Optimal Combination of Galileo Inter-Frequencies

B. Li, Y. Shen
Department of Surveying and Geo-informatics, Tongji University, 1239 Siping Road, Shanghai 200092, P.R. China

Abstract. There are four different frequencies in Galileo system, which can form more combinations than that of GPS. Therefore, it is worth, not only theoretically but also practically, to find the best or at least good combination for fast positioning. In this paper we first introduce the criteria for optimally combining the four frequencies based on 5 kinds of cost functions or constraints, then solve the correspondent coefficients for combinations. At last, we propose an algorithm for ambiguity resolution using optimally combined observables.

Keywords. Galileo, GPS, combination, optimization

1 Introduction

The linear combinations of dual-frequency GPS observables have played an important role in computing GPS baseline for a long time, because the combined observables have many advantages such as longer wavelength, reduced ionospheric delay and so on (Melbbourne, 1985; Han, 1995; Teunissen, 1997; Zhang et al., 2003; Schlötzer and Martin, 2005).

Galileo system will be established by European Union in recent years. Since 4 frequencies of Galileo observables can form more combinations than that of GPS, the phase ambiguities can be determined much more efficiently, and the systemic delay of some combined observables can also be significantly reduced. A lot of papers have discussed the combinations of Galileo inter-frequencies. For example, Wang and Liu (2003) gave out several combinations with relatively longer wavelength or smaller ionospheric delay, unfortunately most of the combinations possessing longer wavelength will have either bigger ionospheric delay or larger combined noise. Nevertheless, the best combination of Galileo inter-frequencies is still not discussed in the literatures until now. In the present contribution, we will investigate the optimal combinations based on different criteria and subjected to different constraints. These optimally combined observables are used to fix the phase ambiguities of uncombined observables. This paper is organized as follows. Section 2 will present the general form of inter-frequency combination. Section 3 will put forward 5 criteria of optimal combinations and solve combined coefficients. Section 4 will discuss the phase ambiguity resolution with optimal combined observables.

2 General Form of Inter-Frequency Linear Combination

2.1 Definition of Combined Observables

In the dual-difference model, the observation equation of the ith frequency can be expressed as,

$$\varphi_i = \rho/\lambda_i - N_i + T/\lambda_i - I_i/\lambda_i + \varepsilon_i/\lambda_i \qquad (1)$$

where, φ_i is differenced phase observable in cycle; ρ is the satellite-to-receiver range; I_i is ionospheric delay; T is tropospheric delay; λ_i, N_i and ε_i denote the wavelength, ambiguity and noise of observable, respectively. The general form of the combined observables can be represented as,

$$\varphi_C = n_1\varphi_1 + n_2\varphi_2 + n_3\varphi_3 + n_4\varphi_4 \qquad (2)$$

where, φ_C is combined observable; n_1, n_2, n_3 and n_4 are the coefficients of combination. The subscript "$_C$" here and in the following sections denotes combined variables. Substituting $\varphi_i = f_i t$ into (2), we obtain the following equation (Han, 1995; Hofmann et al., 2001),

$$\varphi_C = n_1 f_1 t + n_2 f_2 t + n_3 f_3 t + n_4 f_4 t = f_C t \qquad (3)$$

with combined frequency,

$$f_C = n_1 f_1 + n_2 f_2 + n_3 f_3 + n_4 f_4. \qquad (4)$$

The combined ambiguity is integer, if and only if the combination coefficients are integers, and the combined ambiguity is as follows,

$$N_C = n_1 N_1 + n_2 N_2 + n_3 N_3 + n_4 N_4 \quad (5)$$

If the combined frequency is not equal to zero, the wavelength of combined observable can be computed as,

$$\lambda_{Cm} = \frac{c}{f_C} = \frac{c}{n_1 f_1 + n_2 f_2 + n_3 f_3 + n_4 f_4} \quad (6)$$

where, c denotes the speed of light, the additional subscript "$_m$" indicates the unit in meter.

2.2 Definition of Basic Combined Variables Constrained to Non-Zero Combined Frequency

The ionospheric delay, tropospheric delay and the noise of combined observables can be derived from the original observables constrained to the condition that the combined frequency is not equal to zero. Because the ionospheric delays are inversely proportion to square of frequencies, they can be expressed as,

$$I_i = f_1^2 I \big/ f_i^2 \ (i{=}1, 2, 3, 4) \quad (7)$$

where, I is the ionospheric delay of the observable in first frequency. Therefore, we can derive the ionospheric delay of combined observable as follows,

$$I_{Cm} = \frac{n_1 + n_2 f_1/f_2 + n_3 f_1/f_3 + n_4 f_1/f_4}{n_1 + n_2 f_2/f_1 + n_3 f_3/f_1 + n_4 f_4/f_1} I \quad (8)$$

We can similarly derive the tropospheric delay of combined observable as,

$$T_{Cm} = \frac{n_1 f_1 + n_2 f_2 + n_3 f_3 + n_4 f_4}{n_1 f_1 + n_2 f_2 + n_3 f_3 + n_4 f} T = T \quad (9)$$

This indicates that the tropospheric delay is invariable for the non-zero combined frequency.

If the observation noise is one percent of wavelength (Hofmann et al., 2001; Wang and Liu, 2003), and σ_1, σ_2, σ_3 and σ_4 denote the noises of four frequencies in meter, respectively, i.e. $\sigma_i = \lambda_i \big/ 100 (i = 1, 2, 3, 4)$. According to the law of error propagation, we obtain the noise of combined observable as follows,

$$\sigma_{Cm} = \frac{c\sqrt{n_1^2 + n_2^2 + n_3^2 + n_4^2}}{100\,(n_1 f_1 + n_2 f_2 + n_3 f_3 + n_4 f_4)} \quad (10)$$

2.3 Ratio Factors for Evaluating Combined Observables

Ratios of the combined variables over corresponding original ones are used as the factors to evaluate the quality of combined observables. We define four kinds of ratio factors as follows,

$$\beta_\lambda = \frac{\lambda_{Cm}}{\lambda_4} = \frac{f_4}{n_1 f_1 + n_2 f_2 + n_3 f_3 + n_4 f_4} \quad (11)$$

$$\beta_T = T_{Cm} \big/ T \equiv 1 \quad (12)$$

$$\beta_I = \frac{I_{Cm}}{I} = \frac{n_1 + n_2 f_1/f_2 + n_3 f_1/f_3 + n_4 f_1/f_4}{n_1 + n_2 f_2/f_1 + n_3 f_3/f_1 + n_4 f_4/f_1} \quad (13)$$

$$\beta_\sigma = \frac{\sigma_{Cm}}{\sigma_4} = \frac{f_4\sqrt{n_1^2 + n_2^2 + n_3^2 + n_4^2}}{n_1 f_1 + n_2 f_2 + n_3 f_3 + n_4 f} \quad (14)$$

where, β_λ, β_T, β_I and β_σ represent the ratio factors of wavelength, troposheric, ionospheric and noise, respectively.

3 Optimal Combinations Based on Different Criteria

The optimal criterion is usually to minimize or maximize the cost function subjecti to some constraints. Three types of optimal combination models will be presented in this section in the following three cases: (1) combinations with non-zero combined frequency; (2) combinations with free tropospheric delay; and (3) combinations with much smaller ionosphere delay. Corresponding to each of the three kinds of constraints, not only the optimal combination but also the combinations with relatively better properties are given out in order to provide alternative choice for specific application.

3.1 Optimal Combination Models with Non-Zero Combined Frequency

In the first optimal combination model, the cost function is to maximize the wavelength of combined observable subjecting to the following three constraints. The first one is the wavelength of combined observable is greater than any of original wavelengths (See equations (6) and (10), and the second one is that the ratio of wavelength over noise is greater than 3. The criterion of the first optimal problem is represented as follows,

Table 1. Three combinations with the longest wavelength and relatively smaller noises

n_1	n_2	n_3	n_4	β_I	σ_{Cm}	λ_{Cm}/σ_{Cm}
0	–2	7	–5	–3.72	2.59	11.3
0	1	–3	2	–0.77	1.10	26.7
0	4	–13	9	2.17	4.78	6.13

$$\max: \lambda_{Cm} = c/(n_1 f_1 + n_2 f_2 + n_3 f_3 + n_4 f_4) \tag{15a}$$

$$constraints: \begin{cases} \lambda_{Cm} > \lambda_4 \\ \lambda_{Cm}/\sigma_{Cm} > 3 \\ n_1, n_2, n_3, n_4 \in Z \end{cases} \tag{15b}$$

There are enormous combinations fitting to the constraints (15b), and some of which combinations can get the same maximal wavelength 29.305 m. In other words, the solution of the optimal problem (15) is not unique. Fortunately, there is only one combination that has the smallest ionospheric delay and the smallest noise. Therefore, if the smallest ionospheric delay and the smallest noise are introduced as the additional constraint, it exists unique solution as shown in the second row of Table 1. In order to provide an alternative selection, we list 3 combinations with the longest wavelength and relatively smaller noises in Table 1.

In the second optimal combination model, the cost function is to maximize the ratio of combined wavelength over the noise subject to the same constraints as that in the first one. The cost function is as follows,

$$\max: \lambda_{Cm}/\sigma_{Cm} = 100 \Big/ \sqrt{n_1^2 + n_2^2 + n_3^2 + n_4^2} \tag{16}$$

The solution of this optimal problem is also not unique, and there are total 3 solutions with the maximal ratio $\lambda_{Cm}/\sigma_{Cm} = 70.71$ as listed in Table 2. Introducing an additional condition of maximizing the ratio λ_{Cm}/β_I, we get the unique solution as shown in the first row of Table 2.

In the third optimal combination model, the cost function is to minimize the ratio factor of ionospheric delay subject to the same constraints as the first and second problem. The cost function reads,

Table 2. Three combinations with the maximal ratio factor of wavelength

n_1	n_2	n_3	n_4	λ_{Cm}	β_I	λ_{Cm}/β_I	σ_{Cm}
0	0	1	–1	9.768	–1.75	5.59	0.138
0	1	–1	0	4.187	–1.61	2.60	0.059
0	1	0	–1	2.931	–1.65	1.78	0.041

Table 3. Three combinations with smaller ionospheric delay

n_1	n_2	n_3	n_4	λ_{Cm}	β_I	σ_{Cm}	$\frac{\lambda_{Cm}}{\sigma_{Cm}}$
3	–6	–11	14	1.221	–0.00068	0.232	5.26
5	–19	10	4	0.837	–0.00278	0.188	4.46
6	–19	0	13	0.666	0.00244	0.159	4.20

$$\min: \beta_I = \frac{n_1 + n_2 f_1/f_2 + n_3 f_1/f_3 + n_4 f_1/f_4}{n_1 + n_2 f_2/f_1 + n_3 f_3/f_1 + n_4 f_4/f_1} \tag{17}$$

There is unique solution with the coefficients of combination (3, -6, -11, 14) and the ratio factor is $\beta_I = -0.00068$. Three kinds of combinations with relatively smaller ionospheric delay are enumerated in Table 3. The optimal combination in the first row of Table 3 significantly reduces the ionospheric delay.

3.2 Optimal Combination Models with Free Tropospheric Delay

If the frequency of combined observable equals to zero, the wavelength will be infinite and the tropospheric delay will be completely eliminated. In fourth optimal combination model, the first constraint is that the frequency of combined observable is equal to zero. In this case, the noise of the combined observable can only be expressed in cycle since the wavelength is infinite. The second constraint is that the noise of combined observable is smaller than 0.3 cycles. Since the noise can be expressed as,

$$\sigma_C = \sqrt{n_1^2 + n_2^2 + n_3^2 + n_4^2} \Big/ 100 \tag{18}$$

The criterion of the fourth optimal problem is as follows,

$$\begin{cases} f_c = 0 \\ \sqrt{n_1^2 + n_2^2 + n_3^2 + n_4^2} < 100 \times 0.3 \end{cases} \tag{19}$$

There are also a lot of combinations solved by the optimal problem (19). Among these combinations, we need to select some better combinations with comparatively smaller ratio factor of ionospheric delay. In order to compare the ratio factors among different combinations, an arbitrary wavelength, which is normally selected as λ_1, is used to substitute the infinite wavelength. Then, the ratio factor of

Table 4. Three better troposphere-free combinations

N_1	n_2	n_3	n_4	λ_C	β_I	σ_C
–4	5	–3	3	∞	2.26	0.077
1	–6	7	–2	∞	0.07	0.095
1	–3	–3	5	∞	0.08	0.066

ionospheric delay can be derived and expressed as follows,

$$\beta_I = n_1 + n_2 f_1 / f_2 + n_3 f_1 / f_3 + n_4 f_1 / f_4 \quad (20)$$

We enumerate 3 combinations with comparatively smaller noise and ionosphere delay in Table 4. The combination in the last row of Table 4 has bigger ratio factor of ionospheric delay, but much smaller noise, than that in the second row. Therefore, the combination in the last row of Table 4 is taken as the optimal combination.

3.3 Optimal Combination Models with Smaller Ionospheric Delay

In the fifth optimal combination model, the cost function is to maximize the wavelength of combined observable subject to two constraints. The first one is that the ratio factor of ionospheric delay is less than 0.001, which indicates that the ionospheric delay of combined observable is so small that it can be neglected. The second constraint is that the ratio factor of wavelength must be greater than 3. Therefore, the criterion of the fifth optimal combination model can be summarized as,

$$\max : \lambda_{Cm} = c / n_1 f_1 + n_2 f_2 + n_3 f_3 + n_4 f_4 \quad (21a)$$

$$\text{constraints} : \begin{cases} -0.001 < \beta_I < 0.001 \\ \lambda_{Cm} / \sigma_{Cm} > 3 \\ n_1, n_2, n_3, n_4 \in Z \end{cases} \quad (21b)$$

There is the unique solution for the optimal problem (21) with the wavelength as long as 1.221 meters shown in the first row of Table 5. Although there are many combinations fitting the constraints (21b), all of the wavelengths are less than 15 centimeters except for the optimal one. The other two comparatively better combinations with relatively smaller ionospheric delay and relatively longer wavelength are also listed in the table.

Table 5. The combinations with relatively greater ratio factor of wavelength

n_1	n_2	n_3	n_4	λ_{Cm}	I_{Cm}	λ_{Cm}/σ_{Cm}
3	–6	–11	14	1.22	–0.068	5.26
2	3	10	–14	0.116	–0.060	5.68
3	3	0	–5	0.112	0.020	15.33

4 Ambiguity Resolution with the Combined Observables

In this section, we will discuss the application of the above optimal combinations in fixing the phase ambiguities, and these combinations are listed in the Table 6. The tropospheric delay can be efficiently corrected via models, and the ionospheric delay could be significantly reduced via proper combination of observables. The ionospheric delay of the first three combinations in Table 6 have been greatly reduced so that their influences can be ignored, and the ratios of λ_{Cm}/σ_{Cm} are all larger than 5. Although the fourth combination in Table 6 will enlarge the ionospheric delay, it amplifies the wavelength much more significantly, its ratio of phase wavelength over β_I is equal to 29.33, which is much larger than 3.53 for the wide-lane combination in GPS. Therefore, the ambiguities of the combined observables can be determined much easier than that of the wide-lane combination in GPS. The algorithm for fixing phase ambiguities can be summarized as follows,

1. Computing four combined phase observables in Table 6, e.g. $l_{1,-3,-3,5}$, $l_{3,3,0,-5}$, $l_{0,1,-1,0}$ and $l_{0,0,1,-1}$, according to $l_{i,j,k,m} = i\varphi_1 + j\varphi_2 + k\varphi_3 + m\varphi_4$.
2. The dual-difference observation equation for the ith satellite at the jth epoch is given as,

$$\mathbf{l}_i^j = \begin{pmatrix} \mathbf{A}_i^j & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{b} \\ \mathbf{a}_i \end{pmatrix} \quad (22)$$

Table 6. The combinations used to fix phase ambiguity

n_1	n_2	n_3	n_4	λ_{Cm}	β_I	λ_{Cm}/σ_{Cm}
1	–3	–3	5	∞	0.0800	15.08
3	3	0	–5	0.112	0.002	15.33
0	1	–1	0	4.187	–1.61	70.71
0	0	1	–1	9.768	–1.7500	70.71

where, $\mathbf{b} = \begin{pmatrix} x & y & z \end{pmatrix}^T$ is the baseline vector; $\mathbf{a}_i = \left(N^i_{1,-3,-3,5}\, N^i_{3,3,0,-5}\, N^i_{0,1,-1,0}\, N^i_{0,0,1,-1}\right)^T$ is the vector of 4 dual-difference combined ambiguities, $\mathbf{A}^j_i$ is the 4×3 design matrix for the baseline coordinates and coefficient matrix $\mathbf{I}$, respecting to combined ambiguities, is a 4×4 identity matrix. $\mathbf{l}^j_i = \begin{pmatrix} l_{1,-3,-3,5}\, l_{3,3,0,-5}\, l_{0,1,-1,0}\, l_{0,0,1,-1} \end{pmatrix}^T$ is the vector of combined observables. If there are $n+1$ satellites simultaneously tracked by two stations for m epochs, we get the observation equation as,

$$\mathbf{L} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} \begin{pmatrix} \mathbf{b} \\ \mathbf{a} \end{pmatrix} \tag{23}$$

where, $\mathbf{A} = \begin{pmatrix} (\mathbf{A}^1_1)^T & (\mathbf{A}^1_2)^T & \cdots & (\mathbf{A}^m_n)^T \end{pmatrix}^T$, $\mathbf{B} = \mathbf{e}_m \otimes \mathbf{I}_{4n}$, $\mathbf{e}_m$ is m column unit vector and $\mathbf{I}_{4n}$ is $4n \times 4n$ identity matrix, ; $\mathbf{a} = \begin{pmatrix} \mathbf{a}^T_1 & \cdots & \mathbf{a}^T_n \end{pmatrix}^T$, and $\mathbf{L} = \begin{pmatrix} \mathbf{l}^1_1 & \mathbf{l}^1_2 & \cdots & \mathbf{l}^m_n \end{pmatrix}^T$.

3. Solving (23) with the least squares adjustment. The estimates and their covariance matrix are,

$$\begin{pmatrix} \hat{\mathbf{b}} \\ \hat{\mathbf{a}} \end{pmatrix} \begin{pmatrix} \mathbf{Q}_{\hat{b}} & \mathbf{Q}_{\hat{b}\hat{a}} \\ \mathbf{Q}_{\hat{a}\hat{b}} & \mathbf{Q}_{\hat{a}} \end{pmatrix} \tag{24}$$

where, $\mathbf{Q}_{\hat{b}}$ and $\mathbf{Q}_{\hat{a}}$ are respectively the covariance matrix of $\hat{\mathbf{b}}$ and $\hat{\mathbf{a}}$, and $\mathbf{Q}_{\hat{b}\hat{a}}$ is the covariance between $\hat{\mathbf{b}}$ and $\hat{\mathbf{a}}$. The integer ambiguities of combined observables are searched based on the following optimal problem,

$$\min : \left(\mathbf{a} - \hat{\mathbf{a}}\right)^T \mathbf{Q}^{-1}_{\hat{a}} \left(\mathbf{a} - \hat{\mathbf{a}}\right) \text{ with } \mathbf{a} \in Z^n \tag{25}$$

Since the optimally combined observables have better properties than the original observables, the covariance matrix $\mathbf{Q}_{\hat{a}}$ of combined ambiguities is less correlated and therefore the combined ambiguities can be solved much more efficiently than original ambiguities. Once four combined ambiguities for each satellite are obtained, the original ambiguities can be easily calculated by inversely multiplying the 4×4 squared matrix of combination coefficients. This procedure is just like the decorrelation algorithm (Teunissen, 1995, 1997; de Jonge and Tiberius, 1996; Xu, 2001, 2006). In fact, the decorrelation is to find better combination for all ambiguities to be searched, while the combinations discussed in this paper are among the different frequency observables from the same station to the same satellite.

5 Concluding Remarks

The 4 optimal combinations are derived for Galileo 4 frequency observables based on the 5 optimal criteria. An algorithm is proposed for ambiguity resolution with the optimally combined observables. We will carry out simulation experiments to numerically study the efficiency of ambiguity resolution with the combined observables and compare this algorithm with LAMBDA or decorrelation algorithm in our next contribution.

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