A New Way to Fix Carrier-Phase Ambiguities

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GPS double-difference, carrier-phase measurements are ambiguous by an unknown integer number of cycles. High-precision, relative GPS positions can be obtained from a short time span of data (from seconds to a few minutes) if the integer double-difference ambiguities can be determined efficiently and reliably. We have developed a procedure — the Least-squares Ambiguity Decorrelation Adjustment (Lambda) method — that can quickly and accurately estimate the integer ambiguities. In this article, we review our method and its underlying principles and also present some numerical results illustrating its performance.

Why Fix Ambiguities?

High-precision, relative GPS positioning is based on the least-squares adjustment of precise carrier-phase measurements. With short observation time spans, however, if the ambiguities are treated as real-valued numbers (or floating-point numbers in computer parlance), they are difficult to separate from the receiver baseline components. That is due to the very high-altitude orbits of the GPS satellites, which results in the relative positions of the satellites with respect to the receivers changing very slowly.

A least-squares adjustment that ignores the intrinsic integer nature of the ambiguities would, therefore, produce highly correlated and imprecise ambiguity and baseline estimates. To increase baseline precision, particularly for short observation times spans, we can fix the ambiguities at their integer values; this enables us to treat the carrier-phase measurements as essentially pseudorange measurements. As a result, we can estimate the baseline coordinates with much higher precision as the carrier-phase measurements possess.

Integer Least Squares

To fix the ambiguities at their correct integer values, we first need a criterion that determines which integer the correct ones. We generally assume that we have obtained the most-likely real-value ambiguities from a least-squares adjustment of the result of which is often called the fix solution. It seems reasonable to consider integers that are nearest to the real-valued estimates as most likely being the correct integer values.

As a measure of nearness, we take the weighted sum of squared difference between the real-valued estimates and their integer counterparts. The weighting takes care of the existing correlation and variance precision of the real-valued ambiguity estimates.

\[ \chi^2(a) = (a - \hat{a})^T Q_\alpha^{-1} (a - \hat{a}) \in \mathbb{R}^+ \]

In Equation 1, the vector \(a\) and matrix \(Q_\alpha\) are determined by the float solution. The integer vector \(a\), however, is unknown. The most likely integer ambiguity vector is the vector \(\hat{a}\) that minimizes the value of \(\chi^2(a)\). We will denote it as \(\Delta\).

Because the minimization of \(\chi^2(a)\) amounts to a minimization of a sum of squares over the set of integers, we will refer to the solution as the integer least-squares estimate of the ambiguities.

\[ \Delta (a) = (a - \hat{a})^T Q_\alpha^{-1} (a - \hat{a}) \]
space of integers \( Z^2 \) with a smaller subset that still contains the solution. For the subset, we take all integer vectors \( \mathbf{a} \) that satisfy the inequality:

\[
(\mathbf{a} - \mathbf{d})^T \mathbf{Q}_a (\mathbf{a} - \mathbf{d}) \leq \chi^2
\]

in which \( \chi^2 \) is a suitably chosen positive con-
tant that ensures that the subset contains at
least one integer vector \( \mathbf{a} \).

Geometrically, the inequality in Equation 2 describes an \( n \)-dimensional hyperellip-
ipsoid region centered on \( \mathbf{d} \). We will refer to this hyperellipsoid (or just ellipsoid for short) region as the ambiguity search space.

Its orientation (rotation with respect to the
grid axes) and elongation (ratio of the largest
axis length to the smallest axis length) are governed by \( \mathbf{Q}_a \), and its size is controlled by the value of \( \chi^2 \). Figure 1 shows a two-dimen-
sional view of the ambiguity search space. As
the grid spacing in the figure equals one
cycle, the admissible locations for the integer
vector \( \mathbf{a} \) are given by the grid intersection
inside the ellipse.

To determine \( \mathbf{d} \), we must perform a search
through the ellipsoidal region. Different
search procedures are possible and have been
implemented in analysis software. Unfortu-
nately, they are all inefficient when applied to
rotated and extremely elongated search spaces — spaces that are typical for GPS
double-differenced, carrier-phase data from
short observation sessions. For example, for
dual-frequency data collected over a 1-sec-
ond observation time span, an elongation of the
order of \( 3 \times 10^3 \) is not uncommon.

Therefore, if the minor axis of the search
space is \( 1 \) centimeter long, its major axis
would be 300 meters long!

THE IDEAL SITUATION

To understand how we can lighten the burden of the search, it helps if we first ask ourselves the question. What should the structure of \( \chi^2(\mathbf{a}) \) be to make the search as efficient as possible? Clearly, the search becomes trivial when all ambiguities are fully decorrelated.

In that case, the variance–covariance matrix
of the ambiguities, \( \mathbf{Q}_a \), is diagonal, and \( \chi^2(\mathbf{a}) \) reduces to a sum of independent squares. That implies that we can find the minimum of \( \chi^2(\mathbf{a}) \) by minimizing each of the \( n \) individual squares in \( \chi^2(\mathbf{a}) \) separately. Therefore, the integer least-squares solution follows simply
from rounding the individual, real-valued
ambiguity estimates to their nearest integers.

A diagonal matrix \( \mathbf{Q}_a \) also implies that the
axes of the ambiguity search space are
aligned with the grid axes. One way we can

![Figure 1. In the simplified two-dimensional case, the ambiguity search space is an ellipse centered on the real-valued estimates of the ambiguity \( \mathbf{d} \). The grid spacing is one carrier cycle.](image1)

![Figure 2. If the search space is rotated so that the axes of the search space are parallel to the grid axes, the ambiguities will be fully decorrelated. However, the integer nature of the ambiguities is destroyed in the process.](image2)

achieve that alignment is by rotating the search
space (see Figure 2). Computationally, that would correspond to what is known as an eigenvalue decomposition of \( \mathbf{Q}_a \), using the matrix of normalized eigenvectors as the rotation matrix. Unfortunately, such a rotation destroys the integer nature of the trans-
formed ambiguities and cannot be used here.

DECORRELATED AMBIGUITIES

Instead of using a rotation of the search
space, we can also achieve a full decorrela-
tion of the ambiguities by squeezing the
search space along the grid axes. Consider
the two-dimensional ambiguity search space of original ambiguities \( \alpha_1 \) and \( \alpha_2 \). This ellipse will be elongated, and its principal axes will not coincide with the grid axes (see Figure 3a). But by pushing the two horizontal tangents of the ellipse inward, while at the same time keeping the area of the ellipse and its two vertical tangents fixed, we will end up with an ellipse that is perfectly aligned with the grid axes. The transformed ambiguities

![Figure 3. Full decorrelation of the ambiguities can also be achieved by pushing tangents. In (a), the two horizontal tangents are pushed inward; in (b), the two vertical tangents are pushed inward. This technique does not preserve the integer nature of the ambiguities.](image3)
the approach of pushing tangents, as depicted in Figure 3. To maintain the integer nature of the ambiguities, the tangents are not pushed to the limit. Instead, the vertical tangents are pushed upward to a position that guarantees the integer nature of the transformed ambiguities (see Figure 4a). As a result of this transformation, we obtain a one-elongated search space and two transformed ambiguity estimates $f_1$ and $d_2$ that are less correlated.

Now that $a_1$ has been replaced by $t_1$ through the pushing of vertical tangles, we can continue in an analogous way and replace $a_2$ with $t_2$ by pushing the horizontal tangents (see Figure 4b). As a result, we obtain the transformed ambiguity estimates $t_1$ and $t_2$, which are much less correlated than the original ambiguity estimates $d_1$ and $d_2$ and also have an ambiguity search space that is more spatially.

For the two-dimensional example shown in Figure 4, we have also given the variance-covariance matrix of the ambiguities before and after the transformation, $Q_1$ and $Q_2$. Note the improvement in precision (given by the standard deviation, $\sigma$) and the decrease in both correlation (given by the correlation coefficient, $\rho$) and elongation (given by the ratio of the lengths of the largest to the smallest axes of the ellipse, $a$). The elongation has been pushed toward its minimum value of 1.0. With the new ambiguity estimates $t_1$ and $t_2$ and their variance-covariance matrix $Q_2$, we now perform the ambiguity search much more efficiently.

THE n-DIMENSIONAL CASE

When the aforementioned principles are generalized to the n-dimensional case, the correlation of the least-squares ambiguities results in the following n-by-n transformation from the original ambiguity vector $\mathbf{d}$ to the new ambiguity vector $\mathbf{t}$:

$$ \mathbf{t} = Z \mathbf{d} $$

The variance-covariance matrix of the transformed ambiguities follows from the application of the error-propagation law to Equation 3, resulting in:

$$ Q_2 = Z^T Q_1 Z $$

As a consequence, the original search space, as presented by Equation 2 is replaced by the transformed search space:

$$ (\mathbf{x} - \mathbf{d})^T Q_1^{-1} (\mathbf{x} - \mathbf{d}) \approx \chi^2 $$

We then use the search space represented by Equation 5 to search for the integer least-squares ambiguity vector $\mathbf{Q}$, which is much more diagonal than the original variance-covariance matrix $Q_1$, so this search is much more efficient than the search based on the original search space. Because the decorrelating transformation given by Equation 3 preserves both the volume of the search space and the integer nature of the ambiguities, the original and transformed search spaces contain the same number of grid points. Moreover, the correspondence between the original and transformed ambiguities is one-to-one, making it easy for us to transform the solution back to the original as necessary to obtain the integer least-squares solution for the original ambiguities.

TEST RESULTS

We analyzed the performance of our method by employing a representative example that utilizes dual-frequency, carrier-phase measurements taken from a seven-satellite configuration. The sampling interval was 1 second. The a priori standard deviation of the phase observations was set to 3 millimeters. In Figure 5, the elongation of the ambiguity search space for this data set is given as a function of the observation time span as 3 ranges from 1 to 30 seconds.

Reduction in Elongation. For a 1-second observation time span, the elongation is reduced in the transformation by three orders of magnitude. Also, we can see that the elongation of the transformed search space is nearly independent of the observation time span, whereas the elongation before the transformation decreases with an increase in observation time span. This characteristic is caused by the change in receiver-satellite geometric configuration. Even for a 1-hour observation time span, the elongation before transformation is still more than twice as large as the elongation after transformation.

In conclusion, Figure 6 shows the increase in the ambiguities' precision, which occurs from using the transformation of Equation 3. In Figure 6a, the standard deviations, expressed in cycles, are given for the
Figure 6. The precision of the 12 ambiguities estimated from a test data set before transformation (a) ranges from about 50 to 200 cycles for an observation time span of 1 second. The transformed ambiguities (b) have precisions ranging from 0.1 to 0.25 cycles for the same time span.

Further Reading
For a detailed introduction to the carrier phase observable, see
There is an extensive literature on carrier phase ambiguity fixing. For an introduction to the subject, see
"Robust Techniques for Determining GPS Phase Ambiguities," by C. Gold in the Proceedings of the Sixth International Geodetic Symposium on Satellite Positioning, held in Columbus, Ohio, in March 1982, pp. 245-246.
For further details of the authors' Lambda method, see

The computation time is the time needed for computing the integer least-squares estimates. Before transformation refers to the time needed for the search based on the original ambiguities; after transformation refers to the search based on the transformed ambiguities Z. For the latter, the time needed for constructing the transformation matrix Z is included.

CONCLUDING REMARKS
Our least-squares Ambiguity Determination Adjustment method very quickly estimates integer least-squares ambiguities, particularly for short observation time spans. For example, typical computation times on a 486 personal computer are much less than 1 second. Furthermore, the method can be applied to data obtained from a wide variety of receivers because, in principle, it is independent of whether pseudorange data, in addition to carrier-phase data, are available or whether single- or dual-frequency measurements are used. When dual-frequency data are used, the method can be applied to other types of ambiguities, such as wide-lane ambiguities. For more details on the Lambda method, readers should consult the references listed in the sidebar. They may also contact the Delft Geodetic Computing Centre by post: Thijssenweg 11, NL-2629 JA Delft, The Netherlands; by fax: 011 431 15 783711; or by e-mail: lgr@rugv1.tudelft.nl.

Table 1. Ambiguity-fixing performance characteristics of a seven-satellite test case

<table>
<thead>
<tr>
<th>Characteristic</th>
<th>Before</th>
<th>After</th>
</tr>
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<tbody>
<tr>
<td>Minimum</td>
<td>71.0</td>
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<tr>
<td>Maximum</td>
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<td>0.24</td>
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<tr>
<td>Precision (cycles)</td>
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<td>7.5</td>
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<tr>
<td>Computation time (sec)</td>
<td>1,342.8</td>
<td>0.064</td>
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</tbody>
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