 \mid^3 \end{array}$$

or

$$\boxed{\mid Q_{\hat{a}} \mid = [m(\frac{\sigma_{\phi_1}\sigma_{\phi_2}}{k\lambda_1\lambda_2})^{m-1}]^2\ (1 + \frac{\sigma^2_{p_w}}{\sigma^2_{\phi_w}})^3 \qquad (\mathrm{cycles}^{4(m-1)})} \tag{30}$$

Compare this result with (19) of the geometry-free model, to see the improvement obtained with the time-averaged model and compare it to (25) to see the impact of using data on the second frequency. If we assume the phase data to be equally precise, the code data to be equally precise and recognize that $\sigma_\phi \ll \sigma_p$, then the generalized ambiguity variance can be approximated as

$$\mid Q_{\hat{a}} \mid^{\frac{1}{2(m-1)}} \approx\ m^{\frac{1}{m-1}}\ \frac{\sigma_\phi\sigma_p}{k\lambda_1\lambda_2}\ (\frac{\sigma^2_\phi}{\sigma^2_p})^{\frac{m-4}{2(m-1)}} \qquad (\mathrm{cycles}^2) \tag{31}$$

5 The geometry-based model

The geometry-based model is the most commonly used model in GPS surveying. When dual frequency data are available, it reads

$$\begin{array}{rcll} \phi_1(i) &=& A(i)b + \lambda_1 a_1 & \\ \phi_2(i) &=& A(i)b + \lambda_2 a_2 & \\ p_1(i) &=& A(i)b & \\ p_2(i) &=& A(i)b & \qquad i = 1, \ldots, k \end{array} \tag{32}$$

One can expect that the change in receiver-satellite geometry will now also have an impact on the precision of the least-squares ambiguities. In the geometry-free model this change was absent per se, since the receiver-satellite geometry was dispensed with and in the time-averaged model it was neutralized, because of the time-averaging of the observation equations. Due to the change in receiver-satellite geometry, one will now also be able to solve for the ambiguities when using phase data only. Hence, code data are now not needed per se. We therefore first consider the phase-only case and then the phase and code case. But before doing so, we will first derive a relation between the determinant of the ambiguity variance matrix and the determinant of the variance matrix of the baseline.

When we assume the baseline to be known, the variance matrix of the ambiguities becomes a conditional variance matrix, which reads as

$$Q_{\hat{a}|b} = Q_{\hat{a}} - Q_{\hat{a}\hat{b}}Q_{\hat{b}}^{-1}Q_{\hat{b}\hat{a}}$$

Taking the determinant, after post multiplying with the inverse of the ambiguity variance matrix, gives

$$\begin{aligned} \mid Q_{\hat{a}|b} Q_{\hat{a}}^{-1} \mid &= \mid I - Q_{\hat{a}\hat{b}} Q_{\hat{b}}^{-1} . Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} \mid \\ &= \mid I - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} Q_{\hat{a}\hat{b}} Q_{\hat{b}}^{-1} \mid \\ &= \mid Q_{\hat{b}} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} Q_{\hat{a}\hat{b}} \mid\mid Q_{\hat{b}}^{-1} \mid \end{aligned}$$

In the second equality, we have again made use of the fact that $\mid I + PQ \mid = \mid I + QP \mid$ for any two matrices P and Q of appropriate order. In the last equation, we recognize the conditional variance matrix $Q_{\hat{b}|a}$, which is the variance matrix of the fixed baseline. As a result, we have the determinantal relation

$$\mid Q_{\hat{a}} \mid = \mid Q_{\hat{a}|b} \mid \; \frac{\mid Q_{\hat{b}} \mid}{\mid Q_{\hat{b}|a} \mid} \tag{33}$$

This equation, which was already introduced in [Teunissen, 1995], is of interest in its own right. The ratio of determinants of the two baseline variance matrices, the one of the floated solution and the one of the fixed solution, can be seen to measure the improvement in baseline precision which is reached through the fixing of the ambiguities. Equation (33) thus shows, through the determinants of the variance matrices, how the precision of the ambiguities is related to this gain in baseline precision. A higher gain in baseline precision corresponds with a poorer precision of the ambiguities and a lower gain corresponds with a better precision of the ambiguities. The determinantal relation (33) can also be used to show how the volume of the ambiguity confidence ellipsoid is related to the ratio of the volumes of the two baseline confidence ellipsoids.

5.1 Phase-only case

First we will consider the single-frequency case. That is, we will first base our results on either the first set of observation equations or on the second set of observation equations of (32). We will skip the index, showing whether it is L_1 or L_2 that we are dealing with. When we assume the baseline vector to be known, the conditional variance matrix of the least-squares ambiguities, follows as $Q_{\hat{a}|b} = \frac{\sigma_\phi^2}{k\lambda^2} D^T D$. Hence, its determinant reads

$$\mid Q_{\hat{a}|b} \mid = m \; (\frac{\sigma_\phi^2}{k\lambda^2})^{m-1} \tag{34}$$

In order to determine the second part of (33), which is the determinant ratio of the baseline variance matrices, we consider the generalized eigenvalue problem

$$\mid Q_{\hat{b}} - \gamma Q_{\hat{b}|a} \mid = 0 \tag{35}$$

The three eigenvalues γ_i, $i = 1, 2, 3$, are known as the *gain numbers*. They measure the gain in baseline precision that is experienced when one fixes the ambiguities. The gain numbers lie in the interval $[1, \infty)$. There is no gain in baseline precision, when the gain numbers reach their minimum value of one. In that case the fixing of the ambiguities would not result in an improved baseline, precision wise. The gain numbers increase in value, when the observation time span gets shorter and they become infinite, when the observation time span is zero, that is, when only one observation epoch is used. In that case of course, when use is made of phase data only, also no floated baseline solution exists. The gain numbers can be shown to follow an *inverse square law* in the observation time span T. Thus $\gamma_i \sim \frac{1}{T^2}$.

Apart from measuring the gain in baseline precision, the gain numbers γ_i also measure in the phase-only case, the *change* the receiver-satellite geometry undergoes when time progresses. The gain numbers can be shown to be related to the *principal angles* of the receiver-satellite geometry as $\gamma_i = \frac{1}{\cos^2(\omega_i)}$. The canonical theory for single GPS baselines, which is based on the gain number concept, has been introduced and developed in [Teunissen, 1996b]. Since the gain numbers are the solution of the characteristic equation (35), it follows that

$$\frac{\mid Q_{\hat{b}} \mid}{\mid Q_{\hat{b}|a} \mid} = \prod_{i=1}^{3} \gamma_i \tag{36}$$

Substitution of (34) and (36) into (33), gives

$$\boxed{\mid Q_{\hat{a}} \mid = m \; (\frac{\sigma_\phi^2}{k\lambda^2})^{m-1} \prod_{i=1}^{3} \gamma_i \qquad (\text{cycles}^{2(m-1)})} \tag{37}$$

Note that this result indeed shows that the ambiguities can not be solved for, when only a single observation epoch is used. In that case, the gain numbers become infinite and so does the determinant of the ambiguity variance matrix. The result also shows the impact of the receiver-satellite geometry. As the observation time span gets longer, the gain numbers get smaller and hence, the precision of the least- squares ambiguities gets better. When we compare (37) with (25), we note that the two determinants only differ in their factors $(1+\frac{\sigma_p^2}{\sigma_\phi^2})^3$ and $\prod_{i=1}^3 \gamma_i$. This shows that the role of the code data, which is needed per se in the time-averaged model, is taken over by the change in the receiver-satellite geometry, in case of the phase-only geometry-based model.

Let us now consider the dual-frequency case. The results are now based on using both sets of phase observation equations of (32). The determinantal ratio of the baseline variance matrices remains unchanged. Only the determinant of the conditional ambiguity variance matrix changes. Instead of (34), we now have

$$\mid Q_{\hat{a}|b} \mid = [m(\frac{\sigma_{\phi_1}\sigma_{\phi_2}}{k\lambda_1\lambda_2})^{m-1}]^2 \tag{38}$$

In the dual-frequency, phase-only case, the determinant of the ambiguity variance matrix thus becomes

$$\boxed{\mid Q_{\hat{a}} \mid = [m(\frac{\sigma_{\phi_1}\sigma_{\phi_2}}{k\lambda_1\lambda_2})^{m-1}]^2 \prod_{i=1}^{3} \gamma_i \qquad (\text{cycles}^{4(m-1)})} \tag{39}$$

Compare this result with the dual-frequency result (30) of the time-averaged model.

5.2 Phase and code case

Again we start with the single-frequency case. Thus now we are using either the first and third set, or, the second and fourth set of observation equations of (32). The conditional variance matrix of the ambiguities is in this case identical to the one of the single-frequency phase-only case. The determinantal ratio of the baseline variance matrices, changes however. It can still be expressed in the gain numbers γ_i however, if we relate the floated and fixed baseline variance matrices to their counterparts of the single-frequency phase-only case. They are related as follows

$$Q_{\hat{b}} = [Q_{\hat{b}}(p)^{-1} + Q_{\hat{b}}(\phi)^{-1}]^{-1} \ , \ Q_{\hat{b}|a} = [Q_{\hat{b}}(p)^{-1} + Q_{\hat{b}|a}(\phi)^{-1}]^{-1} \ , \ Q_{\hat{b}}(p) = \frac{\sigma_p^2}{\sigma_\phi^2} Q_{\hat{b}|a}(\phi)$$

where the argument indicates on which type of data they are based. Substitution of these expressions into the characteristic equation $\mid Q_{\hat{b}} - \delta Q_{\hat{b}|a} \mid = 0$, allows one to relate δ to γ,

$$\delta = \frac{(1+\sigma_\phi^2/\sigma_p^2)\gamma}{1+(\sigma_\phi^2/\sigma_p^2)\gamma} \tag{40}$$

This result shows by how much the gain in baseline precision changes, when single-frequency code data are added to the single-frequency phase data. Note that the gain in baseline precision when code data are used, is smaller than when only phase data are used, $\delta \leq \gamma$. They become identical of course, when σ_p goes to infinity.

Based on (40), the determinant of the ambiguity variance matrix becomes now

$$\boxed{\mid Q_{\hat{a}} \mid = m\ (\frac{\sigma_\phi^2}{k\lambda^2})^{m-1}\ (1+\frac{\sigma_p^2}{\sigma_\phi^2})^3 \prod_{i=1}^{3} \frac{(\sigma_\phi^2/\sigma_p^2)\gamma_i}{1+(\sigma_\phi^2/\sigma_p^2)\gamma_i} \qquad (\text{cycles}^{2(m-1)})} \tag{41}$$

Compare this result with (25). Note that the two results become identical when the gain numbers γ_i go to infinity. This is understandable, since the gain numbers go to infinity when only one observation epoch is used. In that case, there is no change in receiver- satellite geometry, just like there is no change in receiver-satellite geometry taken into account in the time-averaged model.

Let us now consider the dual-frequency case. Thus now we are using all of the four sets of observation equations of (32). The conditional variance matrix of the ambiguities is in this case identical to the one of the dual-frequency phase-only case. In order to determine the determinantal ratio of the two baseline variance matrices, we again express them in terms of their counterparts of the single-frequency, phase-only case,

$$\begin{aligned} Q_{\hat{b}} &= [Q_{\hat{b}}(p_1,p_2)^{-1} + Q_{\hat{b}}(\phi_1,\phi_2)^{-1}]^{-1}\,, \quad & Q_{\hat{b}}(p_1,p_2) &= \tfrac{\sigma^2_{p_w}}{\sigma^2_{\phi_1}} Q_{\hat{b}|a}(\phi_1)\,, \quad Q_{\hat{b}}(\phi_1,\phi_2) = \tfrac{\sigma^2_{\phi_w}}{\sigma^2_{\phi_1}} Q_{\hat{b}}(\phi_1) \\ Q_{\hat{b}|a} &= [Q_{\hat{b}}(p_1,p_2)^{-1} + Q_{\hat{b}|a}(\phi_1,\phi_2)^{-1}]^{-1}\,, \quad & Q_{\hat{b}|a}(\phi_1,\phi_2) &= \tfrac{\sigma^2_{\phi_w}}{\sigma^2_{\phi_1}} Q_{\hat{b}|a}(\phi_1) \end{aligned}$$

Substitution of these expressions into $\mid Q_{\hat{b}} - \epsilon Q_{\hat{b}|a} \mid = 0$, allows one to relate ϵ to γ,

$$\epsilon = \frac{(1+\sigma^2_{\phi_w}/\sigma^2_{p_w})\gamma}{1+(\sigma^2_{\phi_w}/\sigma^2_{p_w})\gamma} \tag{42}$$

Compare this with (40). This result shows by how much the gain in baseline precision changes, when dual-frequency phase and code data are used instead of only single-frequency phase and code data. The two expressions become identical in case the precision of the L_1 data is equal to that of the L_2 data. Based on (42), the determinant of the ambiguity variance matrix becomes now

$$\boxed{\mid Q_{\hat{a}} \mid = [m(\frac{\sigma_{\phi_1}\sigma_{\phi_2}}{k\lambda_1\lambda_2})^{m-1}]^2\ (1+\frac{\sigma^2_{p_w}}{\sigma^2_{\phi_w}})^3 \prod_{i=1}^{3} \frac{(\sigma^2_{\phi_w}/\sigma^2_{p_w})\gamma_i}{1+(\sigma^2_{\phi_w}/\sigma^2_{p_w})\gamma_i} \qquad (\text{cycles}^{4(m-1)})} \tag{43}$$

Compare this result with its counterpart (30) of the time- averaged model. Note that it clearly shows the presence and impact of all relevant parameters. It shows the number of satellites tracked (m), the number of observation epochs used (k) and the change in the receiver-satellite geometry which is experienced, (γ_i). And through the square of the first term in (43), it shows the impact of using dual-frequency data, and through the third power of the second term in (43), it shows the impact of having a stationary baseline.

6 Summary

In this contribution we have derived closed form expressions for the determinants of the ambiguity variance matrices. This has been done for the three single baseline models: the geometry-free model, the time- averaged model and the geometry-based model. In order to have an easy reference, the results are summarized in table 1. These closed form expressions give an indepth understanding of the various factors that contribute to the determination of the precision of the least-squares ambiguities. These factors are: (1) *data redundancy*, i.e. the use of single-frequency data versus dual-frequency data, and the use of phase- only data or phase and code data; (2) *satellite redundancy*, it equals $(m-2)$ in case of the geometry-free model and $(m-4)$ in case of the time-averaged model and the geometry-based model; (3) *number of observation epochs* k; and (4) *change in receiver-satellite geometry*, which is measured through the gain numbers γ_i.

Since the determinant is invariant against volume preserving transformations, the determinants of the ambiguity variance matrices are also invariant to any admissible ambiguity transformation possible. The closed form expressions derived, therefore also hold true for the decorrelated ambiguities which are obtained with the LAMBDA method. Hence, instead of the trace for instance, the given expressions for the determinants truely measure the intrinsic precision of the ambiguities.

The closed form expressions also provide for an easy way of computing the volume of the various ambiguity search spaces. And since the volume is a good indicator for the number of grid points that are located in the ambiguity search space, it can be used to down size the search space, such that an abundance of unnecessary grid points is avoided.

Acknowledgement

This work was finalized during the author's sabattical stay at the University of New Brunswick, Department of Geodesy and Geomatics Engineering (Fredericton, Canada), with professor A. Kleusberg as his host.

$\mid Q_{\hat{a}} \mid$	GEOMETRY-FREE ($m \geq 2$)	TIME-AVERAGED ($m \geq 4$)	GEOMETRY-BASED ($m \geq 4$)
Φ_1			$m(\frac{\sigma^2_{\phi_1}}{k\lambda^2_1})^{m-1} \prod_{i=1}^3 \gamma_i$
Φ_1/Φ_2			$[m(\frac{\sigma_{\phi_1}\sigma_{\phi_2}}{k\lambda_1\lambda_2})^{m-1}]^2 \prod_{i=1}^3 \gamma_i$
Φ_1/P_1	$m(\frac{\sigma^2_{\phi_1}}{k\lambda^2_1})^{m-1}(1+\frac{1}{\epsilon_1})^{m-1}$	$m(\frac{\sigma^2_{\phi_1}}{k\lambda^2_1})^{m-1}(1+\frac{1}{\epsilon_1})^3$	$m(\frac{\sigma^2_{\phi_1}}{k\lambda^2_1})^{m-1}(1+\frac{1}{\epsilon_1})^3 \prod_{i=1}^3 \frac{\epsilon_1\gamma_i}{\epsilon_1\gamma_i+1}$
$\Phi_1/\Phi_2/P_1/P_2$	$[m(\frac{\sigma_{\phi_1}\sigma_{\phi_2}}{k\lambda_1\lambda_2})^{m-1}]^2(1+\frac{1}{\epsilon_{12}})^{m-1}$	$[m(\frac{\sigma_{\phi_1}\sigma_{\phi_2}}{k\lambda_1\lambda_2})^{m-1}]^2(1+\frac{1}{\epsilon_{12}})^3$	$[m(\frac{\sigma_{\phi_1}\sigma_{\phi_2}}{k\lambda_1\lambda_2})^{m-1}]^2(1+\frac{1}{\epsilon_{12}})^3 \prod_{i=1}^3 \frac{\epsilon_{12}\gamma_i}{\epsilon_{12}\gamma_i+1}$

Phase-code variance ratio's for L_1 and L_1/L_2 : $\epsilon_1 = \frac{\sigma^2_{\phi_1}}{\sigma^2_{p_1}}$ $\epsilon_{12} = \frac{\sigma^2_{\phi_1}\sigma^2_{\phi_2}/(\sigma^2_{\phi_1}+\sigma^2_{\phi_2})}{\sigma^2_{p_1}\sigma^2_{p_2}/(\sigma^2_{p_1}+\sigma^2_{p_2})}$	Gain in baseline precision (phase-only case) : $\mid Q_{\hat{b}} - \gamma_i Q_{\check{b}} \mid = 0$

Table 1: The determinants of the single and dual frequency ambiguity variance matrices, for the geometry-free model, the time-averaged model and the geometry-based model.

7 References

Blewitt, G. (1989): Carrier phase ambiguity resolution for the Global Positioning System applied to geodetic baselines up to 2000 km. *Journal of Geophysical Research*, Vol. 94, No. B8, pp. 12.187-12.203.

Dedes, G., C. Goad (1994): Real-Time cm-level GPS posiyioning of Cutting Blade and Earth Movement Equipment, *Proceedings of the 1994 National Technical Meeting ION*, San Diego, California, pp. 587-594.

Euler, H.-J., C. Goad (1990): On optimal filtering of GPS dual frequency observations without using orbit information. *Bulletin Geodesique*, Vol. 65, pp. 130-143.

Euler, H.-J., R. Hatch (1994): Comparison of several AROF kinematic techniques. *Proceedings ION-94.* pp. 363-370.

Frei, E., G. Beutler (1990): Rapid static positioning based on the Fast Ambiguity Resolution Approach FARA: Theory and first results. *Manuscripta Geodaetica*, 15:325-356.

Hatch, R. (1982): The synergism of GPS code and carrier phase measurements. *Proceedings 3th International Geod. Symp. on Satellite Positioning*, Las Vegas, New Mexico, 8-12 February 1982, Vol. 2, pp. 1213-1231.

Hatch, R. (1994): Instantaneous ambiguity resolution. In: *Kinematic Systems in Geodesy, Surveying and Remote Sensing*, IAG Symposia 127, Springer Verlag, New York, pp. 299-308.

Jonge, de P.J., C.C.J.M. Tiberius (1996): The LAMBDA method for integer ambiguity estimation: implementation aspects. In: *Delft Geodetic Computing Centre LGR Series*, No. 12, 45 pp.

Kleusberg, A., P.J.G. Teunissen (Eds.) (1996): *GPS for Geodesy.* Lecture Notes in Earth Sciences, Vol. 60, Springer Verlag.

Teunissen, P.J.G. (1993): *Least-squares estimation of the integer GPS ambiguities.* Invited lecture, Section IV Theory and Methodology, IAG General Meeting, Beijing, China, August 1993. Also in *Delft Geodetic Computing Centre LGR Series*, No. 6, 16 pp.

Teunissen, P.J.G. (1995): The least-squares ambiguity decorrelation adjustment: a method for fast GPS integer ambiguity estimation. *Journal of Geodesy*, Vol. 70, No. 1-2, pp. 65-82.

Teunissen, P.J.G. (1996a): An analytical study of ammbiguity decorrelation using dual frequency code and carrier phase. *Journal of Geodesy*, 70:515-528.

Teunissen, P.J.G. (1996b): A canonical theory for short GPS baselines, parts I-IV, submitted to *Journal of Geodesy*.

Teunissen, P.J.G., P.J. de Jonge and C.C.J.M. Tiberius (1996): The volume of the GPS ambiguity search space and its relevance for integer ambiguity resolution. *Proceedings ION-GPS'96*, 10 pp.

Tiberius, C.C.J.M., P.J. de Jonge (1995): Fast positioning using the LAMBDA-method. *Proceedings of DSNS'95*, Bergen, Norway, April 24- 28, Paper 30, 8 pp.