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ORIGINAL ARTICLE

On the estimability of parameters in undifferenced, uncombined GNSS network and PPP-RTK user models by means of $\mathcal{S}$-system theory

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Abstract The concept of integer ambiguity resolution-enabled Precise Point Positioning (PPP-RTK) relies on appropriate network information for the parameters that are common between the single-receiver user that applies and the network that provides this information. Most of the current methods for PPP-RTK are based on forming the ionosphere-free combination using dual-frequency Global Navigation Satellite System (GNSS) observations. These methods are therefore restrictive in the light of the development of new multi-frequency GNSS constellations, as well as from the point of view that the PPP-RTK user requires ionospheric corrections to obtain integer ambiguity resolution results based on short observation time spans. The method for PPP-RTK that is presented in this article does not have above limitations as it is based on the undifferenced, uncombined GNSS observation equations, thereby keeping all parameters in the model. Working with the undifferenced observation equations implies that the models are rank-deficient; not all parameters are unbiasedly estimable, but only combinations of them. By application of $\mathcal{S}$-system theory the model is made of full rank by constraining a minimum set of parameters, or S-basis. The choice of this S-basis determines the estimability and the interpretation of the parameters that are transmitted to the PPP-RTK users. As this choice is not unique, one has to be very careful when comparing network solutions in different $\mathcal{S}$-systems; in that case the S-transformation, which is provided by the $\mathcal{S}$-system method, should be used to make the comparison. Knowing the estimability and interpretation of the parameters estimated by the network is shown to be crucial for a correct interpretation of the estimable PPP-RTK user parameters, among others the essential ambiguity parameters, which have the integer property which is clearly following from the interpretation of satellite phase biases from the network. The flexibility of the $\mathcal{S}$-system method is furthermore demonstrated by the fact that all models in this article are derived in multi-epoch mode, allowing to incorporate dynamic model constraints on all or subsets of parameters.

Keywords GNSS · Undifferenced model · Network · PPP-RTK theory · $\mathcal{S}$-system theory · Rank-deficient model · Dynamic model

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1 Introduction

The concept of Precise Point Positioning (PPP) has been described first in Heroux and Kouba (1995) and Zumberge et al. (1997). In this technique a user who employs a single GNSS receiver that collects carrier-phase and code (pseudo-range) observations, is able to determine its position with centimeter to decimeter accuracy. For this purpose the user needs precise satellite orbit and clock products, which are obtained from an external provider, such as the International GNSS Service (IGS; Dow et al. 2009). The accuracy that can be achieved with PPP strongly depends on the observation time that is accumulated. Single-frequency GPS PPP, relying on the availability of ionospheric corrections (obtained from, e.g., Global Ionospheric Maps, see Ovstedal 2002), has a positioning accuracy which is typically at the level of few decimeters after about 15 min (Banville et al. 2014). In the absence of such ionospheric corrections, but based on the ionosphere-free combination of dual-frequency observations, the accuracy of GPS PPP can reach centimeter level, but this requires a much longer observation time span (at least one hour, see e.g. Banville et al. 2014).

Although PPP is based on the very precise carrier-phase data (next to the less precise code data), this high precision cannot be exploited as the carrier-phase ambiguities are not estimable as integers. In relative positioning techniques, where use is made of data of a reference station, such as with Real-Time Kinematic (RTK) positioning, the carrier-phase ambiguities are estimable as double differences and can be resolved to integers, allowing for mm–cm level positioning accuracy based on the high precision of the phase observations. Special algorithms have been developed for the crucial integer ambiguity resolution process, of which the LAMBDA method (Teunissen 1995) is nowadays the defacto standard. However, fast (i.e., instantaneous or near-instantaneous) integer ambiguity resolution is only feasible for relatively short (i.e., less than 10 km) distances between rover and reference receiver, based on the assumption that the differential ionospheric delays can be ignored.

During the last decade, several authors have proposed methods for integer ambiguity resolution-enabled PPP. For a critical review, see Teunissen and Khodabandeh (2015). These methods aim to obtain RTK-like positioning accuracy by resolving the phase ambiguities in the single-receiver observations (in addition to the corrections that enable PPP), where the user incorporates information on the satellite phase and code biases (Ge et al. 2008; Collins et al. 2010; Laurichesse et al. 2009; Geng et al. 2011). In this sense PPP can be considered as a relative technique as well, as these orbit, clock and hardware bias corrections are determined by a global or regional network of reference receivers.

The methods described in the literature have restrictions in their applicability, as most of them are restricted to dual-frequency observations and based on ionosphere-free combinations, thereby a priori eliminating the ionospheric delays. Although the method described by Geng and Bock (2013) is suitable to process triple-frequency observations, it is still based on forming ionosphere-free combinations. As with RTK, the forming of ionosphere-free combinations is, however, clearly unfavorable for PPP, as integer ambiguity resolution within reasonably short time spans is not possible (Banville et al. 2014). In addition, methods based on the ionosphere-free combination are not suitable for single-frequency applications.

In this article, instead of forming linear combinations of observables, we will present a network model that is based on the *undifferenced* and *uncombined* observation equations for (multi-frequency) phase and code data, allowing for generation of ionospheric corrections, such that the single-receiver GNSS user applying these corrections does not have to form ionosphere-free combinations. This concept was earlier applied to dual-frequency GPS data in Teunissen et al. (2010), Zhang et al. (2011), Li et al. (2011, 2014) and Odijk et al. (2012). The method presented in this article is, however, suitable for *any* number of frequencies. Our method, comprising a *network* component for the generation of corrections and a *user* component that applies these corrections, is referred to as "PPP-RTK" (conform the terminology of Wübbena et al. 2005), where "RTK" emphasizes the ability of the user to perform integer ambiguity resolution.

The advantages of an undifferenced model formulation have already been recognized for a long time in case of relative GPS positioning (Lindlohr and Wells 1985; Goad 1985; Teunissen 1995; de Jonge 1998; Schönemann et al. 2011; Lannes and Prieur 2013). With an undifferenced formulation one has the advantages of being able to use the simplest observational variance matrix and having all the parameters remain available for a possible further model strengthening. Parameters that are then not considered of interest can then easily be eliminated through the reduction of the normal equations, instead of performing an a priori elimination at the observational level that usually introduces a more complicated structure of the observational variance matrix.

Working with the undifferenced observation equations implies in case of GNSS that one has to account for rank deficiencies as not all unknown parameters can be estimated unbiasedly. We apply $\mathcal{S}$-system theory, originally developed for terrestrial geodetic networks (Baarda 1973; Teunissen 1985), to solve for the rank-deficient system of GNSS observation equations and to allow for a proper interpretation of the estimable *network* and *user* parameters.

We believe that a proper application of this theory is crucial in the near-future multi-frequency, multi-GNSS landscape. In a multi-GNSS landscape one will witness an increase in the number and types of network correction providers (e.g., providing PPP-RTK corrections using local, regional and/or global networks), as well as an increase in diversity of users (e.g., from mass market single-frequency receiver users to high-grade multi-frequency receiver users, using the GNSSs either stand alone or in combined configurations). As the underlying network- and user models are rank-deficient, and even vary in their rank deficiencies depending on the chosen measurement setup and/or modeling (e.g., regional versus global networks; linking or unlinking of ionospheric models; varying choices of dynamic models to capture temporal variations, etc.), a proper understanding of the *estimability* of the computed parameters is essential. Different sets of estimable parameters, each with their own interpretation, exist, and each such set is defined by the chosen *singularity*-basis or $\mathcal{S}$-basis.

For instance, after resolving the rank-deficient GNSS network system of equations, one cannot speak anymore of *the* satellite clock, or *the* satellite phase bias, or *the* receiver code bias. These parameters, although existing in their original physical form, can simply not be estimated as such. What can be estimated are certain functions of the parameters; functions that then can be treated *as if* they are a satellite clock, a satellite phase bias, or a receiver code bias. Many such functions exist; however, one should have a clear understanding of their interpretation when using and/or combining them. One can namely not simply combine or equate such parameters. Thus although there are many different estimable functions that can be treated as if they are, for instance, a satellite clock, this does not mean that they can be set equal. Just like the minimum norm coordinate solutions of two overlapping survey networks cannot be equated in the overlap (Teunissen 1985), also solutions of, for instance, the zero-mean enforced satellite hardware biases (DCBs) cannot be equated directly. They will namely be defined in different $\mathcal{S}$-systems once the zero-mean enforcement is based on another set of satellites.

With a careful application of $\mathcal{S}$-system theory we are in the position to give a clear description of the estimable parameters that are involved in the different network and user models. And by means of the S-transformation, the relation between the original "absolute" parameters and the estimable parameters is then established. Due to the generality of $\mathcal{S}$-system theory, *any* existing or future PPP-RTK model formulation can be cast in this framework, thereby directly providing the interpretation that should be given to the resolved parameters of the chosen formulation. Such interpretation is essential for gaining a proper insight into PPP-RTK in general, and into the role of the PPP-RTK corrections in particular.

Some examples of the theory's applicability to GNSS can be found in de Jonge (1998) and Odijk (2002), while examples for PPP-RTK can be found in Teunissen et al. (2010), Zhang et al. (2011), Lannes and Teunissen (2011), Odijk et al. (2012), Khodabandeh and Teunissen (2014) and Teunissen and Khodabandeh (2015). In these articles, the models were restricted to single- or dual-frequency and a single-epoch ("epoch-by-epoch") formulation. In the current contribution we will use $\mathcal{S}$-system theory to extend these models to a general multi-frequency, multi-epoch formulation, thereby creating a high level of flexibility. The derived models are therefore valid for any multi-frequency GNSS constellation incorporating the Code Division Multiple Access (CDMA) technology, such as GPS, BeiDou or Galileo and future GLONASS as well.

This article is organized as follows. Section 2 reviews $\mathcal{S}$-system theory for a general model formulation. In Sect. 3, the rank deficiencies are identified for the multi-epoch, multi-frequency undifferenced GNSS network model. Based on the identified rank deficiencies, the null space of the network design matrix is constructed. With the potential of a Kalman filter application in mind, we also have temporal constraints on the parameters included in our null-space analysis.

With the identified null space, the choice of S-bases becomes possible. Section 4 presents two different common-clocks S-bases, together with the interpretation of their estimable network parameters. Both S-bases are chosen such that the resulting estimable receiver and satellite clocks become common for all phase and code observables. The S-transformation linking the two sets is given as well. Section 5 addresses the additional rank deficiencies that occur if the assumptions underlying the undifferenced network model change. These changes concern the parametrization of the ionospheric delays, the size of the network, as well as the presence of the temporal constraints on the parameters. The effect of these changes on the null space and on the interpretation of the estimable parameters is discussed.

In Sect. 6, first the network-derived PPP-RTK corrections are discussed, followed by the rank-deficient user model of observation equations. Its null space and estimable parameters are also identified. It shows how the interpretation of the estimable user parameters is inherited from the network via the PPP-RTK corrections. This is then followed by an analysis of the effect on the estimable parameters of the user when certain network corrections are not applied, when the network is of regional size, or when slant instead of vertical ionospheric delays are used in the network. Finally, Sect. 7 contains the summary and conclusions.

2 Review of $\mathcal{S}$-system theory

In this section, we briefly review for our contribution relevant aspects of $\mathcal{S}$-system theory (Baarda 1973; Teunissen 1985).

2.1 Choosing the S-basis

In general notation, a linear(ized) model of observation equations can be given as:

$$\mathsf{E}\{y\} = Ax \qquad (1)$$

Here $\mathsf{E}\{\cdot\}$ denotes the expectation, y the observable vector of dimension m, x the unknown parameter vector of dimension n, and A an $m \times n$ design matrix of rank$(A) = \dim \mathcal{R}(A) = r \leq n$, with $\mathcal{R}(\cdot)$ denoting the range or column space. This means that matrix A is rank-deficient and that this rank deficiency is of size $\dim \mathcal{N}(A) = n - r$. The *null space*, denoted as $\mathcal{N}(\cdot)$, of design matrix A is defined as $\mathcal{N}(A) = \mathcal{R}(V)$, with V an $n \times (n - r)$ matrix as basis of $\mathcal{N}(A)$, such that

$AV = 0$. Although the null space $\mathcal{R}(V)$ is unique, the matrix V containing basis vectors spanning this null space, is *not* unique.

As matrix A is rank-deficient, the observations lack information to solve for *all* unknown parameters. The parameter vector x can then be decomposed into a vector that is *estimable*, denoted as $\tilde{x}_s \in \mathcal{R}(S)$, and an *inestimable part*, denoted as $x_v \in \mathcal{R}(V)$, in which S and V are basis matrices having complementary range spaces, i.e., $\mathbb{R}^n = \mathcal{R}(S) \oplus \mathcal{R}(V)$,

$$x = \underbrace{S\tilde{\alpha}}_{\tilde{x}_s} + \underbrace{V\beta}_{x_v} = \underbrace{[S \;\; V]}_{n\times n}\begin{bmatrix}\tilde{\alpha}\\ \beta\end{bmatrix} \tag{2}$$

Here $\tilde{\alpha}$ denotes the r-vector of estimable parameter functions, corresponding to the $n \times r$ matrix S, whereas β denotes the $(n-r)$-vector of inestimable parameter functions, corresponding to the $n \times (n-r)$ basis matrix of the null space V. It is emphasized that the choice for S is not unique, as there are many (infinite) ways to decompose vector x. The choice of S determines which estimable parameters are solved for. Figure 1 shows two choices of decomposing the parameter vector, resulting in different $\mathcal{S}$-systems, the first denoted as S and the second denoted as S', whereby it holds that $\mathcal{R}(S) \neq \mathcal{R}(S')$.

As matrix $[S \;\; V]$ is a square and invertible matrix of dimension n, the solution for $\tilde{\alpha}$ and β follows as (Teunissen 1985):

$$\begin{bmatrix}\tilde{\alpha}\\ \beta\end{bmatrix} = [S \;\; V]^{-1}x = \begin{bmatrix}\left[(V^{\perp})^T S\right]^{-1}(V^{\perp})^T\\ \left[(S^{\perp})^T V\right]^{-1}(S^{\perp})^T\end{bmatrix}x \tag{3}$$

Here $(\cdot)^T$ denotes the transpose of a vector/matrix. $V^{\perp}$ is the $n \times r$ basis matrix of which the range space is orthogonal to $\mathcal{R}(V)$, i.e., such that $(V^{\perp})^T V = 0$. Vector β is also referred to as *minimum constraints* vector, as it contains the constraints on the parameters that are minimally required to solve for the estimable parameters. The $n \times (n-r)$ matrix $S^{\perp}$ is a basis matrix of the orthogonal complement of $\mathcal{R}(S)$, i.e., $(S^{\perp})^T S = 0$.

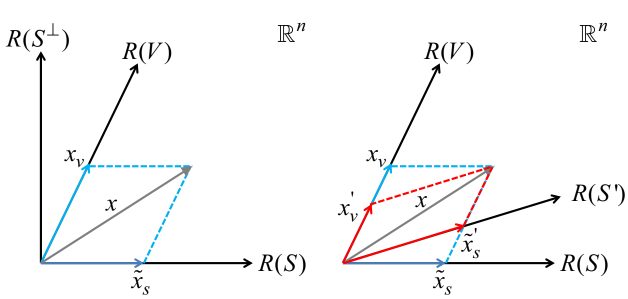

Fig. 1 Two choices of decomposition of parameter vector x: estimable part $\tilde{x}_s$ and inestimable part x_v (*left*), and estimable part $\tilde{x}_s'$ and inestimable part x_v' (*right*)

The *full-rank* model of observation equations, parametrized in $\tilde{\alpha}$, follows now from inserting $x = S\tilde{\alpha} + V\beta$ into the rank-deficient observation model:

$$\mathsf{E}\{y\} = Ax = A(S\tilde{\alpha} + V\beta) = \underbrace{(AS)}_{\tilde{A}}\tilde{\alpha}. \tag{4}$$

Here use is made of $AV = 0$ and $\tilde{A}$ denotes the $m \times r$ design matrix of full rank r.

2.2 S-transformations

Using Eq. (2), the estimable part of parameter vector x can be computed as:

$$\tilde{x}_s = S\tilde{\alpha} = x - V\beta = \mathcal{S}x \tag{5}$$

Here $\mathcal{S}$ denotes the $n \times n$ *S-transformation* matrix, which can be computed, using Eq. (3), as:

$$\mathcal{S} = S[(V^{\perp})^T S]^{-1}(V^{\perp})^T = I_n - V[(S^{\perp})^T V]^{-1}(S^{\perp})^T. \tag{6}$$

It has the following defining properties: $\mathcal{S}V = 0$, $\mathcal{S}S = S$ and $\mathcal{S}\mathcal{S} = \mathcal{S}$ (idempotence). Hence, it is an oblique projector that projects *onto* $\mathcal{R}(S)$ and *along* $\mathcal{R}(V)$ (Teunissen 1985). The rows of $\mathcal{S}$ immediately give the linear functions of the original parameters that are estimable under the S-basis choice.

Another choice of S-basis, e.g., based on matrix S' (see Fig. 1, right), results in another S-transformation matrix:

$$\tilde{x}_s' = S'\tilde{\alpha}' = x - V\beta' = \mathcal{S}'x \tag{7}$$

with:

$$\begin{aligned}\mathcal{S}' &= S'[(V^{\perp})^T S']^{-1}(V^{\perp})^T\\ &= I_n - V[(S'^{\perp})^T V]^{-1}(S'^{\perp})^T.\end{aligned} \tag{8}$$

It is important to realize that solutions for $\tilde{x}_s$ and $\tilde{x}_s'$ cannot be compared directly; after all, they represent different sets of estimable functions. The correct comparison of such solutions can therefore only be done once they are formulated in the same $\mathcal{S}$-system. Since

$$\mathcal{S}\mathcal{S}' = \mathcal{S} \;\text{ and }\; \mathcal{S}'\mathcal{S} = \mathcal{S}', \tag{9}$$

one can transform any solution to the $\mathcal{S}$-system of choice, without the need to know where one transforms from. Thus:

$$\mathcal{S}'\hat{\hat{x}}_s = \hat{\hat{x}}'_s \quad \text{and} \quad \mathcal{S}\hat{\hat{x}}'_s = \hat{\hat{x}}_s \tag{10}$$

with $\hat{\hat{x}}_s$ the least-squares solution in the $\mathcal{S}$-system and $\hat{\hat{x}}'_s$ the least-squares solution in the $\mathcal{S}'$-system. Similarly, we have for their variance matrices:

$$\mathcal{S}' Q_{\hat{\hat{x}}_s\hat{\hat{x}}_s} \mathcal{S}'^T = Q_{\hat{\hat{x}}'_s\hat{\hat{x}}'_s} \quad \text{and} \quad \mathcal{S} Q_{\hat{\hat{x}}'_s\hat{\hat{x}}'_s} \mathcal{S}^T = Q_{\hat{\hat{x}}_s\hat{\hat{x}}_s}. \tag{11}$$

3 Rank-deficient undifferenced GNSS network model and null-space identification

This section presents the multi-epoch, multi-frequency, undifferenced GNSS network model of phase and code observation equations and identifies its rank deficiencies. After that the basis matrix of the null space is constructed, which is needed for the application of $\mathcal{S}$-system theory to the undifferenced GNSS model in Sect. 4.

3.1 GNSS network phase and code observation equations

Starting point are the (linearized) observation equations for multi-frequency phase and code data, that can be given as follows, for an observation epoch i, with $i = 1, \ldots, k$ (Hofmann-Wellenhof et al. 2008; Teunissen and Kleusberg 1998):

$$\begin{aligned}
\mathsf{E}\{\Delta\phi^s_{r,j}(i)\} &= g^s_r(i)^T \Delta x_r(i) + \mathrm{d}t_r(i) + \lambda_j \delta_{r,j}(i) \\
&\quad - \mathrm{d}t^s(i) - \lambda_j \delta^s_j(i) - \mu_j \iota^s_r(i) + \lambda_j z^s_{r,j} \\
\mathsf{E}\{\Delta p^s_{r,j}(i)\} &= g^s_r(i)^T \Delta x_r(i) + \mathrm{d}t_r(i) + d_{r,j}(i) \\
&\quad - \mathrm{d}t^s(i) - d^s_j(i) + \mu_j \iota^s_r(i).
\end{aligned} \tag{12}$$

Here $\Delta\phi^s_{r,j}$ and $\Delta p^s_{r,j}$ denote the undifferenced (though observed-minus-computed) phase and code observables (in meter units). They apply to $r = 1, \ldots, n$ receivers that simultaneously track data of $s = 1, \ldots, m$ satellites, at $j = 1, \ldots, f$ frequencies. The unknown parameters of the GNSS observation Eq. (12) are described below. We remark that the satellite positions are not part of the unknown parameters, as they are computed using precise orbit information. It is furthermore emphasized that it is not an absolute requirement for the application of $\mathcal{S}$-system theory that all n receivers in the network 'see' all m satellites at the same time (this would be impossible, for e.g., a network with globally distributed receivers). However, this assumption is done in this article, as it makes the derivation of the formulas straightforward. More details on the application of the $\mathcal{S}$-system method to networks for which not all receivers track data from all satellites can be found in de Jonge (1998).

The incremental receiver positions, together with zenith tropospheric delay (ZTD) parameters, are contained in the vector $\Delta x_r(i)$. All phase and code observations are a priori corrected for tropospheric delays, for example using Saastamoinen's model (Saastamoinen 1972) or using the empirical GPT (Global Pressure and Temperature) model (Boehm et al. 2007), such that the unknown ZTD parameters become *residual* ZTDs. The vector $\Delta x_r(i)$, together with its coefficient vector $g^s_r(i)$ having the same dimension, is defined as:

$$\Delta x_r(i) = \begin{bmatrix} \Delta b_r(i) \\ \tau_r(i) \end{bmatrix}; \quad g^s_r(i) = \begin{bmatrix} -u^s_r(i) \\ \mathrm{m}^s_r(i) \end{bmatrix}. \tag{13}$$

Here $\Delta b_r(i)$ denotes the three-dimensional (incremental) receiver position vector and $\tau_r(i)$ the ZTD (both in meters). They are multiplied by the receiver–satellite line-of-sight unit vector $u^s_r(i)$ and the tropospheric mapping function coefficient $\mathrm{m}^s_r(i)$. An example of an accurate tropospheric mapping function is given by Niell (1996), or, more recently by Boehm and Schuh (2004). Because of the receiver–satellite geometry appearing in both line-of-sight vector and tropospheric mapping function, the receiver positions and ZTDs are in this article referred to as 'geometry' parameters. For Continuously Operating Reference Station (CORS) networks, the receiver positions are usually fixed to their a priori known values and not estimated in the processing, such that $g^s_r(i) = \mathrm{m}^s_r(i)$ and $\Delta x_r(i) = \tau_r(i)$, i.e., only ZTD parameters are estimated. To flexibly deal with networks that either parameterize receiver positions and/or ZTDs, or networks that keep them fixed, the dimension of vector $\Delta x_r(i)$ is generally denoted as ν. Depending on the type of network, the number of geometry parameters can be either $\nu = 1$ (i.e., positions are not estimated, but a ZTD is estimated per receiver), $\nu = 3$ (only receiver positions are estimated), or $\nu = 4$ (both position and ZTD are parameterized).

The other unknown parameters in Eq. (12) can be listed as follows. Receiver and satellite clock errors are denoted as $\mathrm{d}t_r(i)$ and $\mathrm{d}t^s(i)$, respectively. They are common to both phase and code observation equations. For the phase data the following hardware delays apply: $\delta_{r,j}(i)$, denoting the receiver phase bias and $\delta^s_j(i)$, denoting the satellite phase bias, which are both expressed in cycles. Their counterparts for the code data are $d_{r,j}(i)$ and $d^s_j(i)$, denoting the receiver and satellite code biases, respectively, and expressed in meters. Ionospheric delays (in meters and on the first frequency) are denoted as $\iota^s_r(i)$, having $\mu_j = \lambda^2_j/\lambda^2_1$ as coefficient, with λ_j the wavelength (in meters) corresponding to frequency j. Note that the sign of the ionospheric delays is opposite for phase and code. Finally, $z^s_{r,j}$ denotes the carrier-phase ambiguity which is integer and expressed in cycles. In contrast to all other parameters, note that the ambiguities do not have a time index, as they are time constant (provided that no cycle slips occur).

3.2 Ionosphere-parametric network model

Alternatively, instead of parameterizing the ionospheric delays as *slant* parameters for each receiver–satellite combination as done in Eq. (12), the network model may incorporate a parametric ionosphere model, having fewer parameters than when the ionospheric delays are slant parameters. A simple example of such an ionosphere-parametric model is based on the assumption of a *single layer* for the ionosphere, in which the slant delays corresponding to the same satellite are mapped to one *vertical* ionospheric delay (Schaer 1999):

$$\iota_r^s(i) = \mathrm{f}_r^s(i)\iota^s(i). \tag{14}$$

Here $\mathrm{f}_r^s(i)$ denotes the ionospheric mapping function (which is a function of the zenith angle at the ionospheric piercing point; see Fig. 2) and $\iota^s(i)$ the vertical ionospheric delay parameter corresponding to satellite s. Note that the vertical ionospheric delay does not have a receiver index, which means that it is assumed equal for all receivers in the network with respect to the same satellite. Because of this assumption this simple ionospheric modeling may not be adequate to provide precise ionospheric corrections to users, especially when the distance between the network receivers is large, or in presence of significant horizontal ionospheric gradients. However, this simple ionospheric mapping is used to demonstrate the effect of this ionospheric reparameterization on the rank-deficient network model and the estimability of its parameters, see Sect. 3.3. Later (in Sect. 5.1), we will abandon this assumption and investigate the consequences of a slant ionospheric delay parametrization.

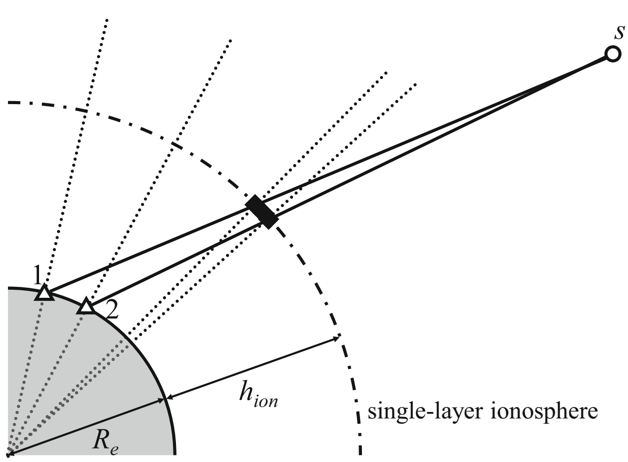

Fig. 2 Geometry of a single-layer ionosphere at a height h_{ion} above the Earth's surface (where R_e denotes the mean Earth's radius). The slant ionospheric delays corresponding to receivers 1 and 2 and the same satellite s are mapped to one common vertical ionospheric delay

3.3 The rank-deficient GNSS network model in presence of random-walk constraints

Our goal is to derive the full-rank, undifferenced GNSS network model and for this we initially assume—for the purpose of rank deficiency identification—that temporal constraints on *all* parameters are incorporated. The ambiguities are assumed to be fully time constant, which holds true as long as no cycle slips occur (in the event of cycle slips it is assumed that these can be repaired, such that their continuity is not disturbed). Afterwards, we will analyze the models and estimability of the parameters in the absence of (some) of these constraints, such that they are *not* connected in time. Receiver clocks for example, are usually treated as unconnected in time, because of their unstable behavior or clock jumps that may occur for some receiver types.

3.3.1 A batch formulation of the Kalman filter setup

The idea of imposing temporal constraints on the parameters is well formulated by the Kalman filter in which the parameters, acting as state vectors, are linked to the observations over time. Despite the recursive nature of the Kalman filter, it is the *batch* formulation of the filter that characterizes the underlying rank deficiency. One therefore needs to form a batch design matrix capturing the structure of the observation equations of all the epochs. To do so, we commence with the measurement model and the dynamic model of the filter that are, respectively, given as:

$$\begin{aligned} \Delta y_i &= A_i\, \Delta x_i + l_i, & i &= 1, \ldots, k \\ \Delta x_i &= \Phi_{i,i-1}\, \Delta x_{i-1} + w_i, & i &= 2, \ldots, k, \end{aligned} \tag{15}$$

where the a priori value x_i^0 is subtracted from the original state vector x_i, giving the state vector $\Delta x_i = x_i - x_i^0$. This in turn makes the original measurement y_i take the form of $\Delta y_i = y_i - A_i x_i^0$, with A_i being the corresponding design matrix. The transition matrix $\Phi_{i,i-1}$ links the state vectors Δx_i to one another over time. The zero-mean measurement and process noises are denoted by l_i and w_i, respectively. Defining the higher-dimensioned vectors $\Delta y = [\Delta y_1^T, \ldots, \Delta y_k^T]^T$ and $\Delta x = [\Delta x_1^T, \ldots, \Delta x_k^T]^T$, a batch formulation for Eq. (15) reads (Teunissen and Khodabandeh 2013)

$$\begin{bmatrix} \mathsf{E}\{\Delta y\} \\ 0 \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix} \mathsf{E}\{\Delta x\} \tag{16}$$

with

$$A = \text{blkdiag}(A_1, \ldots, A_k), \quad B = [B_2^T, \ldots, B_k^T]^T, \tag{17}$$

where $B_i = [0, \ldots, -\Phi_{i,i-1}, I, 0, \ldots, 0]$, $i = 2, \ldots, k$. It is the batch design matrix $[A^T, B^T]^T$ that dictates the rank deficiency underlying the filter.

3.3.2 Random-walk process as special case

In this article, all temporal constraints are modeled as the more commonly used *random-walk* stochastic processes (Chatfield 1994). The choice for a random-walk process is justified by the temporal behavior of the GNSS parameters. During the time span of a satellite's pass in view of a receiver on Earth, from the literature it follows that the temporal behavior of satellite clocks can be modeled as random walk (Herring et al. 1990; Pratt et al. 2013), and this also applies to the behavior of ZTDs (Bar-Sever et al. 1998) (residual ZTD; after correction by an a priori tropospheric model). The receiver and satellite hardware (instrumental) biases are known to slowly vary in time (Wilson and Mannucci 1993). For time spans of 24 h or more, a common assumption is to assume the hardware biases as time constants (Sardon et al. 1994; Komjathy et al. 2005). Otherwise, their temporal behavior can be modeled by a random-walk process, see e.g., Wen et al. (2011). Corrected by an a priori model, time-series of (vertical) ionospheric delays, or its equivalent, the Total Electron Content (TEC), can, to a certain extent, also be assumed as a random-walk process (Wilson and Mannucci 1993).

As stated before, the ambiguities are treated to be time constant, thus no dynamic model needs to be included for them. Let the state vector Δx be further extended as $[\Delta x^T, \Delta z^T]^T$, Δz being the state vector of the ambiguities. Upon the inclusion of the ambiguities, the columns of the design matrix of Eq. (17) are extended as:

$$\begin{bmatrix} A \\ B \end{bmatrix} \mapsto \begin{bmatrix} A & A_z \\ B & 0 \end{bmatrix} \tag{18}$$

with A_z being the corresponding design matrix of the ambiguities.

A random-walk process, in conjunction with the measurement model, follows from Eq. (15) by setting $\Phi_{i,i-1} = I$, with w_i being the (Gaussian) white noise. Under these settings, the structure of matrix B, given in Eq. (16), takes the following form

$$B = D_k^T \otimes I \tag{19}$$

The $(k-1) \times k$ matrix D_k^T is referred to as the between-epoch *differencing* matrix, which contains for each row a "–1" and "+1" (all other entries are zero) to operate the differencing of the states of a certain epoch with respect to those of the *first* epoch, see also Table 1. The matrix Kronecker product is symbolized by ⊗. "Appendix 1" presents its definition and some of its properties. In formulating the network design matrix, for our purpose of rank defect analysis, we make use of the above structure for B, linking the states with respect to the first epoch, see Eq. (20). In a Kalman filter implementation, however, one uses the recursive formulation as given in Eq. (15), linking the states between two *consecutive* epochs.

Table 1 Definition of frequently used scalars, vectors and matrices

Symbol	Definition	Symbol	Definition
k	= # epochs	$e_{(\cdot)}$	$= (1, 1, \ldots, 1)^T$
f	= # frequencies	$I_{(\cdot)}$	$= \text{diag}(1, 1, \ldots, 1)$
n	= # receivers	$D_{(\cdot)}^T$	$= [-e_{(\cdot)-1} \;\; I_{(\cdot)-1}]$
m	= # satellites	μ	$= (\mu_1, \mu_2, \ldots, \mu_f)^T$
ν	= # geometry comp.	Λ	$= \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_f)$
$c_{(\cdot)}$	$= (1, 0, \ldots, 0)^T$	$C_{(\cdot)}$	$= \begin{bmatrix} 0_{1\times\{(\cdot)-1\}} \\ I_{(\cdot)-1} \end{bmatrix}$

Subscript $(\cdot)$ denotes the dimension of the vector/matrix, and diag$(\cdot)$ denotes a diagonal matrix

3.3.3 Network rank-deficient design matrix in batch form

To set up the (rank-deficient) network design matrix, we use the following order of the observables: (1) phase observables, (2) code observables, (3) random-walk constraints. First all (multi-frequency) phase and code observables of the first epoch are collected, then the second epoch, etc., until the last epoch. For each epoch first all observables of the first frequency appear, followed by the second frequency, etc. For each frequency the ordering is first the observables of the first receiver, then of the second receiver, and so forth. And finally, for each receiver the ordering is first the observables of the first satellite, then of the second satellite, etc. After all multi-epoch phase and code data are stored in the above order, the random-walk constraints for the geometry parameters are collected, followed by those for receiver clocks and receiver phase and code hardware biases. Next, the random-walk constraints for satellite clocks and satellite phase and code hardware biases are collected, finally followed by the random-walk constraints for the vertical ionospheric delays.

Based on the above ordering, the GNSS network model's rank-deficient design matrix can be compactly given as follows:

$$A_{\text{net}} = \left[\begin{array}{c|c|c|c|c} F_{\text{geo}} & I_k \otimes \left\{\begin{pmatrix} e_f & \Lambda & 0 \\ e_f & 0 & I_f \end{pmatrix} \otimes (I_n \otimes e_m)\right\} & I_k \otimes \left\{\begin{pmatrix} e_f & \Lambda & 0 \\ e_f & 0 & I_f \end{pmatrix} \otimes (-e_n \otimes I_m)\right\} & F_{\text{ion}} & e_k \otimes \left\{\begin{pmatrix} \Lambda \\ 0 \end{pmatrix} \otimes (I_n \otimes I_m)\right\} \\ \hline D_k^T \otimes (I_n \otimes I_\nu) & 0 & 0 & 0 & 0 \\ 0 & D_k^T \otimes \left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & I_f & 0 \\ 0 & 0 & I_f \end{pmatrix} \otimes I_n\right\} & 0 & 0 & 0 \\ 0 & 0 & D_k^T \otimes \left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & I_f & 0 \\ 0 & 0 & I_f \end{pmatrix} \otimes I_m\right\} & 0 & 0 \\ 0 & 0 & 0 & D_k^T \otimes I_m & 0 \end{array}\right]$$

$$A_{\text{geo}} \quad A_{\text{rec}} \quad A_{\text{sat}} \quad A_{\text{ion}} \quad A_{\text{amb}} \tag{20}$$

Table 1 gives the definition of the vectors and matrices that are used in the above design matrix. Compare the above structure with that of Eq. (18). The above design matrix does not exactly follow Eq. (18). Instead, each partitioned design matrix, e.g. A_{geo}, has the same structure as that of Eq. (18). The reason lies in the way we order the unknown parameters. One can therefore order the unknown parameters in a such way that A_{net} exactly follows the structure of Eq. (18).

The rows above the horizontal line in the design matrix in Eq. (20) correspond to the phase and code and observables, while those beneath correspond to the random-walk constraints. The ordering of the columns that correspond to one parameter group in the above design matrix can be given as follows. The first group of parameters ('geo') is formed by the receiver coordinates (if parameterized) and ZTDs. The second group consists of receiver clocks and receiver hardware biases, referred to as receiver-dependent parameters ('rec'). The third group contains the satellite-dependent clocks and hardware biases ('sat'). The fourth and fifth groups of parameters are the ionospheric delays ('ion') and carrier-phase ambiguities ('amb').

For the geometry parameters the matrix $F_{\text{geo}} = \text{blkdiag}[F_{\text{geo}}(1), \ldots, F_{\text{geo}}(k)]$ is introduced in the design matrix, where blkdiag$(\cdot)$ denotes a block-diagonal matrix in which for every epoch the following $2fnm \times n\nu$ epoch-wise matrix is defined as:

$$F_{\text{geo}}(i) = \begin{pmatrix} e_f \\ e_f \end{pmatrix} \otimes \text{blkdiag}[G_1(i), \ldots, G_n(i)]. \tag{21}$$

Here the $m \times \nu$ matrix G_r is defined as $G_r(i) = [g_r^1(i), \ldots, g_r^m(i)]^T$, with the geometry vectors $g_r^s(i)$ for $s = 1, \ldots, m$ as defined in Eq. (13). Similarly, for the vertical ionospheric parameters $F_{\text{ion}} = \text{blkdiag}[F_{\text{ion}}(1), \ldots, F_{\text{ion}}(k)]$, where the $2fnm \times m$ epoch-wise matrix is defined as:

$$F_{\text{ion}}(i) = \begin{pmatrix} -\mu \\ \mu \end{pmatrix} \otimes \begin{bmatrix} M_1(i) \\ \vdots \\ M_n(i) \end{bmatrix} \tag{22}$$

with $M_r(i) = \text{diag}[\text{f}_r^1(i), \ldots, \text{f}_r^m(i)]$, where $\text{f}_r^s(i)$, $s = 1, \ldots, m$, denotes the ionospheric mapping function, see Eq. (14).

3.4 Identification of rank deficiencies and null space

Given the network's design matrix in Eq. (20), its rank deficiencies can be classified into *five* types. The basis matrix of the null space that corresponds to these five types of rank deficiencies, is denoted as V_{net} and can be constructed as follows:

$$V_{\text{net}} = \left[\begin{array}{c|c|c|c|c} 0 & 0 & 0 & 0 & 0 \\ e_k \otimes \left\{\left(\begin{array}{c|cc} 1 & 0 & 0 \\ 0 & I_f & 0 \\ 0 & 0 & I_f \end{array}\right) \otimes e_n\right\} & e_k \otimes \left\{\begin{pmatrix} -1 \\ \Lambda^{-1} e_f \\ e_f \end{pmatrix} \otimes C_n\right\} & 0 & e_k \otimes \left\{\begin{pmatrix} 0 \\ I_f \\ 0 \end{pmatrix} \otimes -C_n\right\} & 0 \\ e_k \otimes \left\{\left(\begin{array}{c|cc} 1 & 0 & 0 \\ 0 & I_f & 0 \\ 0 & 0 & I_f \end{array}\right) \otimes e_m\right\} & 0 & e_k \otimes \left\{\begin{pmatrix} -1 \\ \Lambda^{-1} e_f \\ e_f \end{pmatrix} \otimes I_m\right\} & 0 & e_k \otimes \left\{\begin{pmatrix} 0 \\ I_f \\ 0 \end{pmatrix} \otimes I_m\right\} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_f \otimes (C_n \otimes e_m) & I_f \otimes (e_n \otimes I_m) \end{array}\right]$$

$$\underbrace{\text{1a 1b}} \quad \text{2a} \quad \text{3a} \quad 4 \quad 5 \tag{23}$$

In the above basis matrix, the $n \times (n-1)$ matrix C_n (see Table 1) can be considered as the n-dimensional identity matrix of which its first column is removed. Note that Λ^{-1} is simply a diagonal matrix containing the inverse wavelengths. The identified basis matrix in Eq. (23) is a valid one, as $A_{\text{net}} V_{\text{net}} = 0$ holds indeed.

The null space matrix V_{net} can be considered to be consisting of five multi-dimensional 'rows' as well as five multi-dimensional 'columns'. Each of these 'rows' represents a group of parameters (i.e., 'geo', 'rec', 'sat', 'ion' and 'amb'), whereas each of the 'columns' [between the vertical bars in Eq. (23)] represents one of the five rank deficiency types. A way of identifying these rank deficiencies is by analyzing which parameter changes leave the observations *invariant*, as was done for dual-frequency observables in Teunissen et al. (2010). For example, the receiver clocks and satellite clocks can both be changed by an identical perturbation that is not "felt" by the phase and code observations, see Eq. (12). This rank deficiency (of size 1) between the receiver and satellite clocks can be eliminated by fixing the receiver clock of one of the receivers (S-basis), such that it is not an unknown parameter anymore. Simple examples of identifying rank deficiencies and constructing the S-basis based on leveling and two-dimensional surveying networks can be found in Teunissen (1985).

The five types of rank deficiencies are identified to occur between the following parameters:

1. *Between receiver- and satellite-dependent parameters.* Given the partial design matrices for the receiver-dependent and satellite-dependent parameters, i.e., A_{rec} and A_{sat}, one can identify two similar types of rank deficiencies:
   - 1a. *Between the receiver and satellite clocks.* This rank deficiency is of size 1.
   - 1b. *Between the receiver and satellite hardware biases.* This rank deficiency is of size $2f$.

2a. *Between receiver clocks and receiver hardware biases.* Another type of rank deficiency, which is of size $n-1$, can be identified between the receiver clocks, the receiver phase and code biases.

3a. *Between satellite clocks and satellite hardware biases.* Similar to the previous type of rank deficiency, there is a rank deficiency (of size m) between the satellite clocks and satellite phase and code biases.

4. *Between receiver hardware biases and ambiguities.* The columns corresponding to the receiver phase biases for a certain frequency, are rank-deficient in combination with the columns of the ambiguities for the same frequency. This rank deficiency is of size $f(n-1)$.
5. *Between satellite hardware biases and ambiguities.* This rank deficiency, of size fm, is the satellite counterpart of the type 4 rank deficiency and applies to the satellite phase biases and ambiguities.

The second and third type of rank deficiency are denoted as '2a' and '3a'. In Sect. 5.1, some additional rank deficiencies are discussed that are closely related to these rank deficiencies and are hence denoted as types '2b' and '3b'. From the null space matrix in Eq. (23) it can also be seen that the rows corresponding to the receiver positions/ZTDs ("1st row") as well as the vertical ionospheric delays ("4th row") are zero, as these parameters are not involved in any of the above rank deficiencies. These parameters are therefore estimable (for this model based on its underlying assumptions) in an absolute, uncombined mode.

The total size of the rank deficiency of the network model, in presence of the temporal constraints on all parameters, can be given as sum of the identified rank deficiencies under the above points 1–5:

$$\#\text{ rank deficiencies} = 1 + 2f + (1+f)(n-1+m). \quad (24)$$

This number thus equals the number of columns of V_{net} in Eq. (23).

In case of relatively short time spans, or short sampling-intervals between the observations, it may be reasonable to keep some parameters as *time constants* in the model. In that case the above rank deficiencies do not alter. Additional rank deficiencies occur, however, not only if the ionospheric delays as modeled as slant parameters (see Sect. 5.1), but also for networks of regional size (see Sect. 5.2), or in the absence of temporal constraints on some or all of the time-varying parameters (see Sect. 5.3).

4 The Common Clocks undifferenced network model

As was discussed in Sect. 2, the choice of the set of network parameters to constrain as S-basis is not unique; there are many choices possible. In this section, we present the functions of estimable parameters for the network model based on two different S-bases. Both can be considered as members of the family of so-called *Common Clocks* (CC) $\mathcal{S}$-systems, which were first described by de Jonge (1998). The S-basis constraints are chosen in such a way, that the resulting estimable receiver and satellite clocks are parameters that are common for all phase and code observables in the model.

4.1 CC-R S-basis vs. CC-S S-basis

To deal with the rank deficiencies that have been identified in the previous section, one has various choices. As a first choice, we follow the approach that was first introduced in

Table 2 CC-R and CC-S S-basis constraints for the network model, together with the types of rank deficiencies (rd #) they eliminate, plus number

CC-R S-basis constraints	rd #	Notation	Number	CC-S S-basis constraints	rd #	Notation	Number
Pivot receiver clock	1a	$dt_1(1)$	1	Mean satellite clock	1a	$\overline{dt}(1)$	1
Pivot receiver hardware biases	1b	$\begin{cases} \delta_{1,j}(1) \\ d_{1,j}(1), \end{cases} j \geq 1$	$2f$	Mean satellite hardware biases	1b	$\begin{cases} \bar{\delta}_j(1) \\ \bar{d}_j(1), \end{cases} j \geq 1$	$2f$
Ionosphere-free receiver code biases	2a	$d_{r,\mathrm{IF}}(1),\ r \geq 2$	$n-1$	Ionosphere-free receiver code biases	2a	$d_{r,\mathrm{IF}}(1),\ r \geq 1$	n
Ionosphere-free satellite code biases	3a	$d_{\mathrm{IF}}^s(1),\ s \geq 1$	m	Ionosphere-free satellite code biases	3a	$d_{\mathrm{IF}}^s(1),\ s \geq 2$	$m-1$
Phase ambiguities of pivot satellite	4	$z_{r,j}^1,\ r \geq 2,\ j \geq 1$	$f(n-1)$	Mean phase ambiguities	4	$\bar{z}_{r,j},\ r \geq 1,\ j \geq 1$	fn
Phase ambiguities of pivot receiver	5	$z_{1,j}^s,\ s \geq 1,\ j \geq 1$	fm	Phase ambiguities of pivot receiver	5	$z_{1,j}^s,\ s \geq 2,\ j \geq 1$	$f(m-1)$

Note I: $d_{r,\mathrm{IF}}(1) = \frac{\mu_2}{\mu_2-\mu_1} d_{r,1}(1) - \frac{\mu_1}{\mu_2-\mu_1} d_{r,2}(1)$ and $d_{\mathrm{IF}}^s(1) = \frac{\mu_2}{\mu_2-\mu_1} d_1^s(1) - \frac{\mu_1}{\mu_2-\mu_1} d_2^s(1)$

Note II: $\overline{(\cdot)} = \frac{1}{m}\sum_{s=1}^m (\cdot^s)$ denotes the satellite mean

Zhang et al. (2011) (albeit in single-epoch mode) and later used in Odijk et al. (2012) as well. In this first approach the idea to constrain the parameters of one of the receiver's in the network to overcome the rank deficiencies, the so-called *pivot* receiver. This S-basis will be referred to as the *Common Clocks (pivot) Receiver* (CC-R) S-basis and in Table 2 (left) the S-basis constraints plus their associated rank deficiencies are given. This S-basis is referred to as *Common Clocks version 1* in Teunissen and Khodabandeh (2015).

Another choice of S-basis is not to constrain parameters corresponding to a certain receiver, but to constrain a certain linear combination, such as the *mean*, of satellite-dependent parameters. This choice of S-basis, which results in estimable satellite-dependent parameters that have a zero mean, will be referred to as the *Common Clocks (mean) Satellite* (CC-S) S-basis. The chosen constraints corresponding to this S-basis are given in the right part of Table 2. In this context, one should realize that the averaging of satellite clocks as well as hardware biases considers the satellites that are tracked at the first epoch. Thus, the S-basis changes (as well as the interpretation of the estimable parameters) the moment the mean is taken at another epoch (or location), based on a *different* set of satellites. Thus, one has to be very careful when parameters are compared that come from different networks that are based on CC-S systems, as the means that are involved may be based on different satellites.

While within the analysis centers of the IGS that generate precise orbit and clock products it is common to fix a clock of one of the receivers (i.e., "clock datum"; Steigenberger et al. 2015), for the generation of code biases the mean of satellite code biases is fixed (Schaer 1999; Montenbruck and Hauschild 2013), resulting in estimable (differential) code biases having a *zero* mean.

Comparing both approaches, whereas the CC-R S-basis constrains the clock, as well as phase and code hardware bias corresponding to the pivot receiver, the CC-S S-basis constrains the mean of the satellite clocks as well as satellite hardware biases, all at the first epoch, to overcome rank deficiency types 1a and 1b. The pivot receiver is here the first receiver conform the ordering in the vector of unknowns. Concerning the rank deficiencies of types 2a and 3a, in both approaches some *ionosphere-free*, dual-frequency, combinations of receiver and satellite code hardware bias parameters are constrained, i.e., $d_{r,\mathrm{IF}}(1)$ and $d_{\mathrm{IF}}^s(1)$; see Table 2. The reason for this is that this will result in estimable parameters that contain ionosphere-free combinations of parameters, making our approaches comparable to the IGS that provides satellite clocks that are ionosphere-free as well (Kouba and Heroux 2001). In terms of the ionospheric coefficients contained in vector μ (see Table 1), the ionosphere-free combination of the first two of f frequencies is represented by the following f-vector (Teunissen and Khodabandeh 2015),

$$\mu_{\mathrm{IF}} = \frac{1}{\mu_2-\mu_1}[\mu_2, -\mu_1, 0, \ldots, 0]^T. \tag{25}$$

Thus, $d_{\mathrm{IF}}^s(i) = \mu_{\mathrm{IF}}^T [d_1^s(i), \ldots, d_f^s(i)]^T = \frac{1}{\mu_2-\mu_1}[\mu_2 d_1^s(i) - \mu_1 d_2^s(i)]$ (similar for the receiver-dependent biases). For GPS, the well-known ionosphere-free vector reads $\mu_{\mathrm{IF}} = (2.5457, -1.5457, 0, \ldots, 0)^T$. The following properties apply to this ionosphere-free vector:

$$\mu_{\mathrm{IF}}^T e_f = 1 \quad \text{and} \quad \mu_{\mathrm{IF}}^T \mu = 0. \tag{26}$$

Thus, the ionosphere-free coefficients add up to 1, whereas they are nullified if multiplied by the ionospheric coefficient vector.

Note from Table 2 that there is a slight difference in ionosphere-free S-basis constraints in both CC-R and CC-S approaches. In case of the CC-R S-basis the ionosphere-free code bias of the pivot receiver is not included, as it can already be computed from the constrained pivot receiver code biases to eliminate rank deficiency 1b. In case of the CC-S S-basis the ionosphere-free code bias of the first (pivot) satellite is not included, as it is (indirectly) present in the mean satellite code bias to eliminate rank deficiency 1b.

To overcome the rank deficiencies of types 4 and 5 that are associated with the ambiguities, in the CC-R approach the ambiguities corresponding to the first (pivot) satellite are constrained, as well as the ambiguities of the pivot receiver. In case of the CC-S approach the mean of the ambiguities per receiver are constrained (conform the mean satellite clock and hardware biases), plus the ambiguities of the pivot receiver.

Considering the total number of S-basis constraints for the two choices of S-basis in Table 2, note that they exactly add up to the size of rank deficiency in the model, which is given in Eq. (24).

4.2 Estimable network parameters

Using the null space matrix V_{net} in Eq. (23), for the two choices of S-basis (see Table 2) in "Appendix 2" the corresponding S-basis ($S^{\perp}$) matrices as well as S-transformation ($\mathcal{S}$) matrices are derived in an analytical manner. The rows of an S-transformation matrix identify the linear combinations of parameters that are estimable in the corresponding $\mathcal{S}$-system, and these are presented in Table 3 for the CC-R S-basis and in Table 4 for the CC-S S-basis. To discriminate the *estimable* parameters from their *original* counterparts, they are denoted using a *single* tilde on top of the estimable parameters of the CC-R system and using a *double* tilde on top of the estimable parameters of the CC-S system. Concerning the units of the estimable parameters, we remark that the position and ZTD, receiver and satellite clock, receiver and satellite code bias, as well as ionospheric parameters are expressed in *meters*, while the estimable phase-specific parameters, i.e., receiver and satellite phase biases, as well as ambiguities, are expressed in *cycles*.

From Tables 3 and 4 it follows that the estimable position/ZTD parameters, as well as the estimable ionospheric parameters, are identical to their original parameters. This is because these parameters were not involved in any of the rank deficiencies. All other estimable parameter types are combinations or *(linear) functions* of the original parameters. For example, in case of the CC-R S-basis, the estimable satellite clock parameter, i.e., $d\tilde{t}^s(i)$, is a function of the true satellite clock $dt^s(i)$, combined with the ionosphere-free satellite code bias $d^s_{\text{IF}}(1)$, as well as the pivot receiver clock $dt_1(1)$ and the ionosphere-free code bias of the pivot receiver $d_{1,\text{IF}}(1)$. In case of the CC-S S-basis, the estimable satellite clock parameter, i.e., $d\tilde{\tilde{t}}^s(i)$, is another linear function, because of the differences in the CC-R and CC-S S-bases. Like in the CC-R system it is function of the true satellite clock $dt^s(i)$, combined with the ionosphere-free satellite code bias $d^s_{\text{IF}}(1)$, but instead of parameters corresponding to the pivot receiver the mean of satellite clocks ($\overline{dt}(1)$) and ionosphere-free satellite code biases ($\overline{d}_{\text{IF}}(1)$) now shows up. The interpretation of the estimable parameters put *conditions* on their existence; the receiver clock in the CC-R system is (at the first epoch) not estimable for the pivot receiver, while in case of the CC-S system the satellite clock is (for the first epoch) not estimable for one of the satellites (in Table 4 this is the first or pivot satellite). This inestimability for the first epoch is because the random-walk constraints on the parameters only play a role from the second epoch onwards, resulting in estimable receiver-dependent parameters for *all* receivers under the CC-R S-basis and estimable satellite-dependent parameters for *all* satellites under the CC-S S-basis from the second

Table 3 Estimable network parameters plus their interpretation using the CC-R S-basis

Estimable parameter	Notation and interpretation	Conditions
Receiver position+ZTD	$\Delta\tilde{x}_r(i) = \Delta x_r(i)$	$i \geq 1,\ r \geq 1$
Receiver clock	$d\tilde{t}_r(i) = \left[dt_r(i) + d_{r,\text{IF}}(1)\right] - \left[dt_1(1) + d_{1,\text{IF}}(1)\right]$	$i = 1:\ r \geq 2$ $i \geq 2:\ r \geq 1$
Receiver phase bias	$\tilde{\delta}_{r,j}(i) = \left[\delta_{r,j}(i) - \frac{1}{\lambda_j} d_{r,\text{IF}}(1)\right] - \left[\delta_{1,j}(1) - \frac{1}{\lambda_j} d_{1,\text{IF}}(1)\right] + \left[z^1_{r,j} - z^1_{1,j}\right]$	$i = 1:\ r \geq 2,\ j \geq 1$ $i \geq 2:\ r \geq 1,\ j \geq 1$
Receiver code bias	$\tilde{d}_{r,j}(i) = \begin{cases} d_{r,\text{GF}}(1) - d_{1,\text{GF}}(1), & \text{for } i = 1 \text{ and } j = 2^{\text{a}} \\ \left[d_{r,j}(i) - d_{r,\text{IF}}(1)\right] - \left[d_{1,j}(1) - d_{1,\text{IF}}(1)\right], & \text{otherwise} \end{cases}$	$i = 1:\ r \geq 2,\ j \geq 2$ $i \geq 2:\ r \geq 1,\ j \geq 1$
Satellite clock	$d\tilde{t}^s(i) = \left[dt^s(i) + d^s_{\text{IF}}(1)\right] - \left[dt_1(1) + d_{1,\text{IF}}(1)\right]$	$i \geq 1,\ s \geq 1$
Satellite phase bias	$\tilde{\delta}^s_j(i) = \left[\delta^s_j(i) - \frac{1}{\lambda_j} d^s_{\text{IF}}(1)\right] - \left[\delta_{1,j}(1) - \frac{1}{\lambda_j} d_{1,\text{IF}}(1)\right] - z^s_{1,j}$	$i \geq 1,\ s \geq 1,\ j \geq 1$
Satellite code bias	$\tilde{d}^s_j(i) = \begin{cases} d^s_{\text{GF}}(1) - d_{1,\text{GF}}(1), & \text{for } i = 1 \text{ and } j = 2^{\text{a}} \\ \left[d^s_j(i) - d^s_{\text{IF}}(1)\right] - \left[d_{1,j}(1) - d_{1,\text{IF}}(1)\right], & \text{otherwise} \end{cases}$	$i = 1:\ s \geq 1,\ j \geq 2$ $i \geq 2:\ s \geq 1,\ j \geq 1$
Ionospheric delay	$\tilde{\iota}^s(i) = \iota^s(i)$	$i \geq 1,\ s \geq 1$
Phase ambiguity	$\tilde{z}^s_{r,j} = [z^s_{r,j} - z^1_{r,j}] - [z^s_{1,j} - z^1_{1,j}]$	$r \geq 2,\ s \geq 2,\ j \geq 1$

Table 4 Estimable network parameters plus their interpretation using the CC-S S-basis

Estimable parameter	Notation and interpretation	Conditions
Receiver position + ZTD	$\Delta\tilde{\tilde{x}}_r(i) = \Delta x_r(i)$	$i \geq 1,\ r \geq 1$
Receiver clock	$\mathrm{d}\tilde{\tilde{t}}_r(i) = \left[\mathrm{d}t_r(i) + d_{r,\mathrm{IF}}(1)\right] - \left[\overline{\mathrm{d}t}(1) + \overline{d}_{\mathrm{IF}}(1)\right]$	$i \geq 1 : r \geq 1$
Receiver phase bias	$\tilde{\tilde{\delta}}_{r,j}(i) = \left[\delta_{r,j}(i) - \frac{1}{\lambda_j} d_{r,\mathrm{IF}}(1)\right] - \left[\overline{\delta}_j(1) - \frac{1}{\lambda_j}\overline{d}_{\mathrm{IF}}(1)\right] + \overline{z}_{r,j}$	$i = 1 : r \geq 1,\ j \geq 1$
Receiver code bias	$\tilde{\tilde{d}}_{r,j}(i) = \begin{cases} d_{r,\mathrm{GF}}(1) - \overline{d}_{\mathrm{GF}}(1), & \text{for } i = 1 \text{ and } j = 2^{\mathrm{a}} \\ \left[d_{r,j}(i) - d_{r,\mathrm{IF}}(1)\right] - \left[\overline{d}_j(1) - \overline{d}_{\mathrm{IF}}(1)\right], & \text{otherwise} \end{cases}$	$i = 1 : r \geq 1,\ j \geq 2$ $i \geq 2 : r \geq 1,\ j \geq 1$
Satellite clock	$\mathrm{d}\tilde{\tilde{t}}^s(i) = \left[\mathrm{d}t^s(i) + d^s_{\mathrm{IF}}(1)\right] - \left[\overline{\mathrm{d}t}(1) + \overline{d}_{\mathrm{IF}}(1)\right]$	$i = 1 : s \geq 2$ $i \geq 2 : s \geq 1$
Satellite phase bias	$\tilde{\tilde{\delta}}^s_j(i) = \left[\delta^s_j(i) - \frac{1}{\lambda_j} d^s_{\mathrm{IF}}(1)\right] - \left[\overline{\delta}_j(1) - \frac{1}{\lambda_j}\overline{d}_{\mathrm{IF}}(1)\right] - \left[z^s_{1,j} - \overline{z}_{1,j}\right]$	$i = 1 : s \geq 2,\ j \geq 1$ $i \geq 2 : s \geq 1,\ j \geq 1$
Satellite code bias	$\tilde{\tilde{d}}^s_j(i) = \begin{cases} d^s_{\mathrm{GF}}(1) - \overline{d}_{\mathrm{GF}}(1), & \text{for } i = 1 \text{ and } j = 2^{\mathrm{a}} \\ \left[d^s_j(i) - d^s_{\mathrm{IF}}(1)\right] - \left[\overline{d}_j(1) - \overline{d}_{\mathrm{IF}}(1)\right], & \text{otherwise} \end{cases}$	$i = 1 : s \geq 2,\ j \geq 2$ $i \geq 2 : s \geq 1,\ j \geq 1$
Ionospheric delay	$\tilde{\tilde{\iota}}^s(i) = \iota^s(i)$	$i \geq 1,\ s \geq 1$
Phase ambiguity	$\tilde{\tilde{z}}^s_{r,j} = [z^s_{r,j} - \overline{z}_{r,j}] - [z^s_{1,j} - \overline{z}_{1,j}]$	$r \geq 2,\ s \geq 2,\ j \geq 1$

"IF" may be replaced by one of the individual frequencies (i.e., "1" or "2"), since the parameters on *both* frequencies are not part of the S-basis. "IF" (Ionosphere-Free) is used though to make the link with the IGS clocks

$d_{r,\mathrm{GF}}(i) = \frac{1}{\mu_2 - \mu_1}[d_{r,2}(i) - d_{r,1}(i)]$, $d^s_{\mathrm{GF}}(i) = \frac{1}{\mu_2 - \mu_1}[d^s_2(i) - d^s_1(i)]$

[a] Only one estimable parameter, which applies to the observations of the first two frequencies

epoch onwards. The most right column of Tables 3 and 4 gives the conditions on the existence of the estimable parameters. Note that in case of the CC-S S-basis the estimable satellite clock (at the first epoch) has zero mean indeed, since

$$\frac{1}{m}\sum_{s=1}^{m} \mathrm{d}\tilde{\tilde{t}}^s(1) = \frac{1}{m}\sum_{s=1}^{m}\left\{\left[\mathrm{d}t^s(1) + d^s_{\mathrm{IF}}(1)\right] - \left[\overline{\mathrm{d}t}(1) + \overline{d}_{\mathrm{IF}}(1)\right]\right\} = 0. \quad (27)$$

This zero-mean property also applies to the estimable satellite phase and code bias parameters, as well the ambiguities under the CC-S S-basis (for the first epoch).

Concerning the estimable receiver and satellite code biases, note from Tables 3 and 4 that in both $\mathcal{S}$-systems a combination of the code bias on a certain frequency minus the ionosphere-free code bias combination appears. This leads to a special situation for the first epoch ($i = 1$) and the first two frequencies ($j = 1, 2$). For example, the estimable satellite code bias parameter can generally be given as, in the CC-R system:

$$\tilde{d}^s_j(i) = [d^s_j(i) - d^s_{\mathrm{IF}}(1)] - [d_{1,j}(1) - d_{1,\mathrm{IF}}(1)]. \quad (28)$$

However, for the *first two frequencies* the code biases can be given as:

$$\left.\begin{aligned} d^s_j(i) &= \mu_j d^s_{\mathrm{GF}}(i) + d^s_{\mathrm{IF}}(i) \\ d_{r,j}(i) &= \mu_j d_{r,\mathrm{GF}}(i) + d_{r,\mathrm{IF}}(i) \end{aligned}\right\} \quad \text{for} \quad j = 1, 2 \quad (29)$$

with $d^s_{\mathrm{GF}}(i)$ and $d_{r,\mathrm{GF}}(i)$ the so-called *geometry-free* combination of dual-frequency code biases, in terms of the f-dimensional coefficient vector (Teunissen and Khodabandeh 2015),

$$\mu_{\mathrm{GF}} = \tfrac{1}{\mu_2 - \mu_1}[-1, 1, 0, \ldots, 0]^T. \quad (30)$$

Thus, $d^s_{\mathrm{GF}}(i) = \mu^T_{\mathrm{GF}}[d^s_1(i), \ldots, d^s_f(i)]^T = -\frac{1}{\mu_2 - \mu_1}[d^s_1(i) - d^s_2(i)]$ (similar for the receiver-dependent biases). For GPS, the well-known geometry-free vector reads $\mu_{\mathrm{GF}} = (-1.5457, 1.5457, 0, \ldots, 0)^T$. It has the following properties

$$\mu^T_{\mathrm{GF}} e_f = 0 \quad \text{and} \quad \mu^T_{\mathrm{GF}}\mu = 1. \quad (31)$$

Thus, the geometry-free coefficients are nullified when added up, whereas multiplied by the ionosphere coefficient vector results in 1. Note that these properties are exactly opposite to those of the ionosphere-free coefficient vector, see Eq. (26).

This leads to the following estimable parameters for the first two frequencies at the first epoch:

$$\tilde{d}^s_j(1) = \mu_j[d^s_{\mathrm{GF}}(1) - d_{1,\mathrm{GF}}(1)], \quad j = 1, 2. \quad (32)$$

From the above expression follows that the code bias of the second frequency is a *scaled version* of the bias on the first frequency, i.e., $\tilde{d}^s_2(1) = \frac{\mu_2}{\mu_1}\tilde{d}^s_1(1)$. This means that only *one* common code bias parameter is estimable for both frequencies. For both CC-R and CC-S systems we opt to parameterize $d^s_{\mathrm{GF}}(1)$ (plus bias depending on the S-basis;

see Tables 3 and 4) for both frequencies, where the coefficients μ_1 and μ_2 are modeled in the design matrix (and are not included in the estimable parameters). In this context, it is noted that this geometry-free combination of satellite code biases is a scaled version of the *Differential Code Bias* (DCB) between the first two frequencies, i.e., $d^s_{\mathrm{GF}}(1) = -\frac{1}{\mu_2-\mu_1}\,\mathrm{DCB}^s_{12}(1)$, with the DCB defined as $\mathrm{DCB}^s_{12}(1) = d^s_1(1) - d^s_2(1)$ (Schaer 1999).

It is also possible to express the estimable code biases at the first epoch for higher frequencies (i.e., $j \geq 3$) as function of the geometry-free code bias corresponding to the first two frequencies:

$$\tilde{d}^s_j(1) = \left[d^s_j(1) - d^s_1(1) + \mu_1 d^s_{\mathrm{GF}}(1)\right] - \left[d_{1,j}(1) - d_{1,1}(1) + \mu_1 d_{1,\mathrm{GF}}(1)\right], \quad j \geq 3. \quad (33)$$

This means that for the third and higher frequencies "modernized" DCBs become estimable, i.e., $d^s_1(1) - d^s_j(1)$, by taking the difference between the (scaled) estimable $d^s_{\mathrm{GF}}(1)$ and $\tilde{d}^s_j(1)$ parameters. This estimability of the modernized DCBs (as well as the traditional DCB) is a direct consequence of the single-layer ionosphere assumption, see Eq. (14). In the absence of this assumption they become inestimable, see Sect. 5.1.

The choice of constraining ambiguities of both pivot receiver and pivot satellite in the CC-R S-basis has as consequence that the estimable ambiguities are *double-differences* and can thus indeed be estimated as integers. Here we mention that in case of a global network, in which not all receivers have the same satellites in view, it is needed to constrain more ambiguities in the S-basis than those corresponding to one receiver and one satellite. In that case the estimable double-differenced ambiguities do not necessarily follow by pivoting a single receiver and a single satellite, but then by pivoting a chain of receivers and satellites. For more details, see de Jonge (1998). Integer ambiguity resolution is essential to obtain network parameter estimates having the best possible precision (Teunissen 1995). Although in case of the CC-S S-basis the estimable ambiguities are double-differenced like as well, they are *not* directly estimable as integers because the satellite mean of the ambiguities is involved in the double difference, and this may destroy the integer nature of the ambiguities. However, it is possible to transform these CC-S ambiguities *afterwards*, such that they are guaranteed to be integers, as follows:

$$\tilde{\tilde{z}}^s_{r,j} + \sum_{s=2}^{m} \tilde{\tilde{z}}^s_{r,j} = \left[z^s_{r,j} - z^1_{r,j}\right] - \left[z^s_{1,j} - z^1_{1,j}\right], \quad r \geq 2,\ s \geq 2. \quad (34)$$

Here use is made of the properties that $\sum_{s=1}^{m}(z^s_{r,j} - \bar{z}_{r,j}) = 0$ and $\sum_{s=2}^{m}\tilde{\tilde{z}}^s_{r,j} = [z^1_{1,j} - \bar{z}_{1,j}] - [z^1_{r,j} - \bar{z}_{r,j}]$. These transformed ambiguities are identical to those estimable with the CC-R S-basis. From this follows that the estimable ambiguities under the two $\mathcal{S}$-systems are related as $\tilde{\tilde{z}}^s_{r,j} + \sum_{s=2}^{m}\tilde{\tilde{z}}^s_{r,j} = \tilde{z}^s_{r,j}$.

4.3 Full-rank systems of network observation equations

Based on the estimable functions of parameters in the $\mathcal{S}$-systems, the *full-rank* system of phase and code observation equations is presented in Table 5 for the parameters in the CC-R system and in Table 6 for the parameters in the CC-S system. Note that although the structure of these full-rank observation equations is quite similar to their original counterparts in Eq. (12), the interpretation of the estimable parameters is completely different from the original (uncombined) parameters. Because of the difference in estimability and existence of the parameters between the first and the other epochs, in Tables 5 and 6 their observation equations are treated separately.

In this context, in case of the CC-S S-basis we even distinct between the observation equations corresponding to the *first* satellite and all other satellites, for the first epoch. This

Table 5 Full-rank system of network multi-frequency phase and code observation equations for $r \geq 1$ and $s \geq 1$, based on the CC-R S-basis

$i = 1$	$\mathsf{E}\{\Delta\phi^s_{r,j}(1)\} = g^s_r(1)^T\Delta\tilde{x}_r(1) \underbrace{+d\tilde{t}_r(1)}_{r\geq 2} \underbrace{+\lambda_j\tilde{\delta}_{r,j}(1)}_{r\geq 2} - d\tilde{t}^s(1) - \lambda_j\tilde{\delta}^s_j(1) \underbrace{+\lambda_j\tilde{z}^s_{r,j}}_{r\geq 2, s\geq 2} - \mu_j \mathrm{f}^s_r(1)\tilde{\iota}^s(1)$	$j \geq 1$
	$\mathsf{E}\{\Delta p^s_{r,j}(1)\} = g^s_r(1)^T\Delta\tilde{x}_r(1) \underbrace{+d\tilde{t}_r(1)}_{r\geq 2} \underbrace{+\mu_j\tilde{d}_{r,2}(1)}_{r\geq 2} - d\tilde{t}^s(1) - \mu_j\tilde{d}^s_1(1) + \mu_j \mathrm{f}^s_r(1)\tilde{\iota}^s(1)$	$j = 1, 2$
	$\mathsf{E}\{\Delta p^s_{r,j}(1)\} = g^s_r(1)^T\Delta\tilde{x}_r(1) \underbrace{+d\tilde{t}_r(1)}_{r\geq 2} \underbrace{+\tilde{d}_{r,j}(1)}_{r\geq 2} - d\tilde{t}^s(1) - \tilde{d}^s_j(1) + \mu_j \mathrm{f}^s_r(1)\tilde{\iota}^s(1)$	$j \geq 3$
$i \geq 2$	$\mathsf{E}\{\Delta\phi^s_{r,j}(i)\} = g^s_r(i)^T\Delta\tilde{x}_r(i) + d\tilde{t}_r(i) + \lambda_j\tilde{\delta}_{r,j}(i) - d\tilde{t}^s(i) - \lambda_j\tilde{\delta}^s_j(i) \underbrace{+\lambda_j\tilde{z}^s_{r,j}}_{r\geq 2, s\geq 2} - \mu_j \mathrm{f}^s_r(i)\tilde{\iota}^s(i)$	$j \geq 1$
	$\mathsf{E}\{\Delta p^s_{r,j}(i)\} = g^s_r(i)^T\Delta\tilde{x}_r(i) + d\tilde{t}_r(i) + \tilde{d}_{r,j}(i) - d\tilde{t}^s(i) - \tilde{d}^s_j(i) + \mu_j \mathrm{f}^s_r(i)\tilde{\iota}^s(i)$	$j \geq 1$

Table 6 Full-rank system of network multi-frequency phase and code observation equations for $r \geq 1$ and $s \geq 1$, based on the CC-S S-basis

$i = 1, s = 1$	$\mathsf{E}\{\Delta\phi^1_{r,j}(1)\} = g^1_r(1)^T \Delta\tilde{\tilde{x}}_r(1) \ + \mathrm{d}\tilde{\tilde{t}}_r(1) \ + \lambda_j \tilde{\tilde{\delta}}_{r,j}(1) \ + \sum_{s=2}^m \mathrm{d}\tilde{\tilde{t}}^s(1) \ + \lambda_j \sum_{s=2}^m \tilde{\tilde{\delta}}^s_j(1) \ \underbrace{-\lambda_j \sum_{s=2}^m \tilde{\tilde{z}}^s_{r,j}}_{r\geq 2} \ -\mu_j \mathrm{f}^1_r(1)\tilde{\tilde{\imath}}^1(1)$	$j \geq 1$
	$\mathsf{E}\{\Delta p^1_{r,j}(1)\} = g^1_r(1)^T \Delta\tilde{\tilde{x}}_r(1) \ + \mathrm{d}\tilde{\tilde{t}}_r(1) \ + \mu_j \tilde{\tilde{d}}_{r,2}(1) \ + \sum_{s=2}^m \mathrm{d}\tilde{\tilde{t}}^s(1) \ + \mu_j \sum_{s=2}^m \tilde{\tilde{d}}^s_1(1) \ + \mu_j \mathrm{f}^1_r(1)\tilde{\tilde{\imath}}^1(1)$	$j = 1, 2$
	$\mathsf{E}\{\Delta p^1_{r,j}(1)\} = g^1_r(1)^T \Delta\tilde{\tilde{x}}_r(1) \ + \mathrm{d}\tilde{\tilde{t}}_r(1) \ + \tilde{\tilde{d}}_{r,j}(1) \ + \sum_{s=2}^m \mathrm{d}\tilde{\tilde{t}}^s(1) \ + \sum_{s=2}^m \tilde{\tilde{d}}^s_j(1) \ + \mu_j \mathrm{f}^1_r(1)\tilde{\tilde{\imath}}^1(1)$	$j \geq 3$
$i = 1, s \geq 2$	$\mathsf{E}\{\Delta\phi^s_{r,j}(1)\} = g^s_r(1)^T \Delta\tilde{\tilde{x}}_r(1) \ + \mathrm{d}\tilde{\tilde{t}}_r(1) \ + \lambda_j \tilde{\tilde{\delta}}_{r,j}(1) \ - \mathrm{d}\tilde{\tilde{t}}^s(1) \ - \lambda_j \tilde{\tilde{\delta}}^s_j(1) \ \underbrace{+\lambda_j \tilde{\tilde{z}}^s_{r,j}}_{r\geq 2} \ -\mu_j \mathrm{f}^s_r(1)\tilde{\tilde{\imath}}^s(1)$	$j \geq 1$
	$\mathsf{E}\{\Delta p^s_{r,j}(1)\} = g^s_r(1)^T \Delta\tilde{\tilde{x}}_r(1) \ + \mathrm{d}\tilde{\tilde{t}}_r(1) + \mu_j \tilde{\tilde{d}}_{r,1}(1) \ - \mathrm{d}\tilde{\tilde{t}}^s(1) \ - \mu_j \tilde{\tilde{d}}^s_1(1) \ + \mu_j \mathrm{f}^s_r(1)\tilde{\tilde{\imath}}^s(1)$	$j = 1, 2$
	$\mathsf{E}\{\Delta p^s_{r,j}(1)\} = g^s_r(1)^T \Delta\tilde{\tilde{x}}_r(1) \ + \mathrm{d}\tilde{\tilde{t}}_r(1) \ + \tilde{\tilde{d}}_{r,j}(1) \ - \mathrm{d}\tilde{\tilde{t}}^s(1) \ - \tilde{\tilde{d}}^s_j(1) \ + \mu_j \mathrm{f}^s_r(1)\tilde{\tilde{\imath}}^s(1)$	$j \geq 3$
$i \geq 2, s \geq 1$	$\mathsf{E}\{\Delta\phi^s_{r,j}(i)\} = g^s_r(i)^T \Delta\tilde{\tilde{x}}_r(i) \ + \mathrm{d}\tilde{\tilde{t}}_r(i) \ + \lambda_j \tilde{\tilde{\delta}}_{r,j}(i) \ - \mathrm{d}\tilde{\tilde{t}}^s(i) \ - \lambda_j \tilde{\tilde{\delta}}^s_j(i) \ \underbrace{+\lambda_j \tilde{\tilde{z}}^s_{r,j}}_{r\geq 2} \ -\mu_j \mathrm{f}^s_r(i)\tilde{\tilde{\imath}}^s(i)$	$j \geq 1$
	$\mathsf{E}\{\Delta p^s_{r,j}(i)\} = g^s_r(i)^T \Delta\tilde{\tilde{x}}_r(i) \ + \mathrm{d}\tilde{\tilde{t}}_r(i) \ + \tilde{\tilde{d}}_{r,j}(i) \ - \mathrm{d}\tilde{\tilde{t}}^s(i) \ - \tilde{\tilde{d}}^s_j(i) \ + \mu_j \mathrm{f}^s_r(i)\tilde{\tilde{\imath}}^s(i)$	$j \geq 1$

is because in this case (i.e., for $i = 1$ and $s = 1$) the satellite-dependent parameters are not estimable as single parameters as is the case for $s \geq 2$, but for the first satellite they are computed from the *sum* of the parameters of all other satellites. For example, the (biased) satellite clock parameter for the first satellite is obtained as follows:

$$-\sum_{s=2}^{m} \mathrm{d}\tilde{\tilde{t}}^s(1) = \left[\mathrm{d}t^1(1) + d^1_{\mathrm{IF}}(1)\right] - \left[\overline{\mathrm{d}t}(1) + \overline{d}_{\mathrm{IF}}(1)\right]. \tag{35}$$

The biased clock for the first satellite is thus equal to the *negative* sum of the clocks of all other satellites, as all clocks should add up to zero. The full-rank design matrix corresponding to the CC-R observation equations in Table 5 is given in Eq. (51), whereas the full-rank design matrix corresponding to the CC-S observation equations in Table 6 is presented in Eq. (56).

4.4 One-to-one transformation between the CC-R and CC-S systems

The total number of estimable parameters is independent of the choice of S-basis. By means of an S-transformation it is possible to directly obtain the estimable parameters from one system into the other and vice versa. The full-rank transformation from the estimable parameters in the CC-S system to their counterparts in the CC-R system is given in Eq. (60), while the inverse transformation is presented in Eq. (61). As the receiver position/ZTD and the ionospheric parameters remain invariant under the S-transformation, they are not included in these transformations.

5 Additional rank deficiencies of the undifferenced network model

This section addresses the additional rank deficiencies that occur if the assumptions underlying the undifferenced network model, as specified in Sect. 3, are changed. These changes concern the parametrization of the ionospheric delays (Sect. 5.1), the size of the network (Sect. 5.2), as well as the presence of the temporal constraints on the parameters (Sect. 5.3).

5.1 Additional rank deficiencies: ionospheric slant delays

In the situation that the network model parameterizes *slant* instead of vertical ionospheric delays, the rank deficiency of the model changes. In the rank-deficient design matrix in Eq. (20) the part corresponding to the ionospheric parameters is replaced by:

$$A_{\mathrm{ion}} = \begin{bmatrix} I_k \otimes \left\{ \begin{pmatrix} -\mu \\ \mu \end{pmatrix} \otimes (I_n \otimes I_m) \right\} \\ 0 \\ 0 \\ 0 \\ D_k^T \otimes (I_n \otimes I_m) \end{bmatrix}. \tag{36}$$

The presence of this alternative partial design matrix in combination with the partial design matrices for the receiver and satellite hardware biases that are inside A_{rec} and A_{sat}, see Eq. (20), causes additional rank deficiencies. In terms of the basis matrix of the null space in Eq. (23), its additional columns to account for these rank deficiencies read:

$$
\left[\begin{array}{c|c}
0 & 0 \\
e_k \otimes \left\{\begin{pmatrix} 0 \\ \Lambda^{-1}\mu \\ -\mu \end{pmatrix} \otimes C_n \right\} & 0 \\
0 & e_k \otimes \left\{\begin{pmatrix} 0 \\ -\Lambda^{-1}\mu \\ \mu \end{pmatrix} \otimes I_m \right\} \\
e_k \otimes (C_n \otimes e_m) & e_k \otimes (e_n \otimes I_m) \\
0 & 0
\end{array}\right]. \quad (37)
$$

$$\text{2b} \qquad\qquad \text{3b}$$

These additional rank deficiencies are categorized as types 2b and 3b and occur between the following parameters:

2b. *Between receiver hardware biases and slant ionospheric delays.* Like type 2a (see Sect. 3.4), the rank deficiency between receiver clocks and receiver hardware biases, this rank deficiency is of size $n - 1$.

3b. *Between satellite hardware biases and slant ionospheric delays.* Like type 3a (see Sect. 3.4), the rank deficiency between satellite clocks and satellite hardware biases, this rank deficiency is of size m.

Whereas the rank deficiencies of types 2a and 3a were, in case of the CC-R and CC-S systems, eliminated by constraining ionosphere-free code biases as part of the S-bases (see Table 2), the $n-1+m$ additional rank deficiencies due to the slant ionospheric delays can be eliminated by constraining the *geometry-free combination* of the receiver and satellite code biases (at the first epoch), see Table 7.

These additional S-basis constraints have as consequence that the estimability and interpretation of the receiver and satellite hardware bias parameters, as well as the ionospheric delay parameters, changes under both CC-R and CC-S systems. All other estimable parameters remain the same. Table 8 presents the affected parameters (denoted using a prime) and shows the change they undergo as consequence of the slant ionospheric delay parametrization. The table also shows how the existence of the parameters changes (see the column 'condition'), but only for the first epoch, as for $i \geq 2$ their existence is not changed with respect to Tables 3 and 4.

An important consequence is that the receiver and satellite code biases are *not* estimable anymore for the first two frequencies (i.e., $j = 1, 2$), as the geometry-free code bias combination that is subtracted in Table 8 is exactly equal to the estimable code bias parameter for the first two frequencies. This means that code biases are only estimable for a (modernized) triple- or higher-frequency constellation (this holds for both CC-R and CC-S systems). As consequence of this model and S-basis, the geometry-free receiver/satellite code biases become lumped to the ionospheric slant delay. The estimable ionospheric delay parameter for both CC-R and CC-S systems is thus biased by the geometry-free receiver and satellite code biases (see Table 8), or, equivalently, the (scaled) receiver and satellite DCBs. This is the same as the well-known lumping of the (receiver and satellite) DCBs to the ionospheric estimates described in the literature (Sardon et al. 1994).

5.2 Additional rank deficiencies: regional-sized networks

So far, there was no rank deficiency identified between the receiver position/ZTD and the other parameters in the network model, resulting in estimable receiver positions and

Table 7 Additional S-basis constraints (CC-R and CC-S) for the network model parameterizing ionospheric slant delays instead of vertical delays

S-basis constraints	rd #	Notation	Number
Geometry-free receiver code biases	2b	$d_{r,\mathrm{GF}}(1)$ $\begin{cases} r \geq 2, \text{ CC-R} \\ r \geq 1, \text{ CC-S} \end{cases}$	$n-1$, CC-R n, CC-S
Geometry-free satellite code biases	3b	$d^s_{\mathrm{GF}}(1)$ $\begin{cases} s \geq 1, \text{ CC-R} \\ s \geq 2, \text{ CC-S} \end{cases}$	m, CC-R $m-1$, CC-S

Table 8 Change of the estimable receiver and satellite hardware bias parameters with respect to Tables 3 and 4 due to the ionospheric slant delays

	Affected estimable CC-R parameters	Condition (1st epoch)	Affected estimable CC-S parameters	Condition (1st epoch)
Receiver hw biases	$\tilde{\delta}_{r,j}(i)' = \tilde{\delta}_{r,j}(i) + \frac{1}{\lambda_j}\mu_j[d_{r,\mathrm{GF}}(1) - d_{1,\mathrm{GF}}(1)]$ $\tilde{d}_{r,j}(i)' = \tilde{d}_{r,j}(i) - \mu_j[d_{r,\mathrm{GF}}(1) - d_{1,\mathrm{GF}}(1)]$	$r \geq 2,\ j \geq 1$ $r \geq 2,\ j \geq 3$	$\tilde{\tilde{\delta}}_{r,j}(i)' = \tilde{\tilde{\delta}}_{r,j}(i) + \frac{1}{\lambda_j}\mu_j[d_{r,\mathrm{GF}}(1) - \bar{d}_{\mathrm{GF}}(1)]$ $\tilde{\tilde{d}}_{r,j}(i)' = \tilde{\tilde{d}}_{r,j}(i) - \mu_j[d_{r,\mathrm{GF}}(1) - \bar{d}_{\mathrm{GF}}(1)]$	$r \geq 1,\ j \geq 1$ $r \geq 1,\ j \geq 3$
Satellite hw biases	$\tilde{\delta}^s_j(i)' = \tilde{\delta}^s_j(i) + \frac{1}{\lambda_j}\mu_j[d^s_{\mathrm{GF}}(1) - d_{1,\mathrm{GF}}(1)]$ $\tilde{d}^s_j(i)' = \tilde{d}^s_j(i) - \mu_j[d^s_{\mathrm{GF}}(1) - d_{1,\mathrm{GF}}(1)]$	$s \geq 1,\ j \geq 1$ $s \geq 1,\ j \geq 3$	$\tilde{\tilde{\delta}}^s_j(i)' = \tilde{\tilde{\delta}}^s_j(i) + \frac{1}{\lambda_j}\mu_j[d^s_{\mathrm{GF}}(1) - \bar{d}_{\mathrm{GF}}(1)]$ $\tilde{\tilde{d}}^s_j(i)' = \tilde{\tilde{d}}^s_j(i) - \mu_j[d^s_{\mathrm{GF}}(1) - \bar{d}_{\mathrm{GF}}(1)]$	$s \geq 2,\ j \geq 1$ $s \geq 2,\ j \geq 3$
Iono slant delays	$\tilde{\iota}^s_r(i)' = \iota^s_r(i) + d_{r,\mathrm{GF}}(1) - d^s_{\mathrm{GF}}(1)$	$r \geq 1,\ s \geq 1$	$\tilde{\iota}^s_r(i)' = \iota^s_r(i) + d_{r,\mathrm{GF}}(1) - d^s_{\mathrm{GF}}(1)$	$r \geq 1,\ s \geq 1$

ZTDs in "absolute" sense (i.e., $\Delta\tilde{x}_r(i)$ in Tables 3 and 4). However, one has to bear in mind that these receiver position parameters are defined in the datum that is fixed by the externally provided satellite positions (e.g., the ITRF).

Moreover, in case the distances between the stations in the network that assumes the satellite positions to be known are limited, the receivers experience more or less *parallel* line-of-sight vectors to the same satellite, as their elevation angles to this satellite are nearly identical. This means that the estimation of "absolute" receiver positions and ZTDs for networks with inter-station distances smaller than about 500 km is not reliable (Rocken et al. 1993). In the limiting case, when it is assumed that the receiver-specific geometry matrices are identical for all receivers in the network, it holds that $g_1^s(i) = \cdots = g_n^s(i) = g^s(i)$, for $s = 1, \ldots, m$. As the ionospheric mapping coefficients also depend on the satellite elevation, it is assumed that $\mathrm{f}_1^s(i) = \cdots = \mathrm{f}_n^s(i) = \mathrm{f}^s(i)$ for receivers not too far apart. Both epoch-specific receiver–satellite geometry matrix $F_{\mathrm{geo}}(i)$ and ionospheric coefficient matrix $F_{\mathrm{ion}}(i)$, see Eqs. (21) and (22), then reduce to:

$$
\begin{aligned}
F_{\mathrm{geo}}(i) &= \begin{pmatrix} e_f \\ e_f \end{pmatrix} \otimes [I_n \otimes G(i)] \\
F_{\mathrm{ion}}(i) &= \begin{pmatrix} -\mu \\ \mu \end{pmatrix} \otimes [e_n \otimes M(i)] \,. \qquad (38)
\end{aligned}
$$

Here $G(i) = [g^1(i), \ldots, g^m(i)]^T$ and $M(i) = \mathrm{diag}[\mathrm{f}^1(i), \ldots, \mathrm{f}^m(i)]$. This so-called "regional network" assumption causes additional rank deficiencies in the following cases:

(i) In the absence of temporal constraints on the receiver positions/ZTDs (geometry) and/or satellite-dependent parameters (including ionospheric parameters);
(ii) In case of a time-constant receiver–satellite geometry;
(iii) In case of a single-epoch model.

In the absence of temporal constraints on geometry parameters and/or satellite-dependent parameters, the following additional rank deficiencies occur:

0a/b. *Between receiver positions/ZTDs and satellite clocks or satellite hardware biases.* This rank deficiency is of size ν and occurs in *presence* of temporal constraints on the geometry parameters (receiver positions/ZTDs), but in *absence* of constraints on either the satellite clocks or satellite hardware biases:

- 0a. *Between receiver positions/ZTDs and satellite clocks,* in the absence of the satellite clock constraints.
- 0b. *Between receiver positions/ZTDs and satellite hardware biases,* in the absence of the satellite hardware bias constraints.

Above types of rank deficiency only occur in case of regional-sized networks and are categorized as a "type 0" rank deficiency. It is overcome by fixing the position/ZTD of one of the receivers in the network as S-basis constraint. Here we constrain the first (pivot) receiver, i.e., $\Delta x_1(1)$. Table 9 shows the estimable parameters under the CC-R and CC-S systems that are changed due to this type 0 rank deficiency. It can be seen that the receiver positions/ZTDs are estimable with respect to the pivot receiver, instead of in "absolute" mode. As a consequence, the satellite-dependent parameters become estimable lumped with the pivot receiver's position + ZTD.

0c. *Between satellite hardware biases and vertical ionospheric delays.* This rank deficiency is of size m and occurs in presence of temporal constraints on the vertical ionospheric parameters, while there are no constraints on the satellite hardware biases. It is overcome by constraining the geometry-free satellite code biases at the first epoch in the S-basis, i.e., $d_{\mathrm{GF}}^s(1)$. As a consequence the estimable ionospheric parameters as well as the estimable satellite hardware bias parameters become lumped by it; see Table 9. This type of rank deficiency is very similar to rank deficiency 3b between the satellite hardware biases and *slant* ionospheric delays (see Sect. 5.1) and hence the satellite code bias at the first epoch is only estimable for more than three frequencies ($j \geq 3$).

The above "type 0" rank deficiency, that only exists under the regional network assumption, is based on the presence of temporal constraints on the geometry parameters. However, in the absence of these constraints, this rank deficiency reoccurs for every epoch, thus in total of size $k\nu$ (this multi-epoch rank deficiency is referred to as type "0(m)"). Also, in the absence of temporal constraints on the vertical ionospheric delays, the type 3b rank deficiency reoccurs for every epoch as well (of size km; referred to as type "3b(m)"). Table 10 presents the additional columns in the basis matrix of the null space in Eq. (23) to account for these additional rank deficiencies.

The previous rank deficiencies of types 0 and 3b occur in a regional network when the random-walk constraints on at least one group of parameters (i.e., receiver positions/ZTDs, satellite clocks, satellite hardware biases or ionospheric delays) are absent. However, additional rank deficiencies can also be identified- even in presence of the random-walk constraints on the mentioned types of parameters—in the case the sampling interval between the observations is so small such that the geometry and ionospheric coefficient matrices are almost the same for subsequent epochs. In the limiting case, assuming a *time-constant* receiver–satellite geometry, $g^s(1) = \cdots = g^s(k) = g^s$ and $\mathrm{f}^s(1) = \cdots = \mathrm{f}^s(k) = \mathrm{f}^s$,

Table 9 Change of the estimable parameters with respect to Table 3 (CC-R S-basis) and Table 4 (CC-S S-basis) for a *regional-sized* network for which additional rank deficiencies occur, of types 0a, 0b, or 0c, depending on the presence/absence of temporal constraints on positions/ZTDs (geometry), satellite clocks, satellite hardware biases and ionospheric delays

rd #	Random-walk constraints	Affected estimable CC-R parameters	Affected estimable CC-S parameters
0a	Rec. position/ZTDs: Y Sat. clocks: N Sat. hardware biases: Y	$\Delta\tilde{x}_r(i)' = \Delta\tilde{x}_r(i) - \Delta x_1(1), \quad \begin{cases} i=1,\ r\geq 2 \\ i\geq 2,\ r\geq 1 \end{cases}$ $\mathrm{d}\tilde{t}^s(i)' = \mathrm{d}\tilde{t}^s(i) - g^s(i)^T\Delta x_1(1)$	$\Delta\tilde{\tilde{x}}_r(i)' = \Delta\tilde{\tilde{x}}_r(i) - \Delta x_1(1), \quad \begin{cases} i=1,\ r\geq 2 \\ i\geq 2,\ r\geq 1 \end{cases}$ $\mathrm{d}\tilde{\tilde{t}}_r(i)' = \mathrm{d}\tilde{\tilde{t}}_r(i) + \overline{g}(1)^T\Delta x_1(1)$ $\mathrm{d}\tilde{\tilde{t}}^s(i)' = \mathrm{d}\tilde{\tilde{t}}^s(i) - [g^s(i) - \overline{g}(1)]^T\Delta x_1(1)$
0b	Rec. position/ZTDs: Y Sat. clocks: Y Sat. hardware biases: N	$\Delta\tilde{x}_r(i)' = \Delta\tilde{x}_r(i) - \Delta x_1(1), \quad \begin{cases} i=1,\ r\geq 2 \\ i\geq 2,\ r\geq 1 \end{cases}$ $\mathrm{d}\tilde{t}^s(i)' = \mathrm{d}\tilde{t}^s(i) - g^s(1)^T\Delta x_1(1)$ $\tilde{\delta}^s_j(i)' = \tilde{\delta}^s_j(i) - \frac{1}{\lambda_j}\left[g^s(i) - g^s(1)\right]^T\Delta x_1(1)$ $\tilde{d}^s_j(i)' = \tilde{d}^s_j(i) - \left[g^s(i) - g^s(1)\right]^T\Delta x_1(1)$	$\Delta\tilde{\tilde{x}}_r(i)' = \Delta\tilde{\tilde{x}}_r(i) - \Delta x_1(1), \quad \begin{cases} i=1,\ r\geq 2 \\ i\geq 2,\ r\geq 1 \end{cases}$ $\mathrm{d}\tilde{\tilde{t}}_r(i)' = \mathrm{d}\tilde{\tilde{t}}_r(i) + \overline{g}(1)^T\Delta x_1(1)$ $\mathrm{d}\tilde{\tilde{t}}^s(i)' = \mathrm{d}\tilde{\tilde{t}}^s(i) - [g^s(1) - \overline{g}(1)]^T\Delta x_1(1)$ $\tilde{\tilde{\delta}}^s_j(i)' = \tilde{\tilde{\delta}}^s_j(i) - \frac{1}{\lambda_j}[g^s(i) - g^s(1)]^T\Delta x_1(1)$ $\tilde{\tilde{d}}^s_j(i)' = \tilde{\tilde{d}}^s_j(i) - [g^s(i) - g^s(1)]^T\Delta x_1(1)$
0c	Ionospheric delays: Y Sat. hardware biases: N	$\tilde{\imath}^s(i)' = \tilde{\imath}^s(i) - \frac{1}{\mathrm{f}^s(1)}[d^s_{\mathrm{GF}}(1) - d_{1,\mathrm{GF}}(1)]$ $\tilde{\delta}^s_j(i)' = \tilde{\delta}^s_j(i) + \frac{1}{\lambda_j}\mu_j\frac{\mathrm{f}^s(i)}{\mathrm{f}^s(1)}[d^s_{\mathrm{GF}}(1) - d_{1,\mathrm{GF}}(1)]$ $\tilde{d}^s_j(i)' = \tilde{d}^s_j(i) - \mu_j\frac{\mathrm{f}^s(i)}{\mathrm{f}^s(1)}[d^s_{\mathrm{GF}}(1) - d_{1,\mathrm{GF}}(1)], \quad \begin{cases} i=1,\ j\geq 3 \\ i\geq 2,\ j\geq 1 \end{cases}$	$\tilde{\tilde{\imath}}^s(i)' = \tilde{\tilde{\imath}}^s(i) - \frac{1}{\mathrm{f}^s(1)}[d^s_{\mathrm{GF}}(1) - \overline{d}_{\mathrm{GF}}(1)]$ $\tilde{\tilde{\delta}}^s_j(i)' = \tilde{\tilde{\delta}}^s_j(i) + \frac{1}{\lambda_j}\mu_j\frac{\mathrm{f}^s(i)}{\mathrm{f}^s(1)}[d^s_{\mathrm{GF}}(1) - \overline{d}_{\mathrm{GF}}(1)]$ $\tilde{\tilde{d}}^s_j(i)' = \tilde{\tilde{d}}^s_j(i) - \mu_j\frac{\mathrm{f}^s(i)}{\mathrm{f}^s(1)}[d^s_{\mathrm{GF}}(1) - \overline{d}_{\mathrm{GF}}(1)], \quad \begin{cases} i=1,\ j\geq 3 \\ i\geq 2,\ j\geq 1 \end{cases}$
0a(m)	Rec. position/ZTDs: N Sat. clocks: N Sat. hardware biases: Y	$\Delta\tilde{x}_r(i)' = \Delta\tilde{x}_r(i) - \Delta x_1(i), \quad i\geq 1,\ r\geq 2$ $\mathrm{d}\tilde{t}^s(i)' = \mathrm{d}\tilde{t}^s(i) - g^s(i)^T\Delta x_1(i)$	$\Delta\tilde{\tilde{x}}_r(i)' = \Delta\tilde{\tilde{x}}_r(i) - \Delta x_1(i), \quad i\geq 1,\ r\geq 2$ $\mathrm{d}\tilde{\tilde{t}}_r(i)' = \mathrm{d}\tilde{\tilde{t}}_r(i) + \overline{g}(1)^T\Delta x_1(1)$ $\mathrm{d}\tilde{\tilde{t}}^s(i)' = \mathrm{d}\tilde{\tilde{t}}^s(i) - [g^s(i)^T\Delta x_1(i) - \overline{g}(1)^T\Delta x_1(1)]$
0b(m)	Rec. position/ZTDs: N Sat. clocks: Y Sat. hardware biases: N	$\Delta\tilde{x}_r(i)' = \Delta\tilde{x}_r(i) - \Delta x_1(i), \quad i\geq 1,\ r\geq 2$ $\mathrm{d}\tilde{t}^s(i)' = \mathrm{d}\tilde{t}^s(i) - g^s(1)^T\Delta x_1(1)$ $\tilde{\delta}^s_j(i)' = \tilde{\delta}^s_j(i) - \frac{1}{\lambda_j}[g^s(i)^T\Delta x_1(i) - g^s(1)^T\Delta x_1(1)]$ $\tilde{d}^s_j(i)' = \tilde{d}^s_j(i) - [g^s(i)^T\Delta x_1(i) - g^s(1)^T\Delta x_1(1)]$	$\Delta\tilde{\tilde{x}}_r(i)' = \Delta\tilde{\tilde{x}}_r(i) - \Delta x_1(i), \quad i\geq 1,\ r\geq 2$ $\mathrm{d}\tilde{\tilde{t}}_r(i)' = \mathrm{d}\tilde{\tilde{t}}_r(i) + \overline{g}(1)^T\Delta x_1(1)$ $\mathrm{d}\tilde{\tilde{t}}^s(i)' = \mathrm{d}\tilde{\tilde{t}}^s(i) - [g^s(1) - \overline{g}(1)]^T\Delta x_1(1)$ $\tilde{\tilde{\delta}}^s_j(i)' = \tilde{\tilde{\delta}}^s_j(i) - \frac{1}{\lambda_j}[g^s(i)^T\Delta x_1(i) - g^s(1)^T\Delta x_1(1)]$ $\tilde{\tilde{d}}^s_j(i)' = \tilde{\tilde{d}}^s_j(i) - [g^s(i)^T\Delta x_1(i) - g^s(1)^T\Delta x_1(1)]$
0c(m)	Ionospheric delays: N Sat. hardware biases: N	$\tilde{\imath}^s(i)' = \tilde{\imath}^s(i) - \frac{1}{\mathrm{f}^s(i)}[d^s_{\mathrm{GF}}(i) - d_{1,\mathrm{GF}}(1)]$ $\tilde{\delta}^s_j(i)' = \tilde{\delta}^s_j(i) + \frac{1}{\lambda_j}\mu_j[d^s_{\mathrm{GF}}(i) - d_{1,\mathrm{GF}}(1)]$ $\tilde{d}^s_j(i)' = \tilde{d}^s_j(i) - \mu_j[d^s_{\mathrm{GF}}(i) - d_{1,\mathrm{GF}}(1)]$	$\tilde{\tilde{\imath}}^s(i)' = \tilde{\tilde{\imath}}^s(i) - \frac{1}{\mathrm{f}^s(i)}[d^s_{\mathrm{GF}}(i) - \overline{d}_{\mathrm{GF}}(1)]$ $\tilde{\tilde{\delta}}^s_j(i)' = \tilde{\tilde{\delta}}^s_j(i) + \frac{1}{\lambda_j}\mu_j[d^s_{\mathrm{GF}}(i) - \overline{d}_{\mathrm{GF}}(1)]$ $\tilde{\tilde{d}}^s_j(i)' = \tilde{\tilde{d}}^s_j(i) - \mu_j[d^s_{\mathrm{GF}}(i) - \overline{d}_{\mathrm{GF}}(1)]$

The rank deficiencies of types 0a(m), 0b(m) and 0c(m) denote the multi-epoch generalizations of rank deficiency types 0a, 0b and 0c, that occur in the absence of the temporal constraints on positions/ZTDs and ionospheric delays. $\overline{g}(1) = \frac{1}{m}\sum_{s=1}^{m} g^s(1)$ is the satellite mean of the LOS direction vectors (at the first epoch; CC-S)

the geometry matrix $F_{\mathrm{geo}}(i)$ and ionospheric coefficient matrix $F_{\mathrm{ion}}(i)$, see Eq. (38), reduce to:

$$F_{\mathrm{geo}}(i) = \begin{pmatrix} e_f \\ e_f \end{pmatrix} \otimes (I_n \otimes G);$$
$$F_{\mathrm{ion}}(i) = \begin{pmatrix} -\mu \\ \mu \end{pmatrix} \otimes (e_n \otimes M) \tag{39}$$

with $G = [g^1, \ldots, g^m]^T$ and $F = \mathrm{diag}(\mathrm{f}^1, \ldots, \mathrm{f}^m)$. This assumption has as consequence that over all epochs the geometry and ionosphere matrices reduce to the following matrices:

$$F_{\mathrm{geo}} = I_k \otimes F_{\mathrm{geo}}(i); \quad F_{\mathrm{ion}} = I_k \otimes F_{\mathrm{ion}}(i) \tag{40}$$

with $F_{\mathrm{geo}}(i)$ and $F_{\mathrm{ion}}(i)$ as given in Eq. (39). In that case an additional rank deficiency of size ν shows up between the partial design matrices of the geometry parameters and satellite clocks or hardware biases, as well as an additional rank deficiency of size m, between the satellite hardware biases and ionospheric delays. The additional columns of the null

Table 10 Columns to be added to the null space matrix V_{net} in Eq. (23) for a *regional-sized* network, in the absence or presence of temporal constraints on geometry and satellite clocks, satellite hardware bias parameters, or ionospheric delays, or in case of a time-constant geometry

Absence of constraints on either *satellite clocks* or *hardware biases* (rd type 0a/b of size ν)	Absence of constraints on *geometry*, *satellite clocks* or *satellite hardware biases* (rd type 0a/b(m) of size $k\nu$)	Absence of constraints on *satellite hardware bias* parameters (rd type 0c of size m)	Absence of constraints on *ionospheric* and *satellite hardware bias* parameters (rd type 0c(m) of size km)	Time-constant receiver–satellite geometry (rd types 0a/b and 0c of size $\nu+m$)
$\begin{bmatrix} e_k \otimes (e_n \otimes I_\nu) \\ 0 \\ \begin{pmatrix} L_1 \otimes G(1) \\ \vdots \\ L_1 \otimes G(k) \end{pmatrix} \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} I_k \otimes (e_n \otimes I_\nu) \\ 0 \\ \begin{pmatrix} L_1 \otimes G(1) & & \\ & \ddots & \\ & & L_1 \otimes G(k) \end{pmatrix} \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ \begin{pmatrix} L_2 \otimes M(1) \\ \vdots \\ L_2 \otimes M(k) \end{pmatrix} \\ e_k \otimes I_m \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ \begin{pmatrix} L_2 \otimes M(1) & & \\ & \ddots & \\ & & L_2 \otimes M(k) \end{pmatrix} \\ I_k \otimes I_m \\ 0 \end{bmatrix}$	$\begin{bmatrix} e_k \otimes (e_n \otimes I_\nu) & 0 \\ 0 & 0 \\ \begin{pmatrix} L_1 \otimes G \\ \vdots \\ L_1 \otimes G \end{pmatrix} & \begin{pmatrix} L_2 \otimes M \\ \vdots \\ L_2 \otimes M \end{pmatrix} \\ 0 & e_k \otimes I_m \\ 0 & 0 \end{bmatrix}$

$L_1 = (1,\ 0,\ 0)^T$ in the absence of satellite clock constraints; $L_1 = (0,\ e_f^T \Lambda^{-1},\ e_f^T)^T$ in the absence of satellite hw bias constraints; $L_2 = (0,\ -\mu^T \Lambda^{-1},\ \mu^T)^T$

space basis matrix V_{net} can then be given as those in the most right column of Table 10. These additional rank deficiencies can be overcome by constraining the pivot receiver's geometry parameters and the geometry-free satellite code biases for the first epoch, i.e., $\Delta x_1(1)$ and $d^s_{\text{GF}}(1)$. The interpretation of the affected estimable parameters is directly obtained from Table 9 by replacing all time-varying vectors $g^s(i)$ and $\text{f}^s(i)$, $i = 1, \ldots, k$, with their time-invariant counterpart g^s and f^s, respectively.

In case of a *single-epoch* model ($k = 1$) all temporal constraints are absent by definition. All epochs of data are namely processed independently of each other, without any link between them. In that case, all above regional-network rank deficiencies become equivalent and reduce to size $\nu + m$. The interpretation of the estimable parameters that are affected is obtained from Table 9 by simply setting $i = 1$.

5.3 Additional rank deficiencies: no temporal constraints on one or more parameter types

In general, irrespective of the 'regional network' assumption as done in the previous section, the rank deficiency of the GNSS network model changes in the absence of random-walk constraints on one or more types of parameters. The amount of change of the rank deficiency hereby depends on the actual types of parameters that are *not* constrained. The additional rank deficiency is, however, always a combination of the rank deficiencies of types 1–3, as explained in Sect. 3.4, as well of type 0 in case of a regional network, but now in their *multi-epoch* mode, i.e., they reappear for every epoch and not only for the first epoch.

Figure 3 presents a scheme from which the additional rank deficiencies can be derived for each situation, depending on the combination of parameter types with or without random-walk constraints. The starting point of this scheme is the size of the rank deficiency in the network model in presence of all random-walk constraints, as given in Eq. (24). The additional multi-epoch rank deficiencies are denoted in the scheme using an asterisk. Not all combinations of parameter types without temporal constraints lead to additional rank deficiencies. For example, in the absence of constraints on geometry, receiver clock and ionospheric delays (while there are random-walk constraints on all other parameter types), no additional rank deficiencies show up. Other examples are discussed in the following subsections.

5.3.1 Example: regional CORS network model

As an example, consider a CORS network of regional size for which all receiver positions are known and fixed. Ionospheric delays are mapped to vertical ionospheric parameters. ZTD parameters are parameterized (i.e., $\nu = 1$) and a random-walk model is assumed to model their temporal behavior. Random-walk constraints are also assumed for the hardware biases. No temporal constraints are, however, assumed for the receiver clocks ('-rcl'), satellite clocks ('-scl') and the ionospheric delays ('-ion'). Following the scheme in Fig. 3, at the starting point the rank deficiency consists of types 1a, 1b, 2a, 3a, 4 and 5. Then, we follow the arrow to '-rcl' (as no dynamic model is assumed on the receiver clocks), which does not yield an additional rank deficiency. From that point the absence of satellite clock constraints ('-scl'), brings, together with the regional network assumption, additional rank deficiency types 0a and 1a*. From there we follow the box to '-ion', which does not bring additional rank deficiencies. The total size of the rank deficiency for this network model is then determined by adding up the sizes which can be found at the bottom of Fig. 3:

$$\#\ \text{rank deficiencies} = 1 + 2f + (1+f)(n-1+m) + \nu + k - 1$$

Compared to Eq. (24), that gives the size of a network's rank deficiency in presence of random-walk constraints on

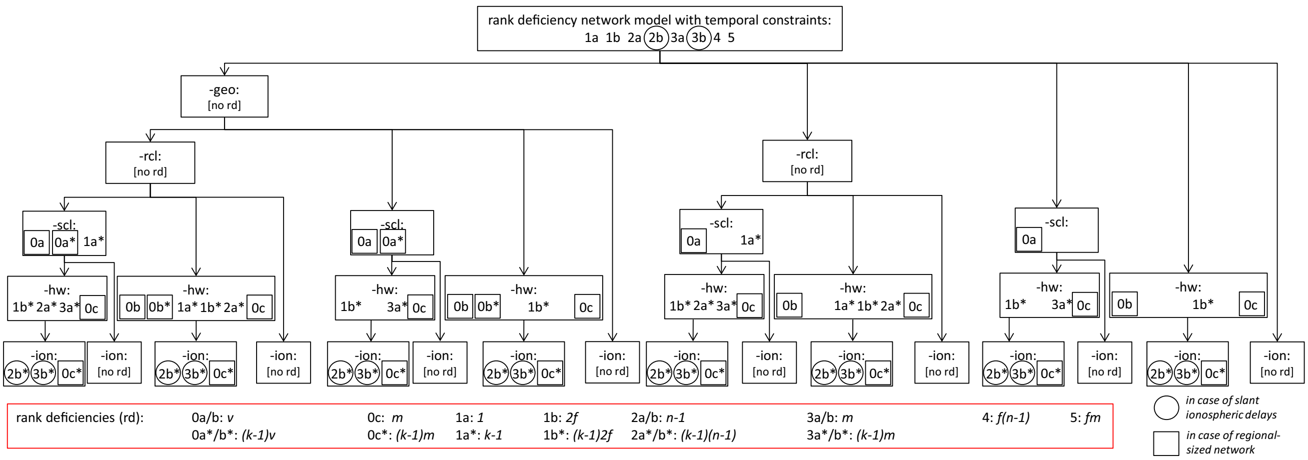

Fig. 3 Scheme to determine the *additional rank deficiencies* in the GNSS network model in the absence of temporal (random-walk) constraints on one or more parameter types. Starting point is the rank deficiency of the multi-epoch model *inside the box at the top*, which corresponds to the model in presence of temporal constraints on all parameters [see Eq. (24)]. The total amount of the additional rank deficiencies now follows from adding up the amounts corresponding to the number (*plus letter* look up at the bottom of the scheme) for the random-walk constraints that are *left out* in the model. *Note I* -geo = no geometry constraints; -rcl = no receiver clock constraints; -scl = no satellite clock constraints; -hw = no receiver and satellite hardware bias constraints; -ion = no ionospheric delay constraints. *Note II* a rank deficiency within a square only occurs in case of a 'regional-sized' network (see Sect. 5.2), whereas a rank deficiency *within a circle* only occurs in case of a slant (instead of *vertical*) ionospheric parametrization (see Sect. 5.1). *Note III* The types of rank deficiencies that are denoted using an *asterisk* (*) are the multi-epoch versions of the rank deficiency types *without an asterisk*. *Note IV* in the absence of ionospheric random-walk constraints, the additional rank deficiencies of types 3b* and 0c* cannot occur at the same time, as they are similar types of rank deficiencies. Thus, in case of a slant ionospheric parametrization type 3b* occurs (and not 0c* even for a regional-sized network), whereas in case of vertical ionospheric parameters type 0c* occurs (and not 3b*), only if the network is regional

all parameters, the additional rank deficiency for the network in this example equals $\nu + k - 1$.

5.3.2 Example: no temporal constraints on all parameters

In the extreme case, if temporal constraints on *none* of the parameters are assumed (except the ambiguities which are time constant), the total rank deficiency of the network model follows simply as the sum of the rank deficiencies that are encountered when following the upper left path in Fig. 3. If the ionospheric delays are modeled as vertical parameters, these rank deficiencies are of the types 1a, 1b, 2a, 3a, 4, 5, 1a*, 1b*, 2a* and 3a*, as well as 0a, 0c, 0a* and 0c* in case of a regional-sized network. This yields a total rank deficiency of:

$$\begin{aligned}\#\ \text{rank deficiencies} = {} & k[1 + 2f + (n-1+m)] \\ & \underbrace{+\,k(\nu+m)}_{\text{regional net}} + f(n-1+m)\end{aligned}$$

This means that the rank deficiencies of the types 1a, 1b, 2a and 3a (plus 0a and 0c for a regional network) reoccur at every epoch, which makes sense as the corresponding parameters are not linked in time. The rank deficiencies 4 and 5 corresponding to the ambiguities show up just once, as the ambiguities are linked in time.

The null space in this case is simply the multi-epoch generalization of the null space in presence of the random-walk constraints, as given in Eq. (23). This means that we only have to change epoch vector e_k to the identity matrix I_k in the first four groups of columns of V_{net}, to obtain the null space matrix in the absence of any temporal constraints. It can be shown that the interpretation of the estimable parameters is identical as given in Tables 3 and 4, with the difference that the parameters relating to the pivot epoch 1 are simply replaced by their counterparts at epoch i. For example, the estimable satellite clock in the CC-R system reduces to:

$$d\tilde{t}^s(i)' = \left[dt^s(i) + d^s_{\text{IF}}(i)\right] - \left[dt_1(i) + d_{1,\text{IF}}(i)\right]. \tag{41}$$

Compared to the satellite clock expression in Table 3, in the absence of temporal constraints, the parameters at the first epoch are replaced by their counterparts at the current epoch i. Therefore, the 'conditions' for the existence of the parameters in the absence of temporal constraints correspond to their existence at the first epoch ($i = 1$; see Tables 3, 4).

6 Undifferenced PPP-RTK user models

Having identified the full-rank, undifferenced network model and its estimable parameters based on different S-basis choices, in this section we will first discuss which of these network parameters enable PPP-RTK (Sect. 6.1). After that, Sect. 6.2 identifies the PPP-RTK user's undifferenced model plus estimable parameters. The following sections (i.e., Sects. 6.3, 6.4, 6.5) then describe the effect on the estimable parameters if the user does not incorporate certain network corrections. The last two subsections discuss the effect on the user's parameters if the network is based on slant instead of vertical ionospheric delays (Sect. 6.6), or if the network is of regional size (Sect. 6.7).

6.1 Network information enabling PPP-RTK

Single-receiver GNSS positioning is possible for users when the network provides parameters that are *common* between the network and the user. These common parameters are the *satellite-dependent* parameters, i.e., satellite clocks, satellite phase and code biases, as well as (vertical) ionospheric delays. Satellite orbits are also needed by the user and here it is assumed that these are provided by an external provider, such as the IGS. Satellite phase biases (for each frequency of the user) are essential to perform integer ambiguity resolution by the single-receiver user, which is the key to PPP-RTK. Ionospheric corrections are crucial to perform *fast* PPP-RTK integer ambiguity resolution; otherwise the ambiguities need a long time to converge (Odijk et al. 2012). In the case ionospheric corrections are applied also satellite code biases (DCBs) are required, even in the single-frequency case, as to remove these biases from the satellite clocks that are based on the ionosphere-free (dual-frequency) combination. For triple- or higher-frequency PPP-RTK satellite code biases are needed for each frequency as well.

Besides the above information provided by the network, we assume for all models that the PPP-RTK user's observations are corrected for the large (i.e., dry) part of the tropospheric delay, phase center offsets, phase wind-up, solid earth tides, ocean loading, etc., similar to standard PPP. More details on how to calculate these corrections can be found in Kouba and Heroux (2001).

As discussed in the previous sections, the network cannot provide the satellite-dependent parameters in their absolute, uncombined form to the user, but only as *combinations* with other parameters, where these combinations depend on the S-basis of the network. For example, in case the network adopts the CC-R system, the satellite-dependent parameters provided to the user are $\mathrm{d}\tilde{t}^s(i)$, $\tilde{\delta}_j^s(i)$, $\tilde{d}_j^s(i)$ and $\tilde{\imath}^s(i)$; see Table 3. However, if the network adopts another $\mathcal{S}$-system, say the CC-S system, the satellite-dependent parameters provided to the user are $\mathrm{d}\tilde{\tilde{t}}^s(i)$, $\tilde{\tilde{\delta}}_j^s(i)$, $\tilde{\tilde{d}}_j^s(i)$ and $\tilde{\tilde{\imath}}^s(i)$, which have another interpretation than the parameters of the CC-R network; see Table 4.

In addition to a different interpretation, the *number* of estimable satellite-dependent parameters varies between both CC-S and CC-R systems. Due to the assumed temporal constraints on the network parameters the number of estimable satellite-dependent parameters only varies for the first epoch ($i = 1$), see Tables 3 and 4: in the CC-R system they are estimable for *all* satellites (i.e., $s \geq 1$), whereas in the CC-S system for all satellites minus the network's pivot satellite (i.e., $s \geq 2$). It can, however, be shown that the PPP-RTK user does not need the satellite-dependent parameters to *all* satellites. It follows that the sufficient information is based on the satellite-dependent parameters that are *differenced* with respect to a pivot satellite, arbitrarily selected by the user (and can thus be different from the network's pivot satellite). Here we denote the pivot satellite selected by the user using the (general) superscript p, whereas the pivot satellite of the network is denoted using superscript "1".

Table 11 Network satellite parameters needed to enable PPP-RTK, depending on the S-basis of the network (here: CC-R or CC-S)

	CC-R network	CC-S network	Condition
Clocks	$\mathrm{d}\tilde{t}^s(i) - \mathrm{d}\tilde{t}^p(1)$	$\mathrm{d}\tilde{\tilde{t}}^s(i) - \mathrm{d}\tilde{\tilde{t}}^p(1)$	$i = 1: s \neq p$ $i \geq 2: s \geq 1$
Phase biases	$\tilde{\delta}_j^s(i) - \tilde{\delta}_j^p(1)$	$\tilde{\tilde{\delta}}_j^s(i) - \tilde{\tilde{\delta}}_j^p(1)$	$i = 1: s \neq p$ $i \geq 2: s \geq 1$
Code biases	$\tilde{d}_j^s(i) - \tilde{d}_j^p(1)$	$\tilde{\tilde{d}}_j^s(i) - \tilde{\tilde{d}}_j^p(1)$	$i = 1: s \neq p$ $i \geq 2: s \geq 1$
Ionospheric delays	$\tilde{\imath}^s(i)$	$\tilde{\tilde{\imath}}^s(i)$	$i \geq 1: s \geq 1$

Note that the provision of ionospheric parameters is optional

The corrections differenced with respect to a pivot satellite are sufficient, as the parameters corresponding to the user's pivot satellite are automatically absorbed by the estimable receiver-dependent parameters, and he does not need to correct for them. This between-satellite differencing should, however, *not* be done for the ionospheric network parameters when the network is based on a vertical ionospheric modeling such as assumed in this article. As there is an ionospheric mapping involved when the user applies the ionospheric correction, the ionospheric parameters provided by the network should be undifferenced, i.e., per satellite. Table 11 summarizes the network information that is required by the PPP-RTK user. It follows that the number of parameters is equal for both network $\mathcal{S}$-systems.

6.2 Estimable PPP-RTK user parameters

When the user applies the satellite-dependent network parameters that are differenced relative to a user-defined pivot satellite p, from Tables 3 and 4, irrespective of whether a CC-R network or CC-S network provides these parameters, exactly the same information is provided to the user, despite a difference in S-bases between both networks. The between-satellite differencing has namely the effect that in case of the CC-R network the network's (pivot) receiver-dependent parameters are eliminated, whereas in case of the CC-S network the network's satellite-mean dependent parameters are eliminated, leaving exactly the same satellite-dependent parameters in the between-satellite differences.

One type of network parameters are, however, *not* eliminated when the user forms between-satellite differences. These are the ambiguities corresponding to the network's pivot receiver, i.e., $z^s_{1,j} - z^p_{1,j}$, which remain in the between-satellite differenced satellite phase biases. This between-satellite differenced ambiguity is, however, essential for the user, as it gets lumped to the between-satellite differenced ambiguity of the user's phase data and consequently results in an estimable *double-differenced* ambiguity for the user, i.e., $[z^s_{u,j} - z^p_{u,j}] - [z^s_{1,j} - z^p_{1,j}]$, $s \neq p$, with the user's parameters denoted using subscript u, and which is *automatically an integer*. Thus, although the user does not need to know *which* of the network's receiver is used as pivot, by applying the (between-satellite differenced) satellite phase biases, the estimable user ambiguities are relative to this pivot receiver.

To construct the PPP-RTK user's full-rank observation equations as well as to identify the estimable parameters based on the undifferenced model, like with the network model first the rank deficiencies need to be identified. As the pure satellite-dependent parameters are absent in the user's model, the only rank deficiencies that occur in the user's model are of types 2a (between receiver clock and receiver hardware biases; rank deficiency of size 1) and 4 (between receiver phase biases and ambiguities; rank deficiency of size f), see Sect. 3.4, where we assume random-walk temporal constraints on the user's receiver clock, as well as receiver hardware bias parameters. To remove these rank deficiencies the user has the freedom to choose an S-basis himself. If the user applies a Common Clocks S-basis with the user's ionosphere-free code biases, i.e., $d_{u,\text{IF}}(1)$, as S-basis constraints (to overcome rank deficiency type 2a), as well as the user's ambiguities of the pivot satellite, i.e., $z^p_{u,j}$ (to overcome rank deficiency type 4), then the estimable user parameters are presented in Table 12. Remark there are no ionospheric parameters for the user, as ionospheric corrections are provided by the network.

It is remarked that the estimable parameters in Table 12 are valid for *single-frequency* ($j = 1$) PPP-RTK users as well. As the estimable parameters are expressed in the ionosphere-free and geometry-free hardware biases, it seems that data from a second frequency is involved here. However, using the following relation $d_{u,\text{IF}}(1) = d_{u,1}(1) - \mu_1 d_{u,\text{GF}}(1)$ (similar for the code bias of the pivot satellite p), we can reparametrize the estimable parameters as function of parameters on the first frequency only (this is similar as removing the DCB from the ionosphere-free clock in case of PPP to obtain the clock on the first frequency). The resulting estimable single-frequency parameters then read as follows:

$$
\begin{aligned}
\mathrm{d}\tilde{t}_u(i) &= [\mathrm{d}t_u(i) + d_{u,1}(1)] - [\mathrm{d}t^p(1) + d^p_1(1)], \;\; i \geq 1 \\
\tilde{\delta}_{u,1}(i) &= [\delta_{u,1}(i) - \frac{1}{\lambda_1} d_{u,1}(1)] \\
&\quad -[\delta^p_1(1) - \frac{1}{\lambda_1} d^p_1(1)] + z^p_{u,1}, \;\; i \geq 1 \\
\tilde{d}_{u,1}(i) &= d_{u,1}(i) - d_{u,1}(1), \;\; i \geq 2.
\end{aligned} \tag{42}
$$

As a consequence, the single-frequency receiver code bias, i.e., $\tilde{d}_{u,1}(i)$, is only estimable from the second epoch onwards.

The PPP-RTK user's full-rank observation equations based on the estimable parameters in Table 12 are easily obtained from Table 5 by bringing the satellite-dependent corrected parameters to the left side of the equal sign and setting $r = u$ (note that the condition $r \geq 2$ in Table 5 does not apply for the user).

6.3 Estimable user parameters in the absence of ionospheric corrections

In the previous subsection it was assumed that ionospheric corrections are provided to the PPP-RTK user. In case these corrections are not precise enough, instead of correcting the user may want to estimate the ionospheric delays himself. In case the user parameterizes *slant* ionospheric delay parameters, an additional rank deficiency needs to be dealt with, occurring between the user's receiver hardware bias para-

Table 12 Estimable ionosphere-corrected PPP-RTK user parameters based on a Common Clocks (CC) S-basis ($j \geq 1$)

Estimable parameter	Notation and interpretation	Conditions
Receiver position + ZTD	$\Delta\tilde{x}_u(i) = \Delta x_u(i)$	$i \geq 1$
Receiver clock	$\mathrm{d}\tilde{t}_u(i) = [\mathrm{d}t_u(i) + d_{u,\text{IF}}(1)] - [\mathrm{d}t^p(1) + d^p_{\text{IF}}(1)]$	$i \geq 1$
Receiver phase bias	$\tilde{\delta}_{u,j}(i) = [\delta_{u,j}(i) - \frac{1}{\lambda_j} d_{u,\text{IF}}(1)] - [\delta^p_j(1) - \frac{1}{\lambda_j} d^p_{\text{IF}}(1)] + z^p_{u,j}$	$i \geq 1: \; j \geq 1$
Receiver code bias	$\tilde{d}_{u,j}(i) = \begin{cases} d_{u,\text{GF}}(1) - d^p_{\text{GF}}(1), & \text{for } i = 1 \text{ and } j = 2^{\text{a}} \\ [d_{u,j}(i) - d_{u,\text{IF}}(1)] - \left[d^p_j(1) - d^p_{\text{IF}}(1)\right], & \text{otherwise} \end{cases}$	$i = 1: \; j \geq 2$ $i \geq 2: \; j \geq 1$
Phase ambiguity	$\tilde{z}^s_{u,j} = [z^s_{u,j} - z^p_{u,j}] - [z^s_{1,j} - z^p_{1,j}]$	$j \geq 1, \; s \neq p$

[a] Only one estimable parameter, which applies to the observations of the first two frequencies

meters and slant ionospheric delays. This rank deficiency is of type 2b and of size 1 (see Sect. 5.1) and is overcome by constraining the geometry-free code bias combination of the user. As consequence, the estimable user's receiver phase and code bias parameters are changed with respect to the situation that ionospheric corrections are present (see Table 12), whereas an ionospheric parameter becomes estimable as well; see Table 13 for the interpretation of these estimable parameters. Note that the for the first epoch the code bias parameter is only estimable for three or more frequencies ($j \geq 3$).

In the *single-frequency* case without ionospheric corrections, the situation is slightly different, as in this case the rank deficiency of type 2b (and size 1) cannot be eliminated by constraining the geometry-free receiver code bias as there is no data of a second frequency. Alternatively, the user can constrain the receiver *phase* bias (of the first epoch) at the single frequency, as the receiver code bias is already constrained to overcome rank deficiency type 2a. The estimable parameters for the single-frequency ionosphere-float case are presented in Table 14. Due to the receiver phase and code bias being part of the S-basis, they are only estimable from the second epoch onwards. Moreover, this has a consequence for the interpretation of the estimable receiver clock and ionospheric delay parameters. An important remark here is that although the receiver clock and ionospheric delay are estimable for the first epoch, due to a lack of observations the single-frequency ionosphere-float PPP-RTK user model is only solvable using at least *two* epochs, whereas with two or more frequencies already a single epoch is sufficient. This single-frequency ionosphere-float case has its analogy with the ionosphere-free combination of phase and code (based on the difference in sign of the ionospheric delay for phase and code), which is known as the "GRAPHIC" (GRoup And PHase Ionospheric Calibration) combination (Yunck 1993).

6.4 Estimable user parameters in the absence of satellite phase biases (standard PPP)

In the absence of corrections for the satellite phase biases, the user needs to estimate them as unknown parameters. However, with the satellite phase biases as additional parameters, there is a rank deficiency of type 5 between these satellite phase biases and the ambiguities in the user's model (see Sect. 3.4). This rank deficiency, of size fm, is overcome by fixing all satellite phase biases as S-basis constraints. In that case, they get lumped to the user ambiguities, such that the resulting estimable ambiguity parameters are then given as:

$$\tilde{z}^s_{u,j}(i)' = [z^s_{u,j} - \delta^s_j(i) + \frac{1}{\lambda_j} d^s_{\text{IF}}(1)] \\ -[z^p_{u,j} - \delta^p_j(1) + \frac{1}{\lambda_j} d^p_{\text{IF}}(1)], \quad s \neq p. \tag{43}$$

Because of these hardware bias terms lumped to the ambiguities, this has as consequence that the estimable ambiguities cannot be estimated as integers. In addition, the estimable ambiguities even become *time-dependent* parame-

Table 13 Estimable ionosphere-float PPP-RTK parameters for users employing *at least a dual-frequency* receiver ($f \geq 2$)

Estimable parameter	Notation and interpretation	Conditions
Receiver position+ZTD	$\Delta\tilde{x}_u(i) \; = \; \Delta x_u(i)$	$i \geq 1$
Receiver clock	$d\tilde{t}_u(i) \; = \; [dt_u(i) + d_{u,\text{IF}}(1)] - [dt^p(1) + d^p_{\text{IF}}(1)]$	$i \geq 1$
Receiver phase bias	$\tilde{\delta}_{u,j}(i)' = [\delta_{u,j}(i) - \frac{1}{\lambda_j}\{d_{u,\text{IF}}(1) - \mu_j d_{u,\text{GF}}(1)\}] - [\delta^p_j(1) - \frac{1}{\lambda_j}\{d^p_{\text{IF}}(1) - \mu_j d^p_{\text{GF}}(1)\}] + z^p_{u,j}$	$i \geq 1: \; j \geq 1$
Receiver code bias	$\tilde{d}_{u,j}(i)' = [d_{u,j}(i) - d_{u,\text{IF}}(1) - \mu_j d_{u,\text{GF}}(1)] - [d^p_j(1) - d^p_{\text{IF}}(1) - \mu_j d^p_{\text{GF}}(1)]$	$i = 1: \; j \geq 3$ $i \geq 2: \; j \geq 1$
Ionospheric delay	$\tilde{\iota}^s_u(i)' \; = \; \iota^s_u(i) + d_{u,\text{GF}}(1) - d^p_{\text{GF}}(1)$	$i \geq 1, \; s \geq 1$
Phase ambiguity	$\tilde{z}^s_{u,j} \; = \; [z^s_{u,j} - z^p_{u,j}] - [z^s_{1,j} - z^p_{1,j}]$	$j \geq 1, \; s \neq p$

Table 14 Estimable ionosphere-float PPP-RTK parameters for users employing a *single-frequency* receiver ($f = 1$)

Estimable parameter	Notation and interpretation	Conditions
Receiver position+ZTD	$\Delta\tilde{x}_u(i) \; = \; \Delta x_u(i)$	$i \geq 1$
Receiver clock	$d\tilde{t}_u(i)' \; = \; [dt_u(i) + \frac{1}{2}\{\lambda_1\delta_{u,1}(1) + d_{u,1}(1) + \lambda_1 z^p_{u,1}\}] - [dt^p(1) + \frac{1}{2}\{\lambda_1\delta^p_1(1) + d^p_1(1)\}]$	$i \geq 1$
Receiver phase bias	$\tilde{\delta}_{u,1}(i)' = \delta_{u,1}(i) - \delta_{u,1}(1)$	$i \geq 2$
Receiver code bias	$\tilde{d}_{u,1}(i)' = d_{u,1}(i) - d_{u,1}(1)$	$i \geq 2$
Ionospheric delay	$\tilde{\iota}^s_u(i)' \; = \; \iota^s_u(i) - \frac{1}{2\mu_1}[\lambda_1\delta_{u,1}(1) - d_{u,1}(1) + \lambda_1 z^p_{u,1}] + \frac{1}{2\mu_1}[\lambda_1\delta^p_1(1) - d^p_1(1)]$	$i \geq 1, \; s \geq 1$
Phase ambiguity	$\tilde{z}^s_{u,1} \; = \; [z^s_{u,1} - z^p_{u,1}] - [z^s_{1,1} - z^p_{1,1}]$	$s \neq p$

ters, if the satellite phase bias are assumed to vary in time. If the satellite phase biases are assumed time constant, the estimable ambiguities are constant in time as well. The other estimable user parameters are not changed in this situation. The resulting user model based on non-integer estimable ambiguities is the well-known model for *standard PPP*.

6.5 Estimable user parameters in the absence of satellite code biases

Satellite code biases are—for the first epoch—estimable by the network from the second frequency onwards, see Tables 3 and 4. If these corrections are not provided by the network, the user needs to estimate them. In a dual-frequency case, if the user also estimates slant ionospheric parameters (see Sect. 6.3), these satellite code biases get lumped to the ionospheric parameters (rank deficiency type 3b of size m, see Sect. 3.4). In that case the user's estimable receiver phase and code bias parameters become equivalent to those presented in Table 13, whereas the estimable ionospheric delay parameter becomes:

$$\tilde{\iota}_u^s(i)' = \iota_u^s(i) + d_{u,\text{GF}}(1) - d_{\text{GF}}^s(1), \quad i \geq 1, \quad s \geq 1. \tag{44}$$

Thus, the estimable ionospheric delay contains the satellite code biases corresponding to the actual satellite s. However, a more serious consequence of the absence of the satellite code biases is that the estimable ambiguity parameters get biased by these satellite code biases as well. In case of corrections from a CC-R network the estimable ambiguity namely becomes equal to

$$\tilde{z}_{u,j}^{s'} = [z_{u,j}^s - \tfrac{\mu_j}{\lambda_j} d_{\text{GF}}^s(1)] - [z_{u,j}^p - \tfrac{\mu_j}{\lambda_j} d_{\text{GF}}^p(1)], \quad s \neq p. \tag{45}$$

Due to this lumping of the DCBs to the ambiguities, the resulting ambiguity parameters cannot be estimated as integers, similar as in the case where satellite phase biases are absent. One has to realize however, that this only occurs because of the different underlying models with respect to the ionosphere between the network (parameterizing vertical ionospheric delays) and the user (parameterizing slant ionospheric delays). Otherwise, the network's satellite phase biases always recover the integerness of the user's ambiguities.

6.6 Estimable user parameters from a network based on slant ionospheric delays

If the network model is based on *slant* instead of *vertical* ionospheric parameters (see Sect. 5.1), this has as consequence that the estimability and interpretation of the satellite phase and code bias corrections changes, as they get biased by the geometry-free combination of code biases (or DCBs); see Table 8. If the user does not correct for the ionospheric delays, this has as consequence that the estimable user's receiver phase and code parameters change, in an identical way as given in Table 13, for the situation that the network is based on vertical ionospheric parameters but the user does not correct for the ionospheric delays. Note that the receiver code bias is then only estimable (at the first epoch) for more than three frequencies. The estimable user's ionospheric delay parameter is, however, different from the expression given in Table 8, as the network based on slant ionospheric delays does not provide the geometry-free satellite code bias (DCB) for the first two frequencies (as they are not estimable). As a consequence the estimable user's ionospheric delay becomes identical to Eq. (44). Thus, compared to Table 8, in this expression the DCB of the pivot satellite gets replaced by the DCB of the current satellite s.

6.7 Estimable user parameters from a regional-based network

The interpretation of the user's parameters as given in Table 12 may change if the network that supplies the correction information is a *regional-based* network, as due to additional rank deficiencies some of the network parameters may change (see Sect. 5.2). The user parameters are only affected in case of the rank deficiencies where the network's receiver position/ZTD parameters are involved, i.e., the types 0a, 0b, 0a(m) and 0b(m) (see Table 9) in the network. The net effect of change in the corrections provided to the user is an additional term $g^s(i)^T \Delta x_1(1)$ (or $g^s(i)^T \Delta x_1(i)$, depending on the presence of dynamic models of the receiver positions/ZTDs and satellite clocks/hardware biases in the network model). This bias gets, however, automatically absorbed by the user's receiver position/ZTD parameters:

$$\Delta\tilde{x}_u(i)' = \Delta\tilde{x}_u(i) - \Delta x_1(1) = \Delta x_u(i) - \Delta x_1(1). \tag{46}$$

Thus, the user's position (and ZTD) become estimable relative to that of the pivot receiver in the network.

7 Summary and conclusions

In this contribution, we used $\mathcal{S}$-system theory to analyze the parameter estimability of the rank-deficient, undifferenced GNSS network and PPP-RTK user models. We believe that a proper application of this theory is crucial in the near-future, complex and diverse, multi-frequency, multi-GNSS landscape. With the existence of diverse network providers and a broad range of different PPP-RTK users, a proper understanding of the estimability of the computed and provided parameter solutions becomes essential. One can for instance not equate resolved satellite biases from different network providers a priori. The computed network parameters are S-basis dependent and will therefore be different for different S-bases. With a careful application of $\mathcal{S}$-system theory, however, one can give a clear interpretation to the different estimable parameters, thus enabling to relate them to one another through the use of the appropriate S-transformation.

The contributions of this work cover both the rank-deficient network model and the PPP-RTK enabled user model. For the undifferenced *network model* the contributions can be summarized as follows:

- For the first time, the rank deficiencies and null space of the multi-epoch, multi-frequency undifferenced GNSS network model have been identified. Based on the identified rank deficiencies, a basis matrix of the null space of the network design matrix was constructed.
- Different sets of estimable parameters, each with their own interpretation, exist, and each such set is defined by the chosen S-basis. We presented two such S-basis choices, both of which are members of the 'Common Clocks (CC) family', i.e., resulting in estimable receiver- and satellite clocks that are common to all observables (phase, code, frequency). They are the CC-R system, which constraints parameters of one receiver, and the CC-S system, which constraints the satellite mean of parameters. We have chosen to discuss these CC-systems as they have a clear link with the GNSS products (ionosphere-free satellite clocks, DCBs) provided by, e.g., the IGS. In a modernized triple-frequency GPS case, the satellite clocks and DCBs that are estimable in the family of CC-systems have the same interpretation as in the dual-frequency case. In practice this is advantageous because of reasons of backward compatibility and consistency with the IGS-chosen S-basis. As shown in this article, for the new third GPS frequency then estimable code biases will enter when using the CC-systems. Other $\mathcal{S}$-systems can be chosen as well of course, see e.g., Teunissen et al. (2010), as they can all be linked with the appropriate S-transformation.
- We identified and described the estimable parameters of the undifferenced network model whereby random-walk dynamic models on all parameters were included. It was demonstrated, that due to a difference in S-basis choice, CC-R versus CC-S, the interpretation of the estimable parameters is quite different, except for those parameters that are not affected by the null space (e.g., receiver position/ZTDs and ionosphere).
- We also demonstrated, as not all GNSS models will have the same rank deficiencies, that the S-basis should be adapted under different model assumptions. It is then important to realize that this has consequences for the estimability and interpretation of its parameters. In this contribution these consequences have been given explicitly for the following changes in the (network) model:
  - When in the absence of an ionospheric model, the ionospheric delays are parametrized as slant delays instead of as, e.g., vertical delays, the DCBs, corresponding to the first two frequencies, become inestimable. Instead they will get lumped into the phase and code biases (for three or more frequencies) as well as into the ionospheric delay parameters.
  - When instead of a large network the size of the network is such that the line-of-sight vectors from the network stations to the same satellite can assumed to be parallel, the pivot receiver position/ZTD will get lumped into the estimable position/ZTDs, as well as into the satellite clocks.
  - We discussed the effects that the presence or absence of dynamic models have on the parameter estimability. We presented a detailed scheme (see Fig. 3) from which it follows how the rank deficiencies change when some parameters have (random-walk) temporal constraints and others not.
- We demonstrated how S-transformations provide the correct method of comparing solutions (their estimates plus precision). We showed how these transformations can be derived once the null space identification has taken place, and for the link between the CC-S system and the CC-R system, the S-transformation was explicitly derived. It was also stressed what can go wrong if one fails to include the proper S-transformation when comparing solutions. As a recent case in point we refer to the results of Zhong et al. (2015), where it was demonstrated that the IGS estimated DCBs show a long term variation in dependence on the set of satellites involved. We believe that this artefact is simply a consequence of comparing solutions in different $\mathcal{S}$-systems, without taking the proper S-transformation into account. The S-basis changes the moment a different set of satellites is involved in the zero-mean enforcing constraint of the DCBs. This phenomenon is completely analogous to the classical surveying situation where the minimum-norm coordinate estimability of a survey network changes once the point set of the network changes. Only with the inclusion of the proper S-transformation can such two minimum-norm solutions be correctly compared.
- Although theoretically there is no preference for a certain S-basis choice, as they are all linked to one another by an S-transformation, in practice depending on the application at hand there may be preference of one over the other. For example, for time transfer purposes where one is interested in the behavior of receiver clocks, the CC-S

system without random-walk constraints as presented in this article need not be a preferred choice, as the estimable receiver clocks would show an undesired jump if a different set of satellites is used to compute the mean satellite clock. The CC-R system is, however, a good choice, as the estimable receiver clocks would only show a jump if the network changes its pivot receiver (usually this does not occur frequently).

For the undifferenced *PPP-RTK enabled user model* our main conclusions can be summarized as follows:

- Irrespective of the network's $\mathcal{S}$-system (for example CC-R or CC-S), the between-satellite differenced corrections (i.e., clocks and hardware biases) essentially contain the same information for the user's position/ZTD parameters and the integer-valued ambiguities.
- Keeping undifferenced observation equations, the user has more freedom to incorporate dynamic models into its measurement setup. However, the user needs to account for rank deficiencies that are of receiver-dependent types. They comprise the linear dependency between the receiver biases and the ambiguities. In case ionospheric corrections are not applied, an additional rank deficiency enters, namely, the one between the receiver biases and the slant ionospheric delays.
- Applying satellite phase bias corrections, the estimable user ambiguities become of the double differenced forms, thus of an integer nature. These differences are formed with respect to user's pivot satellite and a network's pivot receiver. The user does, however, not need to know which receiver is taken as a pivot. In the absence of these satellite phase bias corrections, the estimable user ambiguities remain non-integer. In this case, the user measurement setup is specialized to a standard PPP setup.
- Provided that ionospheric corrections and the corresponding DCBs are applied, the derived PPP-RTK user's model holds valid for the single-frequency users as well. In the absence of ionospheric corrections, single-frequency PPP-RTK can be achieved as well, but not with fast ambiguity resolution.

In addition to the above network and user related claims, we believe that the presented undifferenced GNSS model, combined with $\mathcal{S}$-system theory, indeed provides for the most flexible choice of both network and PPP-RTK user processing, as:

- All parameters are kept in the model, including those that need to be transmitted to the user to enable PPP-RTK (i.e., satellite orbits, satellite clocks, satellite hardware biases and ionosphere). Elimination of parameters, if needed, can simply take place through reduction of the normal equations.
- There is no need for an a priori formation of observation combinations or parameter combinations (e.g., ionosphere-free, Melbourne–Wübbena, widelane, extra-widelane, narrowlane, etc.).
- The models that we presented are general and valid for any number of frequencies. Without modification our results are directly employable for modernized GPS, BDS, Galileo, future GLONASS, as well as regional constellations (QZSS, IRNSS, etc.).
- Dynamic models on (subsets of) parameters can easily be incorporated. Although we restricted attention to the more commonly used random-walk model, the principle used is generally applicable and applies therefore to other dynamic models as well.
- The parameter estimability of any other current or future PPP-RTK model can be cast in our versatile $\mathcal{S}$-system framework, irrespective of whether undifferenced/differenced and/or uncombined/combined approaches are used.

Finally we would like to remark, although this work has its $\mathcal{S}$-system theory focussed on positioning, that due to its generality it can be applied to non-positioning applications as well, such as (as earlier mentioned) time transfer, as well as GNSS-based ionospheric studies or receiver bias calibration.

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Appendix 1: The Kronecker product

If $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$ is an $m \times n$ matrix and B is a $p \times q$ matrix, then their *Kronecker product* $A \otimes B$ is defined as the following $mp \times nq$ matrix (Rao 1973):

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \quad (47)$$

Note: $A \otimes B \neq B \otimes A$. Some of its properties are (assuming that all matrices involved have appropriate dimensions):

$$\begin{aligned} (A + C) \otimes B &= A \otimes B + C \otimes B \\ A \otimes (B \otimes C) &= (A \otimes B) \otimes C \\ (A \otimes B)(C \otimes D) &= (AC) \otimes (BD) \\ (A \otimes B)^T &= A^T \otimes B^T \\ (A \otimes B)^{-1} &= A^{-1} \otimes B^{-1} \end{aligned} \quad (48)$$

In the last equation both A and B are assumed to be square and invertible.

Appendix 2: Full-rank design matrices and S-transformation matrices for the network model

In this section, the full-rank design matrix as well as the S-transformation matrix for the different network S-basis choices in this article are derived, where in all cases it is assumed that random-walk temporal constraints are incorporated for all parameters (except the ambiguities that are assumed to be time constant).

CC-R S-basis

In case of the CC-R S-basis, the S-basis constraints can be casted in the matrix $(S_{\text{CC-R}}^{\perp})^T$ as follows, where each of the five 'columns' represents a parameter group (i.e., receiver positions/ZTDs, receiver clocks/hardware biases, satellite clocks/hardware biases, ionospheric delays and ambiguities):

$$(S_{\text{CC-R}}^{\perp})^T = \begin{bmatrix} 0 & c_k^T \otimes \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_f & 0 \\ 0 & 0 & I_f \end{pmatrix} \otimes c_n^T \right\} & 0 & 0 & 0 \\ 0 & c_k^T \otimes \left\{ \begin{pmatrix} 0 & 0 & \mu_{\text{IF}}^T \end{pmatrix} \otimes D_n^T \right\} & 0 & 0 & 0 \\ 0 & c_k^T \otimes \left\{ \begin{pmatrix} 0 & 0 & \mu_{\text{IF}}^T \end{pmatrix} \otimes \left(-c_n^T \otimes e_m \right) \right\} & c_k^T \otimes \left\{ \begin{pmatrix} 0 & 0 & \mu_{\text{IF}}^T \end{pmatrix} \otimes I_m \right\} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_f \otimes \left(D_n^T \otimes c_m^T \right) \\ 0 & 0 & 0 & 0 & I_f \otimes \left(c_n^T \otimes I_m \right) \end{bmatrix} \tag{49}$$

Its orthogonal complement $S_{\text{CC-R}}$, such that $(S_{\text{CC-R}}^{\perp})^T S_{\text{CC-R}} = 0$, can be constructed as the following block-diagonal matrix:

$$\begin{aligned} S_{\text{CC-R}} &= \text{blkdiag}\left\{ I_k \otimes (I_n \otimes I_v),\ \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & I_f & 0 \\ 0 & 0 & F_f \end{pmatrix} \otimes C_n,\ I_{k-1} \otimes \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_f & 0 \\ 0 & 0 & I_f \end{pmatrix} \right\} \otimes I_n \right], \right. \\ & \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & I_f & 0 \\ 0 & 0 & F_f \end{pmatrix} \otimes I_m,\ I_{k-1} \otimes \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_f & 0 \\ 0 & 0 & I_f \end{pmatrix} \right\} \otimes I_m \right], \\ & \left. I_k \otimes I_m,\ I_f \otimes (C_n \otimes C_m) \right\} \end{aligned} \tag{50}$$

Here the $f \times (f-1)$ matrix F_f is defined as: $F_f = \begin{pmatrix} \mu_1 & 0 \\ \mu_2 & 0 \\ 0 & I_{f-2} \end{pmatrix}$, for which it holds that $\mu_{\text{IF}}^T F_f = 0$. The *full-rank design matrix*, corresponding to the CC-R S-basis, is obtained by post-multiplying the rank-deficient design matrix A_{net} in Eq. (20) with the above $S_{\text{CC-R}}$ and reads as follows:

$$A_{\text{net}}^{\text{CC-R}} = A_{\text{net}} S_{\text{CC-R}} = \left[\begin{array}{c|c|c|c|c} F_{\text{geo}} & \left[\begin{pmatrix} e_f & \Lambda & 0 \\ e_f & 0 & F_f \end{pmatrix} \otimes (C_n \otimes e_m) \quad I_{k-1} \otimes \left\{ \begin{pmatrix} e_f & \Lambda & 0 \\ e_f & 0 & I_f \end{pmatrix} \otimes (I_n \otimes e_m) \right\} \right] & \left[\begin{pmatrix} e_f & \Lambda & 0 \\ e_f & 0 & F_f \end{pmatrix} \otimes (-e_n \otimes I_m) \quad I_{k-1} \otimes \left\{ \begin{pmatrix} e_f & \Lambda & 0 \\ e_f & 0 & I_f \end{pmatrix} \otimes (-e_n \otimes I_m) \right\} \right] & F_{\text{ion}} & e_k \otimes \left\{ \begin{pmatrix} \Lambda \\ 0 \end{pmatrix} \otimes (C_n \otimes C_m) \right\} \\ \hline D_k^T \otimes (I_n \otimes I_v) & 0 & 0 & 0 & 0 \\ 0 & \left[-e_{k-1} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_f & 0 \\ 0 & 0 & F_f \end{pmatrix} \otimes C_n \quad I_{k-1} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_f & 0 \\ 0 & 0 & I_f \end{pmatrix} \otimes I_n \right] & 0 & 0 & 0 \\ 0 & 0 & \left[-e_{k-1} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_f & 0 \\ 0 & 0 & F_f \end{pmatrix} \otimes I_m \quad I_{k-1} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_f & 0 \\ 0 & 0 & I_f \end{pmatrix} \otimes I_m \right] & 0 & 0 \\ 0 & 0 & 0 & D_k^T \otimes I_m & 0 \end{array} \right] \tag{51}$$

where $F_{\text{geo}} = \text{blkdiag}[F_{\text{geo}}(1), \ldots, F_{\text{geo}}(k)]$ and $F_{\text{geo}}(i)$ given as in Eq. (21), whereas $F_{\text{ion}} = \text{blkdiag}[F_{\text{ion}}(1), \ldots, F_{\text{ion}}(k)]$ and $F_{\text{ion}}(i)$ given as in Eq. (22).

We will now derive the *S-transformation matrix* corresponding to the CC-R S-basis. Multiplication of $(S^{\perp}_{\text{CC-R}})^T$ with the null space matrix V_{net} as given in Eq. (23) yields the following block-diagonal matrix:

$$(S^{\perp}_{\text{CC-R}})^T V_{\text{net}} = \text{blkdiag} \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_f & 0 \\ 0 & 0 & I_f \end{pmatrix}, I_{n-1}, I_m, I_f \otimes I_{n-1}, I_f \otimes I_m \right\}. \tag{52}$$

Note that this result is simply an identity matrix. Its inverse matrix $[(S^{\perp}_{\text{CC-R}})^T V_{\text{net}}]^{-1}$ is therefore an identity matrix as well, such that $V_{\text{net}}[(S^{\perp}_{\text{CC-R}})^T V_{\text{net}}]^{-1} = V_{\text{net}}$ again. Finally, this results in the following S-transformation matrix for the CC-R S-basis:

$$\mathcal{S}_{\text{CC-R}} = I - V_{\text{net}}[(S^{\perp}_{\text{CC-R}})^T V_{\text{net}}]^{-1}(S^{\perp}_{\text{CC-R}})^T = I - V_{\text{net}}(S^{\perp}_{\text{CC-R}})^T =$$

$$\left[\begin{array}{c|c|c|c|c} I_k \otimes (I_n \otimes I_\nu) & 0 & 0 & 0 & 0 \\ \hline 0 & E_k \otimes \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_f & 0 \\ 0 & 0 & I_f \end{pmatrix} \otimes I_n \right\} + e_k c_k^T \otimes \left\{ \begin{pmatrix} 1 & 0 & \mu_{\text{IF}}^T \\ 0 & I_f & -\Lambda^{-1} e_f \mu_{\text{IF}}^T \\ 0 & 0 & I_f - e_f \mu_{\text{IF}}^T \end{pmatrix} \otimes E_n \right\} & 0 & 0 & e_k \otimes \left\{ \begin{pmatrix} 0 \\ I_f \\ 0 \end{pmatrix} \otimes (E_n \otimes c_m^T) \right\} \\ \hline 0 & e_k c_k^T \otimes \left\{ \begin{pmatrix} 1 & 0 & \mu_{\text{IF}}^T \\ 0 & I_f & -\Lambda^{-1} e_f \mu_{\text{IF}}^T \\ 0 & 0 & I_f - e_f \mu_{\text{IF}}^T \end{pmatrix} \otimes (-c_n^T \otimes e_m) \right\} & \left\{ I_k \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_f & 0 \\ 0 & 0 & I_f \end{pmatrix} - e_k c_k^T \otimes \begin{pmatrix} 0 & 0 & -\mu_{\text{IF}}^T \\ 0 & 0 & \Lambda^{-1} e_f \mu_{\text{IF}}^T \\ 0 & 0 & e_f \mu_{\text{IF}}^T \end{pmatrix} \right\} \otimes I_m & 0 & e_k \otimes \left\{ \begin{pmatrix} 0 \\ I_f \\ 0 \end{pmatrix} \otimes (-c_n^T \otimes I_m) \right\} \\ \hline 0 & 0 & 0 & I_k \otimes I_m & 0 \\ \hline 0 & 0 & 0 & 0 & I_f \otimes (E_n \otimes E_m) \end{array}\right] \tag{53}$$

In the above matrix, E_k denotes a matrix of dimension k that is computed as $E_k = C_k D_k^T = I_k - e_k c_k^T$ (similar for E_n and E_m). Some of its properties are that $E_k e_k = 0$, $E_k C_k = C_k$ and $E_k E_k = E_k$. Note that the above matrix indeed fulfills the properties of an S-transformation matrix, i.e., $\mathcal{S}_{\text{CC-R}} V_{\text{net}} = 0$, $\mathcal{S}_{\text{CC-R}} \mathcal{S}_{\text{CC-R}} = \mathcal{S}_{\text{CC-R}}$ and $\mathcal{S}_{\text{CC-R}} \mathcal{S}_{\text{CC-R}} = \mathcal{S}_{\text{CC-R}}$.

CC-S S-basis

In case of the CC-S S-basis, the S-basis constraints can be casted in the matrix $(S^{\perp}_{\text{CC-S}})^T$ as follows, where each of the five 'columns' represents a parameter group (i.e., receiver positions/ZTDs, receiver clocks/hardware biases, satellite clocks/hardware biases, ionospheric delays and ambiguities):

$$(S^{\perp}_{\text{CC-S}})^T = \begin{bmatrix} 0 & 0 & c_k^T \otimes \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_f & 0 \\ 0 & 0 & I_f \end{pmatrix} \otimes \frac{1}{m} e_m^T \right\} & 0 & 0 \\ 0 & c_k^T \otimes \left\{ \begin{pmatrix} 0 & 0 & \mu_{\text{IF}}^T \end{pmatrix} \otimes I_n \right\} & 0 & 0 & 0 \\ 0 & 0 & c_k^T \otimes \left\{ \begin{pmatrix} 0 & 0 & \mu_{\text{IF}}^T \end{pmatrix} \otimes D_m^T \right\} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_f \otimes (I_n \otimes \frac{1}{m} e_m^T) \\ 0 & 0 & 0 & 0 & I_f \otimes (c_n^T \otimes D_m^T) \end{bmatrix} \tag{54}$$

Its orthogonal complement $S_{\text{CC-S}}$, such that $(S^{\perp}_{\text{CC-S}})^T S_{\text{CC-S}} = 0$, can be constructed as the following matrix:

$$S_{\text{CC-S}} = \text{blkdiag} \left\{ I_k \otimes (I_n \otimes I_\nu),\ \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & I_f & 0 \\ 0 & 0 & F_f \end{pmatrix} \otimes I_n,\ I_{k-1} \otimes \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_f & 0 \\ 0 & 0 & I_f \end{pmatrix} \right\} \otimes I_n \right], \right.$$
$$\left. \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & I_f & 0 \\ 0 & 0 & F_f \end{pmatrix} \otimes D_m,\ I_{k-1} \otimes \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_f & 0 \\ 0 & 0 & I_f \end{pmatrix} \right\} \otimes I_m \right],\ I_k \otimes I_m,\ I_f \otimes (C_n \otimes D_m) \right\} \tag{55}$$

where use is made of the property that $e_m^T D_m = 0$. The *full-rank design matrix*, corresponding to the CC-S S-basis, is obtained by post-multiplying the rank-deficient design matrix A_{net} in Eq. (20) with the above $S_{\text{CC-S}}$ and reads as follows:

$$
A_{\text{net}}^{\text{CC-S}} = A_{\text{net}} S_{\text{CC-S}} =
\left[\begin{array}{c|c|c|c|c}
F_{\text{geo}} & \begin{bmatrix} \begin{pmatrix} e_f & \Lambda & 0 \\ e_f & 0 & F_f \end{pmatrix} \otimes (I_n \otimes e_m) & \\ & I_{k-1} \otimes \left\{ \begin{pmatrix} e_f & \Lambda & 0 \\ e_f & 0 & I_f \end{pmatrix} \otimes (I_n \otimes e_m) \right\} \end{bmatrix} & \begin{bmatrix} \begin{pmatrix} e_f & \Lambda & 0 \\ e_f & 0 & F_f \end{pmatrix} \otimes (-e_n \otimes D_m) & \\ & I_{k-1} \otimes \left\{ \begin{pmatrix} e_f & \Lambda & 0 \\ e_f & 0 & I_f \end{pmatrix} \otimes (-e_n \otimes I_m) \right\} \end{bmatrix} & F_{\text{ion}} & e_k \otimes \left\{ \begin{pmatrix} \Lambda \\ 0 \end{pmatrix} \otimes (C_n \otimes D_m) \right\} \\
\hline
D_k^T \otimes (I_n \otimes I_\nu) & 0 & 0 & 0 & 0 \\
0 & \begin{bmatrix} -e_{k-1} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_f & 0 \\ 0 & 0 & F_f \end{pmatrix} \otimes I_n & I_{k-1} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_f & 0 \\ 0 & 0 & I_f \end{pmatrix} \otimes I_n \end{bmatrix} & 0 & 0 & 0 \\
0 & 0 & \begin{bmatrix} -e_{k-1} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_f & 0 \\ 0 & 0 & F_f \end{pmatrix} \otimes D_m & I_{k-1} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_f & 0 \\ 0 & 0 & I_f \end{pmatrix} \otimes I_m \end{bmatrix} & 0 & 0 \\
0 & 0 & 0 & D_k^T \otimes I_m & 0
\end{array}\right] \tag{56}
$$

We will now derive the *S-transformation matrix* corresponding to the CC-S S-basis. Multiplication of $(S_{\text{CC-S}}^{\perp})^T$ with the null space matrix V_{net} as given in Eq. (23) yields the following matrix plus its inverse:

$$
(S_{\text{CC-S}}^{\perp})^T V_{\text{net}} =
\begin{bmatrix}
\begin{pmatrix} 1 & 0 & 0 \\ 0 & I_f & 0 \\ 0 & 0 & I_f \end{pmatrix} & 0 & \begin{pmatrix} -1 \\ \Lambda^{-1} e_f \\ e_f \end{pmatrix} \otimes \frac{1}{m} e_m^T & 0 & \begin{pmatrix} 0 \\ I_f \\ 0 \end{pmatrix} \otimes \frac{1}{m} e_m^T \\
\begin{pmatrix} 0 & 0 & \mu_{\text{IF}}^T \end{pmatrix} \otimes e_n & C_n & 0 & 0 & 0 \\
0 & 0 & D_m^T & 0 & 0 \\
0 & 0 & 0 & I_f \otimes C_n & I_f \otimes \left(e_n \otimes \frac{1}{m} e_m^T\right) \\
0 & 0 & 0 & 0 & I_f \otimes D_m^T
\end{bmatrix};
$$

$$
[(S_{\text{CC-S}}^{\perp})^T V_{\text{net}}]^{-1} =
\begin{bmatrix}
\begin{pmatrix} 1 & 0 & \mu_{\text{IF}}^T \\ 0 & I_f & -\Lambda^{-1} e_f \mu_{\text{IF}}^T \\ 0 & 0 & I_f - e_f \mu_{\text{IF}}^T \end{pmatrix} & \begin{pmatrix} -1 \\ \Lambda^{-1} e_f \\ e_f \end{pmatrix} \otimes c_n^T & 0 & \begin{pmatrix} 0 \\ I_f \\ 0 \end{pmatrix} \otimes -c_n^T & 0 \\
0 & D_n^T & 0 & 0 & 0 \\
\begin{pmatrix} 0 & 0 & \mu_{\text{IF}}^T \end{pmatrix} \otimes e_m & -c_n^T \otimes e_m & C_m - \frac{1}{m} e_m e_{m-1}^T & 0 & 0 \\
0 & 0 & 0 & I_f \otimes D_n^T & 0 \\
0 & 0 & 0 & I_F \otimes (c_n^T \otimes e_m) & I_f \otimes \left(C_m - \frac{1}{m} e_m e_{m-1}^T\right)
\end{bmatrix} \tag{57}
$$

This leads to:

$$
[(S_{\text{CC-S}}^{\perp})^T V_{\text{net}}]^{-1} (S_{\text{CC-S}}^{\perp})^T =
\begin{bmatrix}
0 & c_k^T \otimes \left\{ \begin{pmatrix} 0 & 0 & -\mu_{\text{IF}}^T \\ 0 & 0 & \Lambda^{-1} e_f \mu_{\text{IF}}^T \\ 0 & 0 & e_f \mu_{\text{IF}}^T \end{pmatrix} \otimes c_n^T \right\} & c_k^T \otimes \left\{ \begin{pmatrix} 1 & 0 & \mu_{\text{IF}}^T \\ 0 & I_f & -\Lambda^{-1} e_f \mu_{\text{IF}}^T \\ 0 & 0 & I_f - e_f \mu_{\text{IF}}^T \end{pmatrix} \otimes \frac{1}{m} e_m^T \right\} & 0 & \begin{pmatrix} 0 \\ I_f \\ 0 \end{pmatrix} \otimes -\left(c_n^T \otimes \frac{1}{m} e_m^T\right) \\
0 & c_k^T \otimes \left\{ \begin{pmatrix} 0 & 0 & \mu_{\text{IF}}^T \end{pmatrix} \otimes D_n^T \right\} & 0 & 0 & 0 \\
0 & c_k^T \otimes \left\{ \begin{pmatrix} 0 & 0 & \mu_{\text{IF}}^T \end{pmatrix} \otimes (-c_n^T \otimes e_m) \right\} & c_k^T \otimes \left\{ \begin{pmatrix} 0 & 0 & \mu_{\text{IF}}^T \end{pmatrix} \otimes I_m \right\} & 0 & 0 \\
0 & 0 & 0 & 0 & I_f \otimes \left(D_n^T \otimes \frac{1}{m} e_m^T\right) \\
0 & 0 & 0 & 0 & I_f \otimes (c_n^T \otimes I_m)
\end{bmatrix} \tag{58}
$$

Finally, this results in the following S-transformation matrix for the CC-S S-basis:

$$
\mathcal{S}_{\text{CC-S}} = I - V_{\text{net}} [(S_{\text{CC-S}}^{\perp})^T V_{\text{net}}]^{-1} (S_{\text{CC-S}}^{\perp})^T =
\left[\begin{array}{c|c|c|c|c}
I_k \otimes (I_n \otimes I_\nu) & 0 & 0 & 0 & 0 \\
\hline
0 & \left\{ I_k \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_f & 0 \\ 0 & 0 & I_f \end{pmatrix} - e_k c_k^T \otimes \begin{pmatrix} 0 & 0 & -\mu_{\text{IF}}^T \\ 0 & 0 & \Lambda^{-1} e_f \mu_{\text{IF}}^T \\ 0 & 0 & e_f \mu_{\text{IF}}^T \end{pmatrix} \right\} \otimes I_n & -e_k c_k^T \otimes \left\{ \begin{pmatrix} 1 & 0 & \mu_{\text{IF}}^T \\ 0 & I_f & -\Lambda^{-1} e_f \mu_{\text{IF}}^T \\ 0 & 0 & I_f - e_f \mu_{\text{IF}}^T \end{pmatrix} \otimes \frac{1}{m} e_n e_m^T \right\} & 0 & e_k \otimes \left\{ \begin{pmatrix} 0 \\ I_f \\ 0 \end{pmatrix} \otimes \left(I_n \otimes \frac{1}{m} e_m^T\right) \right\} \\
\hline
0 & 0 & E_k \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_f & 0 \\ 0 & 0 & I_f \end{pmatrix} \otimes I_m + e_k c_k^T \otimes \begin{pmatrix} 1 & 0 & \mu_{\text{IF}}^T \\ 0 & I_f & -\Lambda^{-1} e_f \mu_{\text{IF}}^T \\ 0 & 0 & I_f - e_f \mu_{\text{IF}}^T \end{pmatrix} \otimes Z_m & 0 & e_k \otimes \left\{ \begin{pmatrix} 0 \\ I_f \\ 0 \end{pmatrix} \otimes (-c_n^T \otimes Z_m) \right\} \\
\hline
0 & 0 & 0 & I_k \otimes I_m & 0 \\
\hline
0 & 0 & 0 & 0 & I_f \otimes (E_n \otimes Z_m)
\end{array}\right] \tag{59}
$$

In the above S-transformation matrix, the $m \times m$ matrix $Z_m = I_m - \frac{1}{m} e_m e_m^T$ denotes the "zero-mean" matrix. Some of its properties are that $Z_m e_m = 0$ and $Z_m Z_m = Z_m$. Note that the above matrix indeed fulfills the properties of an S-transformation matrix, i.e., $\mathcal{S}_{\text{CC-S}} V_{\text{net}} = 0$, $\mathcal{S}_{\text{CC-S}} \mathcal{S}_{\text{CC-S}} = \mathcal{S}_{\text{CC-S}}$ and $\mathcal{S}_{\text{CC-S}} \mathcal{S}_{\text{CC-S}} = \mathcal{S}_{\text{CC-S}}$.

Appendix 3: Square and invertible transformation between CC-R and CC-S systems

The full-rank transformation from the estimable parameters in the CC-S system to their counterparts in the CC-R system reads:

$$\underbrace{\left[\begin{array}{c} \begin{pmatrix} \mathrm{d}\tilde{t}_r(1) \\ \tilde{\delta}_r(1) \\ \tilde{d}_r(1) \end{pmatrix} \\ \begin{pmatrix} \mathrm{d}\tilde{t}_r \\ \tilde{\delta}_r \\ \tilde{d}_r \end{pmatrix} \\ \hline \begin{pmatrix} \mathrm{d}\tilde{t}^s(1) \\ \tilde{\delta}^s(1) \\ \tilde{d}^s(1) \end{pmatrix} \\ \begin{pmatrix} \mathrm{d}\tilde{t}^s \\ \tilde{\delta}^s \\ \tilde{d}^s \end{pmatrix} \\ \hline \tilde{z} \end{array}\right]}_{\tilde{x}} = \underbrace{\left[\begin{array}{cc|cc|c} \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_f & 0 \\ 0 & 0 & I_{f-1} \end{pmatrix} \otimes D_n^T & & & & \begin{pmatrix} 0 \\ I_f \\ 0 \end{pmatrix} \otimes (-I_{n-1} \otimes e_{m-1}^T) \\ -e_{k-1} \otimes \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_f & 0 \\ 0 & 0 & F_f \end{pmatrix} \otimes e_n c_n^T \right\} & I_{k-1} \otimes \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_f & 0 \\ 0 & 0 & I_f \end{pmatrix} \otimes I_n \right\} & & & e_{k-1} \otimes \left\{ \begin{pmatrix} 0 \\ I_f \\ 0 \end{pmatrix} \otimes (-C_n \otimes e_{m-1}^T) \right\} \\ \hline \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_f & 0 \\ 0 & 0 & I_{f-1} \end{pmatrix} \otimes (-c_n^T \otimes e_m) & & \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_f & 0 \\ 0 & 0 & I_{f-1} \end{pmatrix} \otimes D_m & & \\ -e_{k-1} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_f & 0 \\ 0 & 0 & F_f \end{pmatrix} \otimes (c_n^T \otimes e_m) & & & I_{k-1} \otimes \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_f & 0 \\ 0 & 0 & I_f \end{pmatrix} \otimes I_m \right\} & \\ \hline & & & & I_f \otimes (I_{n-1} \otimes D_m^T D_m) \end{array}\right]}_{T} \underbrace{\left[\begin{array}{c} \begin{pmatrix} \mathrm{d}\tilde{\tilde{t}}_r(1) \\ \tilde{\tilde{\delta}}_r(1) \\ \tilde{\tilde{d}}_r(1) \end{pmatrix} \\ \begin{pmatrix} \mathrm{d}\tilde{\tilde{t}}_r \\ \tilde{\tilde{\delta}}_r \\ \tilde{\tilde{d}}_r \end{pmatrix} \\ \hline \begin{pmatrix} \mathrm{d}\tilde{\tilde{t}}^s(1) \\ \tilde{\tilde{\delta}}^s(1) \\ \tilde{\tilde{d}}^s(1) \end{pmatrix} \\ \begin{pmatrix} \mathrm{d}\tilde{\tilde{t}}^s \\ \tilde{\tilde{\delta}}^s \\ \tilde{\tilde{d}}^s \end{pmatrix} \\ \hline \tilde{\tilde{z}} \end{array}\right]}_{\tilde{\tilde{x}}} \tag{60}$$

Here we made distinction between the estimable parameters at the first epoch, denoted using time index 1, and those for all other epochs, which are collected in one vector without any time index. For example, vector $\mathrm{d}\tilde{t}^s(1)$ denotes the satellite clock parameters at epoch 1 in the CC-R system, whereas vector $\mathrm{d}\tilde{t}^s$ denotes the satellite clock parameters in the same $\mathcal{S}$-system at all other epochs (2 to k).

The inverse transformation, i.e., the transformation from the estimable parameters in the CC-R system to the CC-S system, reads:

$$\underbrace{\left[\begin{array}{c} \begin{pmatrix} \mathrm{d}\tilde{\tilde{t}}_r(1) \\ \tilde{\tilde{\delta}}_r(1) \\ \tilde{\tilde{d}}_r(1) \end{pmatrix} \\ \begin{pmatrix} \mathrm{d}\tilde{\tilde{t}}_r \\ \tilde{\tilde{\delta}}_r \\ \tilde{\tilde{d}}_r \end{pmatrix} \\ \hline \begin{pmatrix} \mathrm{d}\tilde{\tilde{t}}^s(1) \\ \tilde{\tilde{\delta}}^s(1) \\ \tilde{\tilde{d}}^s(1) \end{pmatrix} \\ \begin{pmatrix} \mathrm{d}\tilde{\tilde{t}}^s \\ \tilde{\tilde{\delta}}^s \\ \tilde{\tilde{d}}^s \end{pmatrix} \\ \hline \tilde{\tilde{z}} \end{array}\right]}_{\tilde{\tilde{x}}} = \underbrace{\left[\begin{array}{cc|cc|c} \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_f & 0 \\ 0 & 0 & I_{f-1} \end{pmatrix} \otimes C_n & & \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_f & 0 \\ 0 & 0 & I_{f-1} \end{pmatrix} \otimes \left(-e_n \otimes \frac{1}{m} e_m^T\right) & & \begin{pmatrix} 0 \\ I_f \\ 0 \end{pmatrix} \otimes \left(C_n \otimes \frac{1}{m} e_{m-1}^T\right) \\ & I_{k-1} \otimes \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_f & 0 \\ 0 & 0 & I_f \end{pmatrix} \otimes I_n \right\} & -e_{k-1} \otimes \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_f & 0 \\ 0 & 0 & F_f \end{pmatrix} \otimes \left(e_n \otimes \frac{1}{m} e_m^T\right) \right\} & & e_{k-1} \otimes \left\{ \begin{pmatrix} 0 \\ I_f \\ 0 \end{pmatrix} \otimes \left(C_n \otimes \frac{1}{m} e_{m-1}^T\right) \right\} \\ \hline & & \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_f & 0 \\ 0 & 0 & I_{f-1} \end{pmatrix} \otimes \left(C_m^T - \frac{1}{m} e_{m-1} e_m^T\right) & & \\ & & -e_{k-1} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_f & 0 \\ 0 & 0 & F_f \end{pmatrix} \otimes \frac{1}{m} e_m e_m^T & I_{k-1} \otimes \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_f & 0 \\ 0 & 0 & I_f \end{pmatrix} \otimes I_m \right\} & \\ \hline & & & & I_f \otimes \left\{I_{n-1} \otimes \left[I_{m-1} - \frac{1}{m} e_{m-1} e_{m-1}^T\right]\right\} \end{array}\right]}_{T^{-1}} \underbrace{\left[\begin{array}{c} \begin{pmatrix} \mathrm{d}\tilde{t}_r(1) \\ \tilde{\delta}_r(1) \\ \tilde{d}_r(1) \end{pmatrix} \\ \begin{pmatrix} \mathrm{d}\tilde{t}_r \\ \tilde{\delta}_r \\ \tilde{d}_r \end{pmatrix} \\ \hline \begin{pmatrix} \mathrm{d}\tilde{t}^s(1) \\ \tilde{\delta}^s(1) \\ \tilde{d}^s(1) \end{pmatrix} \\ \begin{pmatrix} \mathrm{d}\tilde{t}^s \\ \tilde{\delta}^s \\ \tilde{d}^s \end{pmatrix} \\ \hline \tilde{z} \end{array}\right]}_{\tilde{x}} \tag{61}$$

Here use is made of the following properties: $D_m^T C_m = I_{m-1}$, $C_m D_m^T = I_m - e_m c_m^T$, $D_m^T e_m = 0$, $e_m^T D_m = 0$ and $(D_m^T D_m)^{-1} = I_{m-1} - \frac{1}{m} e_{m-1} e_{m-1}^T$.

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