The Distributional Dependence Of The Range On Triple Frequency GPS Ambiguity Resolution

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BIOGRAPHY

Dr. Paul de Jonge obtained a PhD in Geodesy at the Delft University of Technology. He is currently employed at the Scripps Institution of Oceanography. Dr. Peter Teunissen is Professor of Mathematical Geodesy and Positioning. Niels Jonkman and Peter Joosten both graduated at the faculty of Geodesy of the Delft University of Technology. They are currently engaged in the development of GPS data processing strategies for medium scaled networks with an emphasis on ambiguity resolution.

Abstract

Both GPS and the envisioned European satellite navigation system Galileo are intended to transmit signals on more than two navigational frequencies. This increase in the number of signals aims, among other things, at enhancing the possibilities for the most precise form of satellite positioning, relative carrier phase positioning by means of ambiguity resolution. In this contribution the performance of triple-frequency ambiguity resolution will be analysed for the modernized GPS. Emphasis is put on long baselines, i.e. baselines for which the atmospheric disturbances cannot be neglected. The analysis will concentrate in particular on the probability distribution of the "fixed" double difference ranges and on its dependence on ambiguity resolution. We consider the quality and impact of both complete and partial ambiguity resolution. This performance analysis is based on the LAMBDA ambiguity success rate, being the probability of correct integer least-squares estimation. It is shown that fast partial ambiguity resolution is feasible, whereas a complete ambiguity resolution is not. It is also shown that fast, triple-frequency, dual-ambiguity resolution - as opposed to the single-ambiguity resolution variant - will significantly improve the double difference range solution. The important conclusion is therefore reached, that modernized GPS with an appropriate dual-ambiguity resolution in place, has the potential of quickly delivering improved ranges over long baselines.

1 INTRODUCTION

GPS satellites currently transmit signals on two frequencies in order to account for the dispersive part of atmosphere induced distortions, the so-called ionospheric delays. Civil access to the signal on the second of the two GPS frequencies however is restricted by a United States Department of Defense imposed encryption of the signal. Although this encryption can be circumvented through specially adapted receiver design, a second non-encrypted signal has long been advocated for civilian users of the GPS. In response to this lobby, a new Global Positioning System modernization initiative was announced on January 25th 1999 by Vice President Gore of the United States. The announcement contained the promise to implement a second civil signal on the GPS block IIF satellites. In addition to this second civil signal however, the announcement also indicated the implementation of an unencrypted signal on a third frequency.

The reason for incorporating two rather than one additional civil signal in the Block IIF satellite design was hinted at in the announcement of the Vice President as "the new signals will enable unprecedented real-time determination of highly accurate positions anywhere on earth". The third signal appears therefore to be specifically aimed at enhancing the possibilities for the most precise form of GPS positioning, relative carrier phase positioning by means of ambiguity resolution. The signal is apparently expected to shorten the time span necessary for the successful estimation at the correct integer values of these ambiguities or to stretch the baseline length for which a successful resolution is possible in real-time. In this contribution the latter type of improvement will be investigated in more detail.

This contribution is organised as follows. In section 2 we first present some general theory of ambiguity resolution, the main results of which are summarized in a theorem. It includes the exact form of the probability distribution of the 'fixed' GPS baseline, together with its dependence on the ambiguity success rate. In section 3, the theory is applied to the ionosphere-float, triple-frequency, geometry-free GPS model. We present the multi-modal distribution of the 'fixed' double difference range and show that fast ambiguity resolution for long baselines will remain problematic, even when the third frequency is included. In section 4, we relax the integer constraints on the ambiguities and analyse the performance of partial ambiguity resolution. In the triple-frequency case, two stages of partial ambiguity resolution are possible, a first stage in which only one ambiguity is considered integer and a second stage in which two ambiguities are assumed integer. The choice of which ambiguity to consider, is determined by means of the decorrelating
ambiguity transformation of the LAMBDA method. Both the stages of partial ambiguity resolution are analysed by means of the ambiguity success rate and the distribution of the corresponding 'fixed' double difference range.

2 GENERAL THEORY

2.1 Integer ambiguity resolution

Ambiguity resolution applies to a great variety of GPS models which are currently in use for navigation, surveying, geodesy and geophysics. An overview of these models, together with their applications, can be found in textbooks such as [Hofmann-Wellenhof, 1997], [Leick, 1995], [Parkinson and Spilker, 1996], [Strang and Borre, 1997], and [Teunissen and Kleusberg, 1998]. Any GPS model can be cast in the following system of linearized observation equations

\[ y = A\alpha + Bb + e \]  

where \( y \) is the given GPS data vector of order \( m \), \( \alpha \) and \( b \) are the unknown parameter vectors respectively of order \( n \) and \( o \), and \( e \) is the noise vector. The data vector \( y \) will usually consist of the 'observed minus computed' single-, dual- or triple-frequency double-difference (DD) phase and/or pseudo range (code) observations accumulated over all observation epochs. The entries of vector \( \alpha \) are then the DD carrier phase ambiguities, expressed in units of cycles rather than distance. They are known to be integers, \( \alpha \in \mathbb{Z}^n \). The entries of the vector \( b \) will consist of the remaining unknown parameters, such as for instance baseline components (coordinates) and possibly atmospheric delay parameters (troposphere, ionosphere). They are known to be real-valued, \( b \in \mathbb{R}^o \). Although vector \( b \) may contain real-valued parameters other than the baseline components, we will in this contribution, as a matter of terminology, still call its estimator the baseline estimator.

The procedure which is usually followed for solving the GPS model (1), can be divided into three steps [Teunissen, 1993]. In the first step one simply disregards the integer constraints \( \alpha \in \mathbb{Z}^n \) on the ambiguities and performs a standard least-squares adjustment. As a result one obtains the (real-valued) estimates of \( \alpha \) and \( b \), together with their variance-covariance (vc-) matrix

\[ \begin{bmatrix} \hat{\alpha} \\ \hat{b} \end{bmatrix}, \begin{bmatrix} Q_\alpha & Q_{\alpha b} \\ Q_{b\alpha} & Q_b \end{bmatrix} \]  

This solution is referred to as the 'float' solution. In the second step the 'float' ambiguity estimate \( \hat{\alpha} \) is used to compute the corresponding integer least-squares ambiguity estimate \( \tilde{\alpha} \),

\[ \tilde{\alpha} = \arg \min_{z \in \mathbb{Z}^n} || \hat{\alpha} - z ||_Q^2 \]  

Once the integer least-squares ambiguities are computed, they are used in the third step to finally correct the 'float' estimate of \( b \). As a result one obtains the 'fixed' solution

\[ \hat{b} = \hat{b} - Q_{\alpha b} Q_\alpha^{-1}(\tilde{\alpha} - \hat{\alpha}) \]  

In this contribution we will discuss the probabilistic consequences of relation (4). We will refrain however, from discussing the computational intricacies of integer estimation. For a discussion of these aspects, we refer to e.g. [Teunissen, 1993], [de Jonge and Tiberius, 1996a] or to the textbooks [Hofmann-Wellenhof, 1997], [Strang and Borre, 1997], [Teunissen and Kleusberg, 1998]. A very efficient method of solving the integer least-squares problem is provided by the LAMBDA method. A description of the LAMBDA method can be found in the aforementioned publications, while practical results obtained with it can be found in e.g. [Boon and Ambrosius, 1997], [Boon et al., 1997], [Cox and Brading, 1999], [de Jonge and Tiberius, 1996b], [de Jonge et al., 1996], [Han, 1995], [Peng et al., 1999], [Tiberius and de Jonge, 1995], [Tiberius et al., 1997].

2.2 Baseline distribution after ambiguity resolution

Ambiguity resolution is not a goal in itself. The purpose of ambiguity resolution is to obtain an improved estimator of the baseline by using the integer ambiguity constraints. Hence, in order to judge the significance of ambiguity resolution, one needs to be able to infer its impact on the baseline estimator. This is possible once the probability distribution of the fixed baseline estimator is known. This distribution is given in the following theorem.

Theorem [Distribution of the 'fixed' baseline]

Let the 'float' solution be normally distributed as

\[ \begin{bmatrix} \hat{\alpha} \\ \hat{b} \end{bmatrix} \sim N\left( \begin{bmatrix} \alpha \\ b \end{bmatrix}, \begin{bmatrix} Q_\alpha & Q_{\alpha b} \\ Q_{b\alpha} & Q_b \end{bmatrix} \right) \]

and let the 'fixed' solution be defined as

\[ \tilde{\alpha} = \arg \min_{z \in \mathbb{Z}^n} || \hat{\alpha} - z ||_Q^2 \]

\[ \tilde{b} = \hat{b} - Q_{\alpha b} Q_\alpha^{-1}(\tilde{\alpha} - \hat{\alpha}) \]

The distribution of the 'fixed' solution follows then as

\[ P(\tilde{\alpha} = z) = \int_{S_z} p_\tilde{\alpha}(y)dy \]

\[ p_\tilde{\alpha}(x) = \sum_{z \in \mathbb{Z}^n} p_{b|\tilde{\alpha}}(x \mid y = z)P(\tilde{\alpha} = z) \]  

with: \( S_z = \{ y \in \mathbb{R}^n \mid z = \arg \min_{u \in \mathbb{Z}^n} || y - u ||_Q^2 \} \) in which the marginal ambiguity and conditional baseline distribution are given as

\[ p_{\tilde{\alpha}}(y) = \frac{1}{\sqrt{\det Q_\alpha(2\pi)^n}} \exp\left( -\frac{1}{2} || y - \hat{\alpha} ||_Q^2 \right) \]

\[ p_{b|\tilde{\alpha}}(x \mid y = z) = \frac{1}{\sqrt{\det Q_{b|\tilde{\alpha}}(2\pi)^n}} \exp\left( -\frac{1}{2} || x - \hat{b}_{|\tilde{\alpha}=z} ||_{Q_{b|\tilde{\alpha}}}^2 \right) \]

with the conditional mean

\[ \hat{b}_{|\tilde{\alpha}=z} = \hat{b} - Q_{\alpha b} Q_\alpha^{-1}(\alpha - z) \]
and the conditional variance matrix
\[ Q_{\hat{b}|a} = Q_b - Q_{\hat{b}a}Q_a^{-1}Q_{\hat{a}b} \]

This theorem was first introduced and proved in [Teunissen, 1999a]. A generalization of the theorem, determining the 'fixed' baseline distribution for a whole class of integer ambiguity estimators, is given in [Teunissen, 1999b]. As the theorem shows, the 'fixed' baseline distribution equals an infinite sum of weighted conditional baseline distributions. These conditional baseline distributions are shifted versions of one another. The size and direction of the shift is governed by \( Q_{\hat{b}a}Q_a^{-1}z \) where \( z \in \mathbb{Z}^n \). Each of the conditional baseline distributions in the sum is downweighted. These weights are given by the probability masses of the distribution of the integer ambiguity estimator \( \hat{a} \). Note that the probability mass function \( P(\hat{a} = z) \) is symmetric about \( a \) and that the probability density function \( p_b(x) \) is symmetric about \( b \). This implies that the two estimators \( \hat{a} \) and \( \hat{b} \) are unbiased.

Now that we know the distribution of the 'fixed' baseline estimator, we can study its quality by means of the probability that \( \hat{b} \) lies in a certain region \( R \subseteq \mathbb{R}^p \). In general it will be difficult to evaluate this probability exactly. For practical purposes it is therefore of importance to have bounds available for the probability \( P(\hat{b} \in R) \). We will assume \( R \) to be convex and symmetric about \( b \). In that case the integral of \( p_{\hat{b}|a}(x \mid y = z) \) over \( R \) reaches its maximum for \( z = a \). This shows that \( P(\hat{b}_{\hat{a}=a} \in R) \) can be taken as upper bound. A lowerbound is also easily found. Since the entries in the sum of (5) are all nonnegative, any finite sum of nonzero entries can be used to obtain a lower bound. The more nonzero entries are used in this finite sum, the sharper this lower bound becomes. As a result we have obtained the following corollary.

**Corollary**

Let \( R \subseteq \mathbb{R}^p \) be any convex set symmetric about \( b \). The probability \( P(\hat{b} \in R) \) of the 'fixed' baseline estimator, can then be bounded from above and from below as
\[
P(\hat{b}_{\hat{a}=a} \in R)P(\hat{a} = a) \leq P(\hat{b} \in R) \leq P(\hat{b}_{\hat{a}=a} \in R)
\]

(6)

Note that this interval relates the probability of the 'fixed' baseline estimator to that of the conditional estimator and to the probability of correct integer ambiguity estimation, or ambiguity success rate \( P(\hat{a} = a) \). The above bounds become tight when the ambiguity success rate approaches one. This shows that, although the probability of the conditional estimator always overestimates the probability of the 'fixed' baseline estimator, the two probabilities are close for large values of the probability of correct integer estimation. This implies that in case of GPS ambiguity resolution, one should first evaluate \( P(\hat{a} = a) \) and make sure that its value is close enough to one, before making any inferences on the basis of the distribution of the conditional baseline estimator. In other words, the (unimodal) distribution of the conditional estimator is a good approximation to the (multimodal) distribution of the 'fixed' baseline estimator, when the success rate is sufficiently close to one. The value to be chosen for \( P(\hat{a} = a) \) depends on the error one is willing to accept. If one accepts a relative error of
\[
0 \leq \frac{P(\hat{b}_{\hat{a}=a} \in R) - P(\hat{b} \in R)}{P(\hat{b} \in R)} \leq 10^{-s}
\]
then the required probability of correct integer estimation should be set at \( P(\hat{a} = a) = \frac{1}{1 + 10^{-s}} \). In the sections following we will apply the above theory to the triple frequency, geometry-free GPS model.

### 3 TRIPLE FREQUENCY GPS

#### 3.1 The geometry-free model

The geometry-free model is the simplest possible mathematical model for the adjustment of GPS observations that still allows the estimation of integer carrier phase ambiguities. In its most basic form, the model consists of the DD pseudo range and carrier phase observations of two receivers to two satellites, parametrized in terms of an unknown DD satellite-receiver range, unknown DD ambiguities and an unknown DD ionospheric delay.

The pseudo range and carrier phase observations in meters, \( \rho \), \( \alpha \) and \( I \) denote the unknown range, the integer phase ambiguity and the ionospheric delay and \( \lambda \) denotes the wavelength of the carrier. The lower index \( i \) indicates the dependence of the observations and unknowns on the navigation frequency. In case of modernized GPS, we have \( i = 1, 2, 3 \). The current two and future three GPS frequencies are listed in table 1. Note that the above geometry-free model is a particular case of the general GPS model (1). The three integer ambiguities \( \alpha_1, \alpha_2, \) and \( \alpha_3 \) form the entries of the ambiguity vector \( \alpha \), and the real-valued unknown parameters \( \rho \) and \( I \) form the entries of the vector \( b \).

<table>
<thead>
<tr>
<th></th>
<th>F1 (MHz)</th>
<th>F2 (MHz)</th>
<th>F3 (MHz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>GPS2</td>
<td>1575.420</td>
<td>1227.600</td>
<td>-</td>
</tr>
<tr>
<td>GPS3</td>
<td>1575.420</td>
<td>1227.600</td>
<td>1176.450</td>
</tr>
</tbody>
</table>

Table 1. The two frequencies of current GPS (GPS2) and the three frequencies of the modernized GPS (GPS3).

For the case of three frequency observations, the single epoch geometry-free model consists of three pairs of pseudo range and carrier phase observation equations: one pair for each frequency. For any additional epochs of data, the model has to be extended with additional unknowns for the range and ionospheric delay. The ambiguities however remain constant if no loss-of-lock occurs. The above-mentioned equations contain unknowns for the ionospheric delays in the observations. The delays are...
however negligible for short baselines, i.e. baselines up to 10 to 20 kilometers. Hence, for such short baselines the ambiguities could be estimated from a model without ionospheric unknowns. To distinguish this model from the model with ionospheric unknowns, it is usually indicated as the ionosphere-fixed model, whereas the model with ionospheric unknowns is indicated as the ionosphere-float model.

Contrary to the ionosphere-float model however, the ionosphere-fixed model already yields very high success rates if observations on only two frequencies are available [Jonkman, 1998]. With pseudo range and carrier phase observations of moderate precision, the single epoch two frequency success rate will already be close to one, if the ionospheric delays can be neglected, while it is extremely small if the delays have to be included in the model. The impact of the third observation frequency on the ionosphere-float success rates is therefore by far the more important and the analyses in this contribution will therefore be limited to this long baseline case.

In the following it will be assumed that the 'float' solution of the above model is obtained in a standard least-squares sense. The ambiguities are considered to be time-invariant for the duration of the observation period. We also assume that time correlation and cross correlation are absent. Unless otherwise stated, the undifferenced variances of the carrier phase and pseudo range (code) observations are chosen as $\sigma_p^2 = (2.5 \text{mm})^2$ and $\sigma_s^2 = (25 \text{cm})^2$.

3.2 The distribution of the ambiguity fixed DD range

In this section we will study the distribution of the estimator corresponding standard deviation of the DD range reduces dramatically and becomes equal to

$$\sigma_\beta \approx \frac{\sigma_p \sigma_s}{\sigma_p^2 + \sigma_s^2}$$

when the ambiguity success rate equal one. This follows from (9), since $p_\beta(x)$ becomes equal to $p_{\beta|\beta}(x \mid y = \alpha)$ when $P(\tilde{a} = \alpha) = 1$.

The actual single-epoch distributions of $\tilde{\rho}$ (Gaussian) and $\tilde{\rho}$ (multi-modal) are shown in figure 1 (left). The multimodal distribution of $\tilde{\rho}$ has been generated according to equations (5), wherein the weights $P(\tilde{a} = z), z \in Z^n$ have been approximated through simulation. The inset of figure 1 (left) contains a detail of the distribution of $\tilde{\rho}$, clearly illustrating that the spike-like features correspond with downweighted and shifted versions of the conditional distribution $p_{\beta|\beta}(x \mid y = \alpha)$.

Figure 1 (left) shows that the distribution of the 'fixed' DD range does not resemble the conditional distribution $p_{\beta|\beta}(x \mid y = \alpha)$ at all. The distribution of the 'fixed' DD range can not be considered an overall improvement over the distribution of the 'floated' DD range. Although $p_{\beta}(x)$ is more peaked at the centre than $p_\beta(x)$, it still has a significant amount of probability mass at its two main side-lobes.

The explanation for the above shown characteristics of $p_\beta(x)$ lies in the values taken by the probability mass function $P(\tilde{a} = z)$, and in particular in the value taken by the ambiguity success rate $P(\tilde{a} = \alpha)$. Due to the very low value of the ambiguity success rate in the present case only 0.075, many integer vectors other than the correct one, have a significant probability of being selected as the integer least-squares solution. The three spikes in the inset of figure 1 (left) for example, correspond with the integer vectors $(5, -1, 0)^T$, $(4, -1, 0)^T$ and $(3, -1, 0)^T$, whereas the correct integer least squares solution equals $(0, 0, 0)^T$. These three candidates have a probability of being selected of 0.004, 0.005 and 0.006 respectively, which compared to the small success rate of 0.075 is indeed considerable.

The conclusion reads therefore that triple-frequency, instantaneous ambiguity resolution can not be expected to be successful for long baselines. And as shown in table 2, the same conclusion is reached for other measurement precisions of the pseudo range data as well.

<table>
<thead>
<tr>
<th>$\sigma_s$ [cm]</th>
<th>GPS2</th>
<th>GPS3</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>0.0488</td>
<td>0.1455</td>
</tr>
<tr>
<td>25</td>
<td>0.0237</td>
<td>0.0748</td>
</tr>
<tr>
<td>35</td>
<td>0.0134</td>
<td>0.0435</td>
</tr>
</tbody>
</table>

Table 2. Single epoch, ionosphere-float, success rates for GPS2 and GPS3. Success rates refer to undifferenced pseudo range standard deviations of 15, 25 and 35 centimeters. The ratio of undifferenced phase and pseudo range standard deviations was kept fixed to 1/100.

Table 2 also shows by how much the ambiguity success rate improves as a result of including the third GPS frequency. But although modernized GPS will require a much lower number of epochs than the current dual-frequency GPS to achieve a high success rate, the number of epochs needed is still significant. In order to achieve a success rate of 99 percent for example, the number of epochs required lies in the range of 190 to 1040, depending on the precision of the pseudo range observations (see table 3).
4 PARTIAL AMBIGUITY RESOLUTION

4.1 The procedure

So far the aim was to resolve all three ambiguities simultaneously. However, as the results of the previous section have shown, this would require many epochs of data for the case modernized GPS is used to measure long baselines (the ionosphere-float case). In this case it is simply not possible to quickly resolve the complete vector of ambiguities with sufficient probability.

As an alternative of resolving the complete vector of ambiguities, one might consider resolving only a subset of the ambiguities. This idea of partial ambiguity resolution was introduced in [Teunissen et al., 1999], where it was applied to long baselines using the current GPS. The idea is based on the fact that the success rate will generally increase when fewer integer constraints are imposed. Fewer epochs will then be needed for partial ambiguity resolution to be successful. However, in order to apply partial ambiguity resolution, one first will have to determine which subset of ambiguities to choose. It will be clear that this decision should be based on the precision of the ‘float’ ambiguities. The more precise the ambiguities, the larger the ambiguity success rate. When executing this principle however, the following two points should be taken into account. First, one should not restrict ones attention to the DD ambiguities only. After all, there might exist admissible integer linear combinations of the DD ambiguities which have an even better precision than the most precise DD ambiguity. Second, the identification of the most precise ambiguities depends on the model used (observation equations and vc-matrix of the data). That is, one linear combination may be optimal for one model, while another linear combination may be optimal for another model.

Our procedure for partial ambiguity resolution is as follows. We start with the complete vc-matrix of the ‘float’ solution, $Q_0$. This matrix contains all the relevant information. The LAMBDA method is then used to transform the original vector of DD ambiguities, $\hat{a}$, to a new ambiguity vector with corresponding vc-matrix:

$$\tilde{z} = Z^T \hat{a} \quad \text{and} \quad Q_\tilde{z} = Z^T Q_0 Z$$

The reason for applying the LAMBDA method is twofold. First, the decorrelating transformation matrix $Z^T$ returns ambiguities that are generally far more precise than the original DD ambiguities, while at the same time, their vc-matrix is close to being a diagonal matrix. The second reason for applying the LAMBDA method is that the method guarantees that the linear combinations so obtained are admissible. One should be aware of the fact that not every set of integer linear combinations of the DD ambiguities is admissible, see [Teunissen, 1995].

Once the transformed vc-matrix $Q_\tilde{z}$ is obtained, the construction of the subset proceeds in a sequential fashion. We first start with the most precise ambiguity, say $\tilde{z}_1$, and
compute its success rate $P(\hat{z}_1 = z_1)$, with $z_1$ being the expectation of $\hat{z}_1$, $z_1 = E[\hat{z}_1]$. If this success rate is large enough, we continue and determine the most precise pair of ambiguities, say $(\hat{z}_1, \hat{z}_2)$. If their success rate $P((\hat{z}_1, \hat{z}_2)^T = (z_1, z_2)^T)$ is still large enough, we continue again by trying to extend the subset. This procedure continues until we reach a point where the corresponding success rate becomes unacceptably small. When this point is reached, we know that the previously identified ambiguities can be resolved successfully.

Once the subset for partial ambiguity resolution has been identified, one still needs to determine what this will do to improve the estimator of the real-valued parameters in the model, such as the baseline, the ionospheric delays or, in our case, the DD range $\rho$. After all, being able to successfully resolve the ambiguities does not necessarily mean that the 'fixed' solution is significantly better than the 'float' solution. The extra effort needed to perform ambiguity resolution is, therefore, only worthwhile when a significant improvement is obtained.

### 4.2 Triple-frequency, single-ambiguity resolution

We will now apply the above procedure to the triple-frequency, geometry-free GPS model (8). The decorrelating ambiguity transformation obtained with the LAMBDA method reads

$$
\begin{bmatrix}
\hat{z}_1 \\
\hat{z}_2 \\
\hat{z}_3
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & -1 \\
1 & -3 & 2 \\
-24 & 147 & -122
\end{bmatrix}
\begin{bmatrix}
\hat{a}_1 \\
\hat{a}_2 \\
\hat{a}_3
\end{bmatrix}
$$

(11)

The three transformed ambiguities are ordered according to their precision. We start with the best determined ambiguity $\hat{z}_1$. It equals the widelane formed from the second and third DD ambiguity. Since its single-epoch success rate turns out to be large enough, namely $P(\hat{z}_1 = z_1) = 1.000$, instantaneous ambiguity resolution of this single ambiguity will be successful with a high probability.

In order to infer the effect of resolving this single ambiguity, we need the corresponding distribution of the 'fixed' DD range. It reads

$$
p_{\hat{z}_1}(x) = \sum_{u \in Z^2} p_{\hat{z}_1}(x \mid y = u)P(\hat{z}_1 = u)
$$

(12)

where the upper index in $p^{(1)}$ is used to indicate that only a single ambiguity, namely $\hat{z}_1$, is treated as an integer. This distribution is shown in figure 1 (right), together with the distribution of the 'float' solution $p$. Upon comparing the two distributions, we observe that the distribution of $p^{(1)}$ is only slightly more peaked than the one of $p$. The impact of resolving the single ambiguity $z_1$ is therefore almost negligible.

### 4.3 Triple-frequency, dual-ambiguity resolution

We will now try to resolve both $z_1$ and $z_2$. Note that $z_2$ equals the widelane of the first two DD ambiguities minus two times the widelane of the last two DD ambiguities. The single-epoch success rate of resolving both $z_1$ and $z_2$ reads

$$
P((\hat{z}_1, \hat{z}_2)^T = (z_1, z_2)^T) = 0.836.\text{ Although this success rate is still far much larger than the one encountered when resolving all three ambiguities (see section 3.2), for reliable ambiguity resolution its value is unfortunately too small. In}
$$

about 16 per cent of the cases one will still obtain the wrong integers. This effect is also visible when one considers the corresponding distribution of the 'fixed' DD range. This distribution is given as

$$
p_{\hat{z}_1}(x) = \sum_{u \in Z^2} p_{\hat{z}_1}(x \mid y = u)P((\hat{z}_1, \hat{z}_2)^T = u)
$$

(13)

and it is shown in figure 2 (left). The two side-lobes of $p_{\hat{z}_1}(x)$, which were already visible in $p_{\hat{z}_1}(x)$ (see figure 1, left), can now be explained as follows. To a good approximation, these two side-lobes make up for the remaining 16 percent of the ambiguity probability mass function. This implies that apart from the correct integer vector $(z_1, z_2)^T$, only two other integer vectors have a significant change of being selected by the integer least-squares estimator. These two vectors are located reflection-symmetric with respect to the correct integer vector. They are given in the transformed ambiguity space as

$$
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix} =
\begin{bmatrix}
z_1 \\
z_2
\end{bmatrix} + \begin{bmatrix}0 & \pm 1
\end{bmatrix}
$$

(14)

### 4.4 Elimination of the side-lobes

The two side-lobes of $p_{\hat{z}_1}(x)$ in figure 2 (left) will disappear, when the success rate approaches 1.0. This can be achieved, when one may consider a sufficiently improved measurement precision, in particular of the pseudo ranges, or, when a sufficient number of additional epochs is taken into account. Since we would like to stick to the rather conservative value of the pseudo range precision, we will only consider the second option. Although the instantaneous success rate of 0.836 was too small for dual-ambiguity resolution to be reliable, it is large enough to expect that not too many additional epochs are needed to push this success rate close enough to its optimal value of one. And indeed, in the present case, six epochs already suffice to obtain a success rate of 0.999, and with more epochs, it will even get larger still. This shows that fast and reliable ambiguity resolution over long baselines is possible with modernized GPS, provided only two of the three ambiguities, namely $z_1$ and $z_2$, are assumed integer. What remains to be shown, is that it also pays off in the sense of significantly improving the DD range estimator. For that purpose we have shown in figure 2 (right) the distributions of $\hat{\rho}^{(2)}$ and $\hat{\rho}$, both based on ten epochs. The corresponding success rate equals 1.000. The figure clearly shows the absence of the two side-lobes. Their probability masses are now included in the central lobe of $p_{\hat{\rho}}(x)$. As the figure shows, the distribution of $\hat{\rho}^{(2)}$ is significantly more peaked than the distribution of the corresponding 'float' solution. This improvement is of course not as dramatic as it would have been with complete ambiguity resolution. Nevertheless it is still very significant, since it brings the standard deviation of the DD ranges
down from 40 cm (‘float’ solution) to about 15 cm (‘fixed’ solution). The important conclusion is therefore reached, that fast, triple-frequency, dual-ambiguity resolution is feasible and that it will enable one to significantly improve the DD range solution for long baselines.

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