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Simplified equivalent multiple baseline solutions with elevation-dependent weights

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Abstract Since the assumption of all stations tracking the same satellites with identical weights was previously employed by Shen and Xu (GPS Solut 12:99–108, 2008) to derive the simplified GNSS single- and double-differenced equivalent equations, this supplementary paper expands these simplified equations in the case of each station tracking different satellites with elevation-dependent weights. Numerical experiments are performed to demonstrate the computational efficiency of the simplified equivalent algorithm relative to the traditional method in various scenarios of multi-baseline solutions with tracking different satellites. The fast computational speed of the simplified equivalent algorithm will potentially benefit the local, regional and even global GNSS multi-baseline solutions as well as the combined GNSS application.

Keywords GNSS data processing · Multi-baseline solutions · Equivalent representation · Combined GNSS application

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Introduction

The Global Navigation Satellite Systems (GNSS) single- and double-differenced simplified equivalent observation equations were derived by Shen and Xu (2008) by means of adding pseudo-observations; their corresponding unbiased variance estimators of unit weight were derived according to the theorem proposed by Schaffrin and Grafarend (1986) and Xu (2002). Although the stochastic model for the GPS measurements could be more complicated to reflect the reality in the actual applications (Li et al. 2008; Wang et al. 1998, 2002), all formulae developed by Shen and Xu (2008) for the simplified representation are subject to the assumption that all stations track the same satellites and that all observations are independent and equally weighted. This is not the case in real GNSS applications. For example, each satellite can only cover an area less than the hemisphere in global networks, different stations track different satellites because of obstruction, and the variances assigned to the observables should be a function of the satellite elevation angle. Therefore, these simplified observation equations must be expanded for use in real GNSS applications.

The contents of this paper are arranged as follows. Sections "Single-differenced simplified equivalent observation equations" and "Double-differenced simplified equivalent observation equations" develop the single- and double-differenced simplified equivalent observation equations for the case of different stations tracking the different satellites with elevation-dependent weights. In section "Numerical experiments", the numerical experiments are performed to evaluate the computational efficiency of the proposed simplified equivalent algorithm. Its potential applications for future multiple satellite systems with multiple frequencies are also discussed, followed by a summary.

Single-differenced simplified equivalent observation equations

The GNSS observation equations for one epoch can be symbolically expressed as

$$\varepsilon = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{y} + \boldsymbol{C}\boldsymbol{z} - \boldsymbol{l}, \quad \boldsymbol{P} \tag{1}$$

where $\boldsymbol{y}$ and $\boldsymbol{z}$ are the vectors of station and satellite biases, and $\boldsymbol{B}$ and $\boldsymbol{C}$ denote the respective coefficient matrices with full column rank; $\boldsymbol{x}$ is a column vector with t parameters, and $\boldsymbol{A}$ is its coefficient matrix also with full column rank; $\boldsymbol{l}$ and $\boldsymbol{\varepsilon}$ are the column vectors of observables and normally distributed observation errors; $\boldsymbol{P}$ is weight matrix of observations. In this paper, elevation-dependent weights are used and different stations can track the different satellites, but the correlations among the observables (temporal, cross and channel) are not considered. Thus the weight matrix $\boldsymbol{P}$ is diagonal with varying elements. Refer to Leick (2004) for the detailed interpretation of these parameters. If there are total of k stations and each station only tracks the subset of the total n satellites, then $\boldsymbol{y} = (y_1 \quad y_2 \quad \ldots \quad y_k)^{\mathrm{T}}$ and $\boldsymbol{z} = (z_1 \quad z_2 \quad \ldots \quad z_n)^T$. The coefficient matrices, vector of observables and weight matrix are grouped with the following sub-blocks in the order of satellites as

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{A}^1 \\ \boldsymbol{A}^2 \\ \vdots \\ \boldsymbol{A}^n \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} \boldsymbol{B}^1 \\ \boldsymbol{B}^2 \\ \vdots \\ \boldsymbol{B}^n \end{pmatrix}, \quad \boldsymbol{C} = \begin{pmatrix} \boldsymbol{e}_{k_1} & & & \\ & \boldsymbol{e}_{k_2} & & \\ & & \cdots & \vdots \\ & & & \boldsymbol{e}_{k_n} \end{pmatrix},$$

$$\boldsymbol{l} = \begin{pmatrix} \boldsymbol{l}^1 \\ \boldsymbol{l}^2 \\ \vdots \\ \boldsymbol{l}^n \end{pmatrix}, \quad \boldsymbol{P} = \begin{pmatrix} \boldsymbol{P}^1 & & & \\ & \boldsymbol{P}^2 & & \\ & & \ddots & \\ & & & \boldsymbol{P}^n \end{pmatrix}, \tag{2}$$

where

$$\boldsymbol{A}^j = \begin{pmatrix} \boldsymbol{a}^j_{S^j(1)} \\ \boldsymbol{a}^j_{S^j(2)} \\ \vdots \\ \boldsymbol{a}^j_{S^j(k_j)} \end{pmatrix}, \quad \boldsymbol{l}^j = \begin{pmatrix} l^j_{S^j(1)} \\ l^j_{S^j(2)} \\ \vdots \\ l^j_{S^j(k_j)} \end{pmatrix},$$

$$\boldsymbol{P}^j = \begin{pmatrix} \boldsymbol{p}^j_{S^j(1)} & & & \\ & \boldsymbol{p}^j_{S^j(2)} & & \\ & & \ddots & \\ & & & \boldsymbol{p}^j_{S^j(k_j)} \end{pmatrix}.$$

The symbols $\boldsymbol{a}^j_{S^j(i)}$ and $\boldsymbol{l}^j_{S^j(i)}$ denote, respectively, the coefficient row vector and observable of the satellite j tracked by the station $S^j(i)$, and $p^j_{S^j(i)}$ is its weight. The symbol S^j represents the set of all stations that simultaneously track the satellite j and $S^j(i)$is the order of the ith station in the total set. The letter k_j denotes the number of stations that track the satellite j, $\boldsymbol{e}_{k_j} = (1 \quad 1 \quad \ldots \quad 1)^{\mathrm{T}}$ is a k_j vector. The coefficient for the jth satellite is a $k_j \times k$ matrix $\boldsymbol{B}^j$ consisting of k_j canonical row vectors; in each canonical row vector all elements are zeros expect the element associated with the tracking receiver is one. For example, if there are 5 stations and the 2nd station does not track the 3rd satellite, then the matrix $\boldsymbol{B}^3$ is

$$\boldsymbol{B}^3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The satellite-specific parameter vector $\boldsymbol{z}$ can be eliminated by single differencing in the station domain or by right-multiplying the original observation equations with the transformation matrix $\boldsymbol{R}$. The transformation matrix is (Teunissen 1997)

$$\boldsymbol{R} = \boldsymbol{I}_{\Sigma k} - \boldsymbol{C}\left(\boldsymbol{C}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{C}\right)^{-1}\boldsymbol{C}^{\mathrm{T}}\boldsymbol{P} = \begin{pmatrix} \boldsymbol{R}_{k_1} & & & \\ & \boldsymbol{R}_{k_2} & & \\ & & \ddots & \\ & & & \boldsymbol{R}_{k_n} \end{pmatrix}, \tag{3}$$

where the dimension of identity matrix $\boldsymbol{I}_{\Sigma k}$ is $\Sigma k = \sum_{j=1}^{n} k_j$, and

$$\boldsymbol{R}_{k_j} = \boldsymbol{I}_{k_j} - \frac{1}{\sum_{i \in S^j} p^j_i} \boldsymbol{e}_{k_j}\boldsymbol{e}^T_{k_j}\boldsymbol{P}^j = \boldsymbol{I}_{k_j} - \frac{1}{p^{\Sigma j}}\boldsymbol{P}^j \otimes \boldsymbol{e}^T_{k_j} \tag{4}$$

with $p^{\Sigma j} = \sum_{i \in S^j} p^j_i$ being the sum of weights of observables for all stations that tack the jth satellites.

Multiplying Eq. 1 with matrix $\boldsymbol{R}$, we obtain the equivalently transformed observation equations

$$\tilde{\varepsilon} = \tilde{\boldsymbol{A}}\boldsymbol{x} + \tilde{\boldsymbol{B}}\boldsymbol{y} - \tilde{\boldsymbol{l}}, \quad \boldsymbol{P} \tag{5}$$

where $\tilde{\boldsymbol{A}} = \boldsymbol{R}\boldsymbol{A}$, $\tilde{\boldsymbol{B}} = \boldsymbol{R}\boldsymbol{B}$, $\tilde{\boldsymbol{l}} = \boldsymbol{R}\boldsymbol{l}$ and $\tilde{\varepsilon} = \boldsymbol{R}\varepsilon$. As shown in Eq. 3, the matrix $\boldsymbol{R}$ is diagonal with sub-matrix $\boldsymbol{R}_{k_j}$. Therefore, Eq. 5 can be further simplified as

$$\tilde{\varepsilon}^j = \tilde{\boldsymbol{A}}^j\boldsymbol{x} + \tilde{\boldsymbol{B}}^j\boldsymbol{y} - \tilde{\boldsymbol{l}}^j, \quad \boldsymbol{P}^j, \quad j = 1, 2, \ldots, n \tag{6}$$

with

$$\tilde{\boldsymbol{A}}^j = \boldsymbol{R}_{k_j}\boldsymbol{A}^j, \quad \tilde{\boldsymbol{B}}^j = \boldsymbol{R}_{k_j}\boldsymbol{B}^j, \quad \tilde{\boldsymbol{l}}^j = \boldsymbol{R}_{k_j}\boldsymbol{l}^j \tag{7}$$

It is obvious that $\boldsymbol{R}_{k_j}$ has a rank defect of one. This means that one station-specific parameter can be linearly represented with the others, i.e. only $k - 1$ station-specific parameters can be independently parameterized.

In the single-differenced equivalent observation equations, the independent parameterized station-specific parameters are generally merged into $\boldsymbol{x}$, and Eq. 6 becomes

$$\tilde{\varepsilon}^j = \tilde{\boldsymbol{A}}^j \boldsymbol{x} - \tilde{\boldsymbol{l}}^j, \quad \boldsymbol{P}^j, \quad j = 1, 2, \ldots, n \tag{8}$$

where the transformed coefficient matrix and observation vector can also be further simplified as

$$\tilde{\boldsymbol{A}}^j = \boldsymbol{R}_{k_j} \boldsymbol{A}^j = \boldsymbol{A}^j - \frac{1}{p^{\Sigma j}} \boldsymbol{e}_{k_j} \boldsymbol{e}_{k_j}^T \boldsymbol{P}^j \boldsymbol{A}^j = \boldsymbol{A}^j - \delta \boldsymbol{A}^j \tag{9a}$$

$$\tilde{\boldsymbol{l}}^j = \boldsymbol{R}_{k_j} \boldsymbol{l}^j = \boldsymbol{l}^j - \frac{1}{p^{\Sigma j}} \boldsymbol{e}_{k_j} \boldsymbol{e}_{k_j}^T \boldsymbol{P}^j \boldsymbol{l}^j - \boldsymbol{l}^j - \delta \boldsymbol{l}^j \tag{9b}$$

with

$$\delta \boldsymbol{A}^j = \frac{1}{p^{\Sigma j}} \boldsymbol{e}_{k_j} \boldsymbol{e}_{k_j}^T \boldsymbol{P}^j \boldsymbol{A}^j = \frac{1}{p^{\Sigma j}} \boldsymbol{e}_{k_j} \sum_{i \in S^j} \left(p_i^j \boldsymbol{a}_i^j \right) = \frac{1}{p^{\Sigma j}} \boldsymbol{e}_{k_j} \left[\boldsymbol{a}^j \right] \tag{10a}$$

$$\delta \boldsymbol{l}^j = \frac{1}{p^{\Sigma j}} \boldsymbol{e}_{k_j} \boldsymbol{e}_{k_j}^T \boldsymbol{P}^j \boldsymbol{l}^j = \frac{1}{p^{\Sigma j}} \boldsymbol{e}_{k_j} \sum_{i \in S^j} \left(p_i^j l_i^j \right) = \frac{1}{p^{\Sigma j}} \boldsymbol{e}_{k_j} \left[l^j \right] \tag{10b}$$

and $[\boldsymbol{a}^j] = \sum_{i \in S^j} \left(p_i^j \boldsymbol{a}_i^j \right)$ and $[l^j] = \sum_{i \in S^j} \left(p_i^j l_i^j \right)$. Each element of the column vector $\delta \boldsymbol{l}^j$ and each column vector of $\delta \boldsymbol{A}^j$ are the weighted mean of their corresponding column vectors. Therefore, the transformed vector $\tilde{\boldsymbol{l}}^j$ is the centrobaric vector of $\boldsymbol{l}^j$, and the transformed matrix $\tilde{\boldsymbol{A}}^j$ is the column centrobaric matrix of $\boldsymbol{A}^j$. In other words, the equivalent observation equations 8 can also be simply obtained through the centrobaric operation to the column vectors of $\boldsymbol{A}^j$ and $\boldsymbol{l}^j$.

In addition, the expression 8 can alternatively be expanded in the same way as described by Shen and Xu (2008) in the form of pseudo-observations,

$$\tilde{\varepsilon}^j = \boldsymbol{A}^j \boldsymbol{x} - \boldsymbol{l}^j, \quad \boldsymbol{P}^j, \quad j = 1, 2, \ldots, n \tag{11a}$$

$$\left[\varepsilon^j \right] = \left[\boldsymbol{a}^j \right] \boldsymbol{x} - \left[l^j \right], \quad {-1}/{p^{\Sigma j}}, \quad j = 1, 2, \ldots, n \tag{11b}$$

where $[\tilde{\varepsilon}^j]$ denotes the residual of the jth sum pseudo-observation. The same normal equations can be obtained by the equivalent observation equations 8 and 11a, 11b, and the proof is given in Appendix 1. Once the unknown parameter vector $\hat{x}$ is solved, the residual vector is computed by

$$\boldsymbol{v}^j = \tilde{\boldsymbol{A}}^j \hat{\boldsymbol{x}} - \tilde{\boldsymbol{l}}^j, \quad j = 1, 2, \ldots, n \tag{12}$$

Double-differenced simplified equivalent observation equations

If there are more than two stations and each station may track a subset of the total n satellites, the double-differenced equivalent observation equations for multi-baseline solutions will be much more complicated than single-differenced ones. In order to derive the simplified double-differenced equivalent observation equations, we rearrange Eq. 5 with the sub-blocks in the order of receivers and use the same symbols as used in Eq. 5 to represent the rearranged single-differenced observation equations as

$$\tilde{\varepsilon} = \tilde{\boldsymbol{A}} \boldsymbol{x} + \tilde{\boldsymbol{B}} \boldsymbol{y} - \tilde{\boldsymbol{l}}, \quad \boldsymbol{P} \tag{13}$$

where

$$\tilde{\boldsymbol{A}} = \begin{pmatrix} \tilde{\boldsymbol{A}}_1 \\ \tilde{\boldsymbol{A}}_2 \\ \vdots \\ \tilde{\boldsymbol{A}}_k \end{pmatrix}$$

with

$$\tilde{\boldsymbol{A}}_i = \begin{pmatrix} \tilde{\boldsymbol{a}}_i^{S_i(1)} \\ \tilde{\boldsymbol{a}}_i^{S_i(2)} \\ \vdots \\ \tilde{\boldsymbol{a}}_i^{S_i(n_i)} \end{pmatrix} = \begin{pmatrix} \boldsymbol{a}_i^{S_i(1)} - \left[\boldsymbol{a}^{S_i(1)} \right] / p^{\Sigma S_i(1)} \\ \boldsymbol{a}_i^{S_i(2)} - \left[\boldsymbol{a}^{S_i(2)} \right] / p^{\Sigma S_i(2)} \\ \vdots \\ \boldsymbol{a}_i^{S_i(n_i)} - \left[\boldsymbol{a}^{S_i(n_i)} \right] / p^{\Sigma S_i(n_i)} \end{pmatrix},$$

n_i is the number of satellites tracked by the station i and S_i denotes a set comprised of these n_i satellites. $S_i(l)$ is the order of the lth satellite in the total set;

$$\tilde{\boldsymbol{l}} = \begin{pmatrix} \tilde{\boldsymbol{l}}_1 \\ \tilde{\boldsymbol{l}}_2 \\ \vdots \\ \tilde{\boldsymbol{l}}_k \end{pmatrix}$$

with

$$\tilde{\boldsymbol{l}}_i = \begin{pmatrix} \tilde{\boldsymbol{l}}_i^{S_i(1)} \\ \tilde{\boldsymbol{l}}_i^{S_i(2)} \\ \vdots \\ \tilde{\boldsymbol{l}}_i^{S_i(n_i)} \end{pmatrix}, \quad P = \begin{pmatrix} \boldsymbol{P}_1 & & & \\ & \boldsymbol{P}_2 & & \\ & & \ddots & \\ & & & \boldsymbol{P}_k \end{pmatrix}$$

with

$$\boldsymbol{P}_i = \begin{pmatrix} \boldsymbol{p}_i^{S_i(1)} & & & \\ & \boldsymbol{p}_i^{S_i(2)} & & \\ & & \ddots & \\ & & & \boldsymbol{p}_i^{S_i(n_i)} \end{pmatrix}$$

and $\tilde{\boldsymbol{B}} = \begin{pmatrix} \tilde{\boldsymbol{b}}_2 & \tilde{\boldsymbol{b}}_3 & \cdots & \tilde{\boldsymbol{b}}_k \end{pmatrix}$. The first element in $\boldsymbol{y}$ is fixed to zero to achieve independent parameterization. According to Eqs. 6 and 7, we can determine the rearranged column vector $\tilde{\boldsymbol{b}}_i$ as

$$\tilde{\boldsymbol{b}}_i = \Big(-(\boldsymbol{Q}_1 \boldsymbol{G}_i \boldsymbol{\alpha}_i)^{\mathrm{T}} \quad \cdots \quad -(\boldsymbol{Q}_{i-1} \boldsymbol{G}_i \boldsymbol{\alpha}_i)^{\mathrm{T}} \quad (\boldsymbol{Q}_i (\boldsymbol{e}_n - \boldsymbol{\alpha}_i))^{\mathrm{T}} \\ -(\boldsymbol{Q}_{i+1} \boldsymbol{G}_i \alpha_i)^{\mathrm{T}} \quad \cdots \quad -(\boldsymbol{Q}_k \boldsymbol{G}_i \boldsymbol{\alpha}_i)^{\mathrm{T}} \Big)^{\mathrm{T}} \tag{14}$$

where

$$\boldsymbol{\alpha}_i = \begin{pmatrix} \frac{p_i^1}{p^{\Sigma 1}} & \frac{p_i^2}{p^{\Sigma 2}} & \cdots & \frac{p_i^n}{p^{\Sigma n}} \end{pmatrix}^{\mathrm{T}},$$

$\mathbf{G}_i$ is a $n \times n$ diagonal matrix and its diagonal element is equal to either one (corresponding to tracked satellite) or zero (corresponding to non-tracked satellite). The n_i non-zero row vectors of G_i construct the $n_i \times n$ matrix Q_i. If there are 6 satellites and the 3rd station does not track the 2nd and 5th satellites, the matrices G_3 and Q_3 are expressed as

$$\boldsymbol{G}_3 = \begin{pmatrix} 1 & & & & & \\ & 0 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 0 & \\ & & & & & 1 \end{pmatrix}, \qquad (15)$$

$$\boldsymbol{Q}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is obvious that the matrices $\boldsymbol{G}_i$ and $\boldsymbol{Q}_i$ hold true for the following properties

$$\boldsymbol{G}_i = \boldsymbol{Q}_i^T \boldsymbol{Q}_i, \quad \boldsymbol{Q}_i \boldsymbol{G}_i = \boldsymbol{Q}_i, \quad \boldsymbol{G}_i = \boldsymbol{G}_i^T, \quad \boldsymbol{G}_i \boldsymbol{G}_i = \boldsymbol{G}_i. \qquad (16)$$

In order to determine the transformation matrix $\tilde{\boldsymbol{R}}$ for eliminating station-specific parameters, the following matrix should be computed,

$$\tilde{\boldsymbol{B}}^T \boldsymbol{P} \tilde{\boldsymbol{B}} = \begin{pmatrix} \tilde{\boldsymbol{b}}_2^T \boldsymbol{P} \tilde{\boldsymbol{b}}_2 & \tilde{\boldsymbol{b}}_2^T \boldsymbol{P} \tilde{\boldsymbol{b}}_3 & & \tilde{\boldsymbol{b}}_2^T \boldsymbol{P} \tilde{\boldsymbol{b}}_k \\ \tilde{\boldsymbol{b}}_3^T \boldsymbol{P} \tilde{\boldsymbol{b}}_2 & \tilde{\boldsymbol{b}}_3^T \boldsymbol{P} \tilde{\boldsymbol{b}}_3 & & \tilde{\boldsymbol{b}}_3^T \boldsymbol{P} \tilde{\boldsymbol{b}}_k \\ & \cdots & \vdots & \\ \tilde{\boldsymbol{b}}_k^T \boldsymbol{P} \tilde{\boldsymbol{b}}_2 & \tilde{\boldsymbol{b}}_k^T \boldsymbol{P} \tilde{\boldsymbol{b}}_3 & & \tilde{\boldsymbol{b}}_k^T \boldsymbol{P} \tilde{\boldsymbol{b}}_k \end{pmatrix}. \qquad (17)$$

According to Eqs. 14 and 16, the expressions for the submatrices of $\tilde{\boldsymbol{B}}^T \boldsymbol{P} \tilde{\boldsymbol{B}}$, which have been derived in Appendix 2, are

$$\tilde{\boldsymbol{b}}_i^T \boldsymbol{P} \tilde{\boldsymbol{b}}_i = p_{\Sigma i} - \sum_{l \in S_i} \frac{p_i^l p_i^l}{p^{\Sigma l}} \qquad (18a)$$

$$\tilde{\boldsymbol{b}}_i^T \boldsymbol{P} \tilde{\boldsymbol{b}}_j = - \sum_{l \in S_{ij}} \frac{p_i^l p_j^l}{p^{\Sigma l}}, \qquad (18b)$$

where $p_{\Sigma i} = \sum_{j \in S_i} p_i^j$ is the sum of weights of observables for all satellites tracked by the ith station. S_{ij} is a intersection set of S_i and S_j, denoted by $S_{ij} = S_i \cap S_j$, and refers to the set of satellites that are simultaneously tracked by both station i and station j. The matrix $\tilde{\boldsymbol{B}}^T \boldsymbol{P} \tilde{\boldsymbol{B}}$ can be efficiently computed by Eqs. 18a, b, but its inverse is rather complicated and not symbolically expressible. Therefore, the transformation matrix is numerically computed by

$$\tilde{\boldsymbol{R}} = \boldsymbol{I}_{\Sigma k} - \tilde{\boldsymbol{B}} \left(\tilde{\boldsymbol{B}}^T \boldsymbol{P} \tilde{\boldsymbol{B}} \right)^{-1} \tilde{\boldsymbol{B}}^T \boldsymbol{P} = \boldsymbol{I}_{\Sigma k} - \tilde{\boldsymbol{J}} \qquad (19)$$

Analogously, multiplying the transformation matrix $\tilde{\boldsymbol{R}}$ by Eq. 13, the double-differenced equivalent equations are obtained as

$$\bar{\bar{\varepsilon}} = \bar{\bar{\boldsymbol{A}}} \boldsymbol{x} - \bar{\bar{\boldsymbol{l}}}, \quad \boldsymbol{P} \qquad (20)$$

with

$$\bar{\bar{\boldsymbol{A}}} = \tilde{\boldsymbol{R}} \tilde{\boldsymbol{A}} = \tilde{\boldsymbol{A}} - \tilde{\boldsymbol{J}} \tilde{\boldsymbol{A}} \qquad (21a)$$

$$\bar{\bar{\boldsymbol{l}}} = \tilde{\boldsymbol{R}} \tilde{\boldsymbol{l}} = \tilde{\boldsymbol{l}} - \tilde{\boldsymbol{J}} \tilde{\boldsymbol{l}} \qquad (21b)$$

As mentioned in section “Single-differenced simplified equivalent observation equations”, the arrays $\tilde{\boldsymbol{A}}$ and $\tilde{\boldsymbol{l}}$ consist of all sub-matrices $\tilde{\boldsymbol{A}}^j$ and sub-vectors $\tilde{\boldsymbol{l}}^j$ respectively, and can be very efficiently computed by centrobaric operation to their column vectors. The $(k-1) \times (k-1)$ square matrix $\tilde{\boldsymbol{B}}^T \boldsymbol{P} \tilde{\boldsymbol{B}}$ and its inverse are needed to determine the transformation matrix $\tilde{\boldsymbol{R}}$. The matrix $\tilde{\boldsymbol{B}}^T \boldsymbol{P} \tilde{\boldsymbol{B}}$ can be efficiently implemented by Eqs. 18a, b; its inverse matrix can be trivially computed, which is certainly more efficient than computing the weight matrix of double-differenced observables for multi-baseline solutions. Once the least squares solution to parameter vector $\hat{\boldsymbol{x}}$ is obtained, the residuals can be exactly computed by

$$\boldsymbol{v} = \bar{\bar{\boldsymbol{A}}} \hat{\boldsymbol{x}} - \bar{\bar{\boldsymbol{l}}}. \qquad (22)$$

Numerical experiments

The numerical experiments are performed in order to compare the computational efficiency of forming the double-differenced normal equations by the proposed simplified equivalent algorithm or by the traditional method. The traditional method forms the weight matrix for double-differenced observables and the normal equations as described above. All computations are performed with Matlab7.3.0 programs on a Pentium D, 3.2 GHz PC with 1 GB memory running Windows XP professional. Most commercial GNSS softwares currently employ the simple model of single baseline solution because of the complicated transformation of the weight matrix and computational inefficiency in case of multiple baselines. As we enter the multi-GNSS era with 20 or more satellites in view such legacy softwares might require updates. High-performance software will always explore the pattern of matrices to avoid unnecessary computations, e.g. multiplication by zero, and most likely be implemented in C++. In our Matlab implementation we did not take advantage of the pattern of matrices to reduce computational time.

We study the efficiency of the simplified equivalent algorithm for four different satellites and stations scenarios. All experiments include over 1000 epochs to illustrate the time difference between two methods. In Figs. 1, 2 the efficiency in satellite domain is evaluated. Figure 1 illustrates the computational time used by the simplified equivalent algorithm and traditional method as 10 stations track different satellites. The computational time increases with number of tracked satellite. This increase is more significant for traditional method than for the simplified

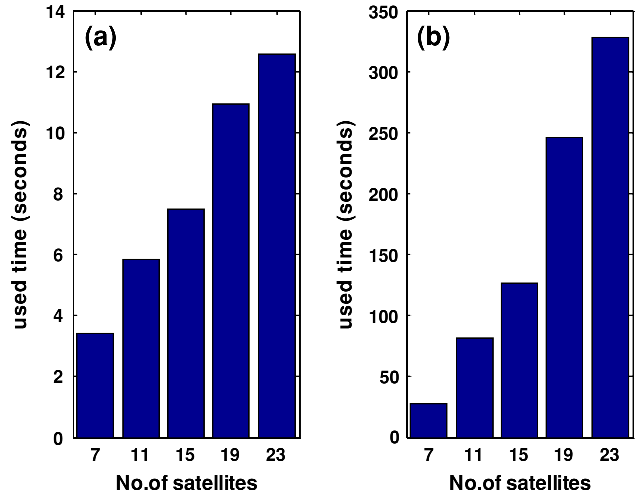

Fig. 1 Computational time for simplified equivalent algorithm (**a**) and traditional method (**b**) as a function of satellites tracked by 10 stations

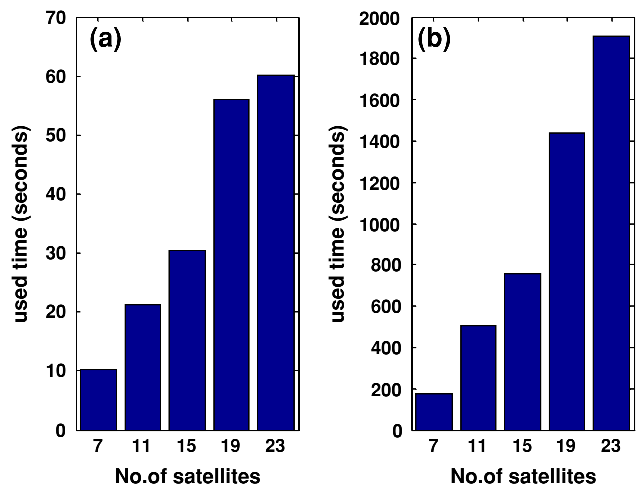

Fig. 2 Computational time for simplified equivalent algorithm (**a**) and traditional method (**b**) as a function of satellites tracked by 20 stations

equivalent algorithm. Figure 2 presents a similar comparison but for 20 tracking stations. When 23 satellites are tracked by 20 stations, the traditional method requires more than 30 min while the simplified equivalent algorithm takes just 1 min. The Figs. 3, 4 demonstrate the efficiency in station domain when 11 and 19 satellites are tracked. Similarly, the computational time increases with the number of the tracking stations. In case of 25 stations it takes about 16(43) min to form the normal equations by the traditional method but only about 37(97) s for the simplified equivalent algorithm in when tracking 11(19) satellites. We have noticed that the influence of number of stations on computation time is more serious than that of number of satellites. This is because the elimination

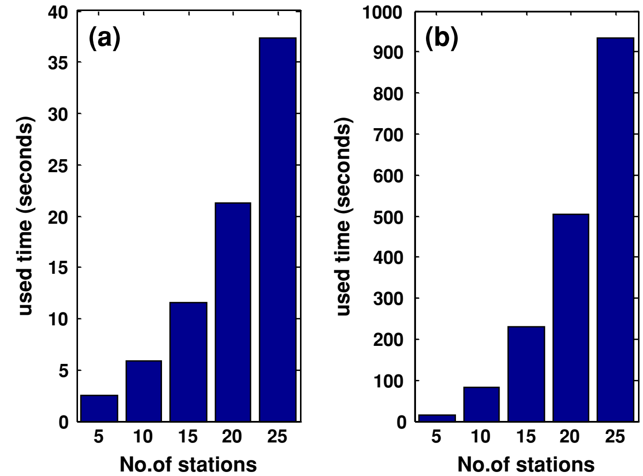

Fig. 3 Computational time for simplified equivalent algorithm (**a**) and traditional method (**b**) as a function of stations tracking 11 satellites

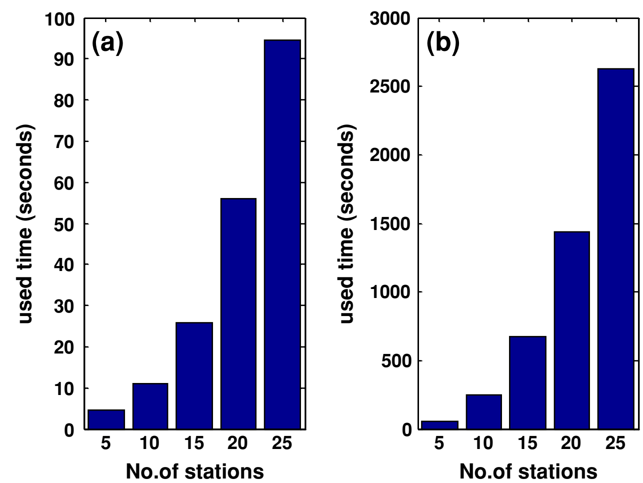

Fig. 4 Computational time for simplified equivalent algorithm (**a**) and traditional method (**b**) as a function of stations tracking 19 satellites

of satellite-specific biased by pseudo-observation equation 11b is so efficient that the time consumed is much less sensitive to the number of satellites, compared with the elimination of the station-specific biases by the Eqs. 17–21a, 21b. Therefore, if there are more stations than satellites, we should first eliminate station-specific biases and then satellite-specific biases to achieve an efficient implementation.

The final expressions for forming the equivalent normal equations are simple and readily programmable, even though the derivation is complicated. The new algorithm is computationally highly efficient, especially for multiple baseline solutions with many satellites and stations. When analyzing the results of our computations we must keep in mind that we did not explore the computational efficiency that potentially becomes available by taking advantage of the pattern of zeros in the matrices. In any case, as multiple GNSS systems would become available in the near future and as a scale factor might be needed to balance the observables from the different systems, the derived formulae still work well. Therefore, the proposed algorithm can provide a theoretical and technical cornerstone for the development the efficient GNSS software.

Concluding remarks

The expressions of the simplified equivalent algorithm derived by Shen and Xu (2008) have been expanded to the case where stations track the different satellites and elevation-dependent weighting is used. The derivation procedure is rather complicated, but the final formulae are not. The experiments show that the simplified equivalent algorithm is significantly faster than the traditional method, especially when a large number of satellites are tracked by multiple stations. This promising finding will aid the development of efficient GNSS software, potentially benefiting the local, regional and even global GNSS multi-baseline solutions.

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Appendix 1

The proof that the single-differenced observation equations can be equivalently achieved by Eqs. 8 and 11a, 11b is provided here. The single-differenced normal equations are obtained from Eqs. 8 and 9a, 9b as

$$\left(\tilde{\boldsymbol{A}}^j\right)^T\boldsymbol{P}^j\tilde{\boldsymbol{A}}^j = \left(\boldsymbol{A}^j\right)^T\boldsymbol{P}^j\boldsymbol{A}^j - \left(\boldsymbol{A}^j\right)^T\boldsymbol{P}^j\delta\boldsymbol{A}^j - \left(\delta\boldsymbol{A}^j\right)^T\boldsymbol{P}^j\boldsymbol{A}^j + \left(\delta\boldsymbol{A}^j\right)^T\boldsymbol{P}^j\delta\boldsymbol{A}^j \tag{23}$$

$$\left(\tilde{\boldsymbol{A}}^j\right)^T\boldsymbol{P}^j\tilde{\boldsymbol{l}}^j = \left(\boldsymbol{A}^j\right)^T\boldsymbol{P}^j\boldsymbol{l}^j - \left(\boldsymbol{A}^j\right)^T\boldsymbol{P}^j\delta\boldsymbol{l}^j - \left(\delta\boldsymbol{A}^j\right)^T\boldsymbol{P}^j\boldsymbol{l}^j + \left(\delta\boldsymbol{A}^j\right)^T\boldsymbol{P}^j\delta\boldsymbol{l}^j \tag{24}$$

Inserting Eq. 10a into Eq. 23 gives

$$\left(\tilde{\boldsymbol{A}}^j\right)^T\boldsymbol{P}^j\tilde{\boldsymbol{A}}^j = \left(\boldsymbol{A}^j\right)^T\boldsymbol{P}^j\boldsymbol{A}^j - \frac{2}{p^{\Sigma j}}\left(\left[\boldsymbol{a}^j\right]\right)^T\left[\boldsymbol{a}^j\right] + \frac{1}{\left(p^{\Sigma j}\right)^2}\left(\left[\boldsymbol{a}^j\right]\right)^T\boldsymbol{e}_{k_j}^T\boldsymbol{P}^j\boldsymbol{e}_{k_j}\left[\boldsymbol{a}^j\right] \tag{25}$$

Inserting Eqs. 10a and 10b into Eq. 24 gives

$$\left(\tilde{\boldsymbol{A}}^j\right)^T\boldsymbol{P}^j\tilde{\boldsymbol{l}}^j = \left(\boldsymbol{A}^j\right)^T\boldsymbol{P}^j\boldsymbol{l}^j - \frac{2}{p^{\Sigma j}}\left(\left[\boldsymbol{a}^j\right]\right)^T\left[l^j\right] + \frac{1}{\left(p^{\Sigma j}\right)^2}\left(\left[\boldsymbol{a}^j\right]\right)^T\boldsymbol{e}_{k_j}^T\boldsymbol{P}^j\boldsymbol{e}_{k_j}\left[l^j\right] \tag{26}$$

Substituting $\boldsymbol{e}_{k_j}^T\boldsymbol{P}^j\boldsymbol{e}_{k_j} = p^{\Sigma j}$ individually into Eqs. 25 and 26, we have

$$\left(\tilde{\boldsymbol{A}}^j\right)^T\boldsymbol{P}^j\tilde{\boldsymbol{A}}^j = \left(\boldsymbol{A}^j\right)^T\boldsymbol{P}^j\boldsymbol{A}^j - \frac{1}{p^{\Sigma j}}\left(\left[\boldsymbol{a}^j\right]\right)^T\left[\boldsymbol{a}^j\right] \tag{27}$$

$$\left(\tilde{\boldsymbol{A}}^j\right)^T\boldsymbol{P}^j\tilde{\boldsymbol{l}}^j = \left(\boldsymbol{A}^j\right)^T\boldsymbol{P}^j\boldsymbol{l}^j - \frac{1}{p^{\Sigma j}}\left(\left[\boldsymbol{a}^j\right]\right)^T\left[l^j\right] \tag{28}$$

It is obvious that the single-differenced normal equations 27 and 28 are exactly equivalent to those from Eqs. 11a and 11b.

Appendix 2

According to the definition of $\tilde{\boldsymbol{b}}_i$ in Eq. 14, the submatrices $\tilde{\boldsymbol{b}}_i^T\boldsymbol{P}\tilde{\boldsymbol{b}}_i$ and $\tilde{\boldsymbol{b}}_i^T\boldsymbol{P}\tilde{\boldsymbol{b}}_j$ of $\tilde{\boldsymbol{B}}^T\boldsymbol{P}\tilde{\boldsymbol{B}}$ can be expanded as follows

$$\tilde{\boldsymbol{b}}_i^T\boldsymbol{P}\tilde{\boldsymbol{b}}_i = \boldsymbol{\alpha}_i^T\boldsymbol{G}_i^T\left(\sum_{l=1}^{k}\boldsymbol{Q}_l^T\boldsymbol{P}_l\boldsymbol{Q}_l\right)\boldsymbol{G}_i\boldsymbol{\alpha}_i - \boldsymbol{\alpha}_i^T\boldsymbol{G}_i^T\boldsymbol{Q}_i^T\boldsymbol{P}_i\boldsymbol{Q}_i\boldsymbol{G}_i\boldsymbol{\alpha}_i + \boldsymbol{\alpha}_i^T\boldsymbol{Q}_i^T\boldsymbol{P}_i\boldsymbol{Q}_i\boldsymbol{\alpha}_i - 2\boldsymbol{e}_n^T\boldsymbol{Q}_i^T\boldsymbol{P}_i\boldsymbol{Q}_i\boldsymbol{\alpha}_i + \boldsymbol{e}_n^T\boldsymbol{Q}_i^T\boldsymbol{P}_i\boldsymbol{Q}_i\boldsymbol{e}_n \tag{29}$$

$$\begin{aligned}\tilde{\boldsymbol{b}}_i^T\boldsymbol{P}\tilde{\boldsymbol{b}}_j = {} & \boldsymbol{\alpha}_i^T\boldsymbol{G}_i^T\left(\sum_{l=1}^{k}\boldsymbol{Q}_l^T\boldsymbol{P}_l\boldsymbol{Q}_l\right)\boldsymbol{G}_j\boldsymbol{\alpha}_j - \boldsymbol{\alpha}_i^T\boldsymbol{G}_i^T\boldsymbol{Q}_j^T\boldsymbol{P}_j\boldsymbol{Q}_j\boldsymbol{G}_j\boldsymbol{\alpha}_j \\ & - \boldsymbol{\alpha}_i^T\boldsymbol{G}_i^T\boldsymbol{Q}_i^T\boldsymbol{P}_i\boldsymbol{Q}_i\boldsymbol{G}_j\boldsymbol{\alpha}_j - \boldsymbol{e}_n^T\boldsymbol{Q}_i^T\boldsymbol{P}_i\boldsymbol{Q}_i\boldsymbol{G}_j\boldsymbol{\alpha}_j \\ & + \boldsymbol{\alpha}_i^T\boldsymbol{Q}_i^T\boldsymbol{P}_i\boldsymbol{Q}_i\boldsymbol{G}_j\boldsymbol{\alpha}_j - \boldsymbol{\alpha}_i^T\boldsymbol{G}_i^T\boldsymbol{Q}_j^T\boldsymbol{P}_j\boldsymbol{Q}_j\boldsymbol{e}_n \\ & + \boldsymbol{\alpha}_i^T\boldsymbol{G}_i^T\boldsymbol{Q}_j^T\boldsymbol{P}_j\boldsymbol{Q}_j\boldsymbol{\alpha}_j\end{aligned} \tag{30}$$

Considering the properties Eq. 16 of matrices $\boldsymbol{G}_i$ and $\boldsymbol{Q}_i$, the expressions 29 and 30, can further be simplified as

$$\tilde{\boldsymbol{b}}_i^T\boldsymbol{P}\tilde{\boldsymbol{b}}_i = \boldsymbol{\alpha}_i^T\boldsymbol{G}_i^T\left(\sum_{l=1}^{k}\boldsymbol{Q}_l^T\boldsymbol{P}_l\boldsymbol{Q}_l\right)\boldsymbol{G}_i\boldsymbol{\alpha}_i - 2\boldsymbol{e}_n^T\boldsymbol{Q}_i^T\boldsymbol{P}_i\boldsymbol{Q}_i\boldsymbol{\alpha}_i + \boldsymbol{e}_n^T\boldsymbol{Q}_i^T\boldsymbol{P}_i\boldsymbol{Q}_i\boldsymbol{e}_n \tag{31}$$

$$\tilde{\boldsymbol{b}}_i^T\boldsymbol{P}\tilde{\boldsymbol{b}}_j = \boldsymbol{\alpha}_i^T\boldsymbol{G}_i^T\left(\sum_{l=1}^{k}\boldsymbol{Q}_l^T\boldsymbol{P}_l\boldsymbol{Q}_l\right)\boldsymbol{G}_j\boldsymbol{\alpha}_j - \boldsymbol{e}_n^T\boldsymbol{Q}_i^T\boldsymbol{P}_i\boldsymbol{Q}_i\boldsymbol{G}_j\boldsymbol{\alpha}_j - \boldsymbol{\alpha}_i^T\boldsymbol{G}_i^T\boldsymbol{Q}_j^T\boldsymbol{P}_j\boldsymbol{Q}_j\boldsymbol{e}_n \tag{32}$$

Apparently,

$$\sum_{l=1}^{k}\boldsymbol{Q}_l^T\boldsymbol{P}_l\boldsymbol{Q}_l = \begin{pmatrix} p^{\Sigma 1} & & & \\ & p^{\Sigma 2} & & \\ & & \ddots & \\ & & & p^{\Sigma n}\end{pmatrix} \tag{33}$$

therefore,

$$\boldsymbol{\alpha}_i^T \boldsymbol{G}_i^T \left(\sum_{l=1}^{k} \boldsymbol{Q}_l^T \boldsymbol{P}_l \boldsymbol{Q}_l \right) \boldsymbol{G}_i \boldsymbol{\alpha}_i = \sum_{j \in S_i} \frac{\left(p_i^j\right)^2}{p^{\Sigma j}},$$
$$\boldsymbol{e}_n^T \boldsymbol{Q}_i^T \boldsymbol{P}_i \boldsymbol{Q}_i \boldsymbol{\alpha}_i = \sum_{l \in S_i} \frac{\left(p_i^l\right)^2}{p^{\Sigma l}}, \quad \boldsymbol{e}_n^T \boldsymbol{Q}_i^T \boldsymbol{P}_i \boldsymbol{Q}_i \boldsymbol{e}_n = p_{\Sigma i} \tag{34}$$

$$\boldsymbol{\alpha}_i^T \boldsymbol{G}_i^T \left(\sum_{l=1}^{k} \boldsymbol{Q}_l^T \boldsymbol{P}_l \boldsymbol{Q}_l \right) \boldsymbol{G}_j \boldsymbol{\alpha}_j = \sum_{l \in S_{ij}} \frac{p_i^l p_j^l}{p^{\Sigma l}},$$
$$\boldsymbol{e}_n^T \boldsymbol{Q}_i^T \boldsymbol{P}_i \boldsymbol{Q}_i \boldsymbol{G}_j \boldsymbol{\alpha}_j = \sum_{l \in S_{ij}} \frac{p_i^l p_j^l}{p^{\Sigma l}}, \quad \boldsymbol{\alpha}_i^T \boldsymbol{G}_i^T \boldsymbol{Q}_j^T \boldsymbol{P}_j \boldsymbol{Q}_j \boldsymbol{e}_n = \sum_{l \in S_{ij}} \frac{p_i^l p_j^l}{p^{\Sigma l}} \tag{35}$$

Inserting Eqs. 34 and 35 into Eqs. 31 and 32, respectively, the final expressions are

$$\tilde{\boldsymbol{b}}_i^T \boldsymbol{P} \tilde{\boldsymbol{b}}_i = p_{\Sigma i} - \sum_{l \in S_i} \frac{\left(p_i^l\right)^2}{p^{\Sigma l}} \tag{36}$$

$$\tilde{\boldsymbol{b}}_i^T \boldsymbol{P} \tilde{\boldsymbol{b}}_j = - \sum_{l \in S_{ij}} \frac{p_i^l p_j^l}{p^{\Sigma l}}. \tag{37}$$

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