Integer Ambiguity Resolution with Nonlinear Geometrical Constraints

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Abstract

Integer ambiguity resolution is the key to obtain very accurate positioning solutions out of the GNSS observations. The Integer Least Squares (ILS) principle, a derivation of the least-squares principle applied to a linear system of equations in which some of the unknowns are subject to an integer constraint, was demonstrated to be optimal among the class of admissible integer estimators. In this contribution it is shown how to embed into the functional model a set of nonlinear geometrical constraints, which arise when considering a set of antennae mounted on a rigid platform. A method to solve for the new model is presented and tested: it is shown that the strengthened underlying model leads to an improved capacity of fixing the correct integer ambiguities.

Keywords

Constrained methods • GNSS • Integer ambiguity resolution

1 Introduction

The GNSS (Global Navigation Satellite System) observations are obtained tracking a number of satellites: both the code and carrier phase data are used to estimate the antennae positions. Because only the fractional part of the phase carrier observations can be measured, an ambiguity must be resolved for each incoming signal in order to fully exploit the capabilities of the GNSS positioning: by resolving the ambiguities one is able to achieve higher accuracies than using only the code data. The set of GNSS observations is usually cast into a (overdetermined) system of linearized equations, and the theory of Integer Least-Squares (ILS) (Teunissen 1993) is applied to solve for the linearized model in a least-squares sense, with a subset of the unknowns being integer-valued, namely the phase carrier ambiguities. An efficient implementation of the ILS was proposed in Teunissen (1994): the LAMBDA (Least-squares AMBiguity Decorrelation Adjustment) method is currently widely used for its high efficiency. For those applications where a subset of the real-valued unknowns is subject to geometrical constraints, one faces a substantial complication for the solution of the constrained ILS

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problem. A modification of the LAMBDA method was recently proposed in Teunissen (2006), Teunissen (2008), Teunissen (2010), Park and Teunissen (2003), Buist (2007), Park and Teunissen (2008), Giorgi et al. (2008) and Giorgi and Buist (2008) to solve for single-baseline constrained problems. We investigate in this contribution how to resolve for the integer ambiguities when a set of two or more antennae are mounted on the same rigid platform, with their relative positions known and constant. The problem was originally addressed in Teunissen (2007): the peculiar set of geometrical constraints posed on the baselines vectors is tackled by introducing a suitable parameterization of the baseline coordinates, and a modified cost function to be minimized in an ILS sense is introduced. It is shown here how to efficiently proceed for the search of the integer minimizer of the modified objective function, and a numerical evaluation of the capabilities of the constrained ILS is given: the single-frequency, single-epoch success rate is investigated.

2 Modeling of the GNSS Observables

Assuming two antennae tracking the same $n+1$ GNSS satellites, the set of single frequency, linearized double difference (DD) GNSS observations for the baseline at a given epoch is described via a Gauss-Markov model (Teunissen and Kleusberg 1998)

$$E(y) = Az + Gb \quad z \in \mathbb{Z}^n; b \in \mathbb{R}^p$$
$$D(y) = Q_y \tag{6.1}$$

where $E(\cdot)$ is the expectation operator, y is the vector of code and carrier phase observables (order $2n$), z contains the n integer-valued ambiguities and b is the vector of remaining p real-valued unknowns. Here, we restrict ourselves to short baseline applications, assuming the three baseline coordinates as the only real-valued unknowns ($p = 3$). A and G are the design matrices which link the observables with the vectors of unknowns: A contains the carrier wavelengths, while G is the matrix of line-of-sight vectors.

$D(\cdot)$ is the dispersion operator: a Gaussian-distributed error is assumed on the vectors of observables, characterized by the variance-covariance (v-c) matrix Q_y.

We consider in this work a set of $m+1$ antennae tracking the same $n+1$ GNSS satellites: we cast the set of GNSS DD observations collected at the different m independent baselines into a unique frame, thus formulating a multivariate model (Teunissen 2007) as

$$E(Y) = AZ + GB \quad Z \in \mathbb{Z}^{n\times m}; B \in \mathbb{R}^{3\times m}$$
$$D(vec(Y)) = Q_Y \tag{6.2}$$

where Y is the $2n$ by m matrix whose columns are the code and phase observations from each baseline, Z is the matrix containing the nm integer-valued ambiguities and B is the matrix of remaining $3m$ real-valued unknowns, i.e. the matrix whose columns are the coordinates of each baseline. The relative distances between the antennae are assumed to be short, so that the deviations between the different line-of-sight vectors as seen from each antenna can be disregarded and the same matrix of line-of-sight vectors G is used. The vec operator is here introduced in order to define the v-c matrix of the observables: it stacks the columns of the $2n$ by m matrix Y into a vector of order $2nm$. The dispersion of the vector $vec(Y)$ is characterized by the v-c matrix Q_Y.

We study in this contribution how to embed a set of nonlinear geometrical constraints posed on the $3m$ real-valued entries of B. We assume that the antennae are firmly mounted on the same rigid platform, and their relative distances are completely known. This results in two types of constraints to be considered: the baseline lengths and their relative orientation are known and constant. The hypothesis of constant length constrains the extremity of each baseline vector to lie on the surface of a sphere of radius equal to the baseline length; this reduces the number of independent baseline coordinates from $3m$ to $2m$. Due to the invariance of the antennae relative positions, the set of admissible baseline coordinates is described by a rigid rotation, and the real-valued unknowns to be determined are drastically reduced to three (two in the case of single-baseline) by virtue of the Euler's rotation theorem (Goldstein 1980). A suitable parameterization for the baseline coordinates is necessary to efficiently describe the characteristics of the baseline-constrained problem. To this purpose we introduce a frame of body axis ($u_1 u_2 u_3$) defined by the antennae placement. The first body axis is aligned with the first baseline, the second body axis is perpendicular to the first, lying in the plane formed by the first two baselines, and the third body axis is directed so that $u_1 u_2 u_3$ form a right-handed orthogonal frame. The relation between the baseline coordinates expressed in the body frame $u_1 u_2 u_3$ (F) and a reference frame $x_1 x_2 x_3$ (B) under the hypothesis of rigid rotations is

$$B = R \cdot F \tag{6.3}$$

where the rotation matrix R, which describes the relative orientation of the two systems, defines a linear transformation $\mathbb{R}^{3\times m} \rightarrow \mathbb{R}^{3\times m}$. Due to the invariance of both the baselines lengths and their relative positions, the relation $B^T B = F^T F$ holds true; multiplying both the terms of (6.3) for B^T, we obtain $B^T B = F^T R^T R F$: hence the matrix R has to be orthogonal ($R^T R = I$). In order to avoid loss of generality when only two or three antennae are available, we define the rotation matrix as (Teunissen 2007)

$$\begin{aligned}
m \geq 3: \quad RF &= [r_1, r_2, r_3] \begin{bmatrix} f_{11} & f_{21} & f_{31} & \cdots & f_{m1} \\ 0 & f_{22} & f_{32} & \cdots & f_{m2} \\ 0 & 0 & f_{33} & \cdots & f_{m3} \end{bmatrix} \\
m = 2: \quad RF &= [r_1, r_2] \begin{bmatrix} f_{11} & f_{21} \\ 0 & f_{22} \end{bmatrix} \\
m = 1: \quad RF &= [r_1] [f_{11}]
\end{aligned} \tag{6.4}$$

with r_i the i-th column of R and f_{ij} (scalar) the entries of F. We introduce for notational convenience the parameter q, to indicate the second dimension of R: $q = m$ for $m < 3$ and $q = 3$ for $m \geq 3$.

By the use of the rotation matrix, the problem of estimating the $3m$ baseline coordinates turns into the problem of estimating the $3q \leq 3m$ entries of an orthogonal matrix R, of which only three (two for a single baseline) are independent. The multivariate constrained model is then formulated as (Teunissen 2007):

$$\begin{aligned}
E(Y) = AZ + GRF \quad & Z \in \mathbb{Z}^{n\times m}; R \in \mathbb{O}^{3\times q} \\
D(vec(Y)) = Q_Y = P_m \otimes Q_y &
\end{aligned} \tag{6.5}$$

where R describes the orientation of the body frame with respect to the frame wherein the GNSS measurements are obtained. The unknowns to be resolved are the nm integer-valued ambiguities and the three (or two in case of single-baseline) real-valued independent entries of R, which must belong to the class of 3 by q orthogonal matrices $\mathbb{O}^{3\times q}$. We assume that the different baseline observations are described by the same v-c matrix Q_y, and the dispersion of the matrix of observables Y is obtained via a Kronecker product between Q_y and the m by m matrix P_m, which defines the correlation between the baselines.

3 Constrained Integer Least-Squares

The Integer Least-Squares estimator for the solution of the system (6.1) was demonstrated to be optimal among the class of admissible integer estimators (Teunissen 1999). A closed-form solution of the ILS is not known: hence, a least-squares minimization implies an exhaustive search over a set of integer candidates. The LAMBDA method is a well-known and efficient implementation of the ILS, introduced in Teunissen (1993) and Teunissen (1995). The nonlinear constraints posed on the baseline coordinates strongly affects the resolution technique to be adopted, and a new formulation of the LAMBDA method is presented here. To express the model (6.5) in a vectorial form, we again make use of the vec operator:

$$\begin{aligned}
E\,(vec(Y)) &= \left[(I_m \otimes A) \; (F^T \otimes G) \right] \begin{pmatrix} vec(Z) \\ vec(R) \end{pmatrix} \\
& \qquad Z \in \mathbb{Z}^{n\times m}; R \in \mathbb{O}^{3\times q} \\
D(vec(Y)) &= P_m \otimes Q_y
\end{aligned} \tag{6.6}$$

We want to solve the system (6.6) in a least-squares sense, therefore minimizing the squared norm of the residuals with respect to the integer-valued matrix Z. The squared norm and its sum-of-squares decomposition reads (Teunissen 2007):

$$\begin{aligned}
& \left\| vec(Y) - (I_m \otimes A) vec(Z) - (F^T \otimes G) vec(R) \right\|^2_{P \otimes Q_y} \\
& = \left\| vec(\hat{E}) \right\|^2_{P_m \otimes Q_y} + \left\| vec\left(Z - \hat{Z}\right) \right\|^2_{Q_{\hat{Z}}} \\
& \quad + \left\| vec\left(\hat{R}(Z) - R\right) \right\|^2_{Q_{\hat{R}(Z)}}
\end{aligned} \tag{6.7}$$

where $\|\cdot\|^2_Q = (\cdot)^T Q^{-1} (\cdot)$ is the weighted squared norm and $\hat{Z}$ and $\hat{R}$ are the float solutions of the unknowns, i.e. the least-squares solution of (6.6) obtained without imposing any constraint on Z or R. $\hat{E}$ is the matrix of least-squares residuals, while $\hat{R}(Z)$ is the float estimator of R given the ambiguity matrix Z known. $Q_{\hat{Z}}$ is the v-c matrix of the float solution $vec(\hat{Z})$, while the v-c matrix $Q_{\hat{R}(Z)}$ defines the dispersion of $vec(\hat{R}(Z))$. Due to the constraints posed on Z and B, the last two terms of (6.7) cannot in general be made zero for any value of Z; thus the minimization problem must be taken with respect to both the integer matrix Z and the orthogonal matrix R:

$$\check{Z} = \arg \min_{Z \in \mathbb{Z}^{n \times m}} C(Z)$$
$$C(Z) = \left\| vec(Z - \hat{Z}) \right\|^2_{Q_{\hat{Z}}} + \left\| vec(\hat{R}(Z) - \check{R}(Z)) \right\|^2_{Q_{\hat{R}(Z)}} \quad (6.8)$$

with

$$vec(\check{R}(Z)) = \arg \min_{R \in \mathbb{O}^{3 \times q}} \left\| vec(\hat{R}(Z) - R) \right\|^2_{Q_{\hat{R}(Z)}} \quad (6.9)$$

The evaluation of the cost function $C(Z)$ involves the computation of two correlated terms: the first is the distance between Z and the float solution $\hat{Z}$, weighted by the v-c matrix $Q_{\hat{Z}}$, and the second is the distance between the conditional solution $\hat{R}(Z)$ and the minimizer of the constrained nonlinear least-squares problem (6.9).

The solution of the minimization problem (6.8) provides the fixed matrix of integer ambiguities $\check{Z}$ by taking advantage of the geometrical constraints expressed by the orthogonality of $\check{R}(Z)$. Solving the problem (6.9) for $Z = \check{Z}$ then gives the least squares estimation of the attitude of the body axis $\check{R}(\check{Z})$, i.e. the orientation of the set of m baselines with respect the frame of axes wherein the GNSS observation are taken. Since no analytical solution for the integer minimizer of (6.8) is known, a direct search method must be employed. The integer matrix which provides the smallest value for $C(Z)$ is exhaustively searched inside the set of integer candidates defined as

$$\Omega\left(\chi^2\right) = \{Z \in \mathbb{Z}^{n \times m} \mid C(Z) \leq \chi^2\} \quad (6.10)$$

where χ is a scalar chosen as to limit the search space $\Omega(\chi^2)$. The shape of set $\Omega(\chi^2)$ is driven by the matrices $Q_{\hat{Z}}$ and $Q_{\hat{R}(Z)}$ in (6.8): if $Q_{\hat{R}(Z)} \to 0$, the set would be ellipsoidal, as follows from the relation $\left\| vec(Z - \hat{Z}) \right\|^2_{Q_{\hat{Z}}} \leq \chi^2$. The tight relation between the two terms of (6.8) complicates the evaluation of the shape of the search space for $Q_{\hat{R}(Z)} \neq 0$.

We now focus on the three steps involved in the computation of the minimizer of (6.8): the derivation of the float solution, the search for the integer minimizer and the computation of the constrained nonlinear least-squares problem (6.9).

3.1 The Float Estimators

The float estimators $\hat{Z}$ and $\hat{R}$ are the least-squares solution of the system (6.6) when disregarding the integerness of the ambiguities and the orthogonality of R. These are obtained by solving the set of normal equations

$$N \begin{pmatrix} vec(\hat{Z}) \\ vec(\hat{R}) \end{pmatrix} = \begin{bmatrix} P_m^{-1} \otimes A^T Q_y^{-1} \\ F P_m^{-1} \otimes G^T Q_y^{-1} \end{bmatrix} vec(Y)$$
$$N = \begin{bmatrix} P_m^{-1} \otimes A^T Q_y^{-1} A & P_m^{-1} F^T \otimes A^T Q_y^{-1} G \\ F P_m^{-1} \otimes G^T Q_y^{-1} A & F P_m^{-1} F^T \otimes G^T Q_y^{-1} G \end{bmatrix} \quad (6.11)$$

The inversion of the normal matrix N provides the v-c matrices of the float solutions $vec(\hat{Z})$ and $vec(\hat{R})$:

$$\begin{bmatrix} Q_{\hat{Z}} & Q_{\hat{Z}\hat{R}} \\ Q_{\hat{R}\hat{Z}} & Q_{\hat{R}} \end{bmatrix} = N^{-1} \quad (6.12)$$

If we assume the matrix of ambiguities known, $\hat{R}(Z)$ and the associated v-c matrix are obtained as

$$vec(\hat{R}(Z)) = vec(\hat{R}) - Q_{\hat{R}\hat{Z}} Q_{\hat{Z}}^{-1} vec(\hat{Z} - Z)$$
$$Q_{\hat{R}(Z)} = Q_{\hat{R}} - Q_{\hat{R}\hat{Z}} Q_{\hat{Z}}^{-1} Q_{\hat{Z}\hat{R}} \quad (6.13)$$

Thus, the knowledge of the fixed matrix of ambiguities improves the precision of $\hat{R}(Z)$: the dispersion is reduced according to (6.13).

3.2 The Search for the Integer Ambiguities

As stated above, the minimization problem (6.8) can in principle be solved with an extensive search in the search space $\Omega\left(\chi^2\right)$: this is a non-trivial task if one aims to have an efficient and fast search. The choice for the scalar χ in (6.10) is critical: it must be large enough to guarantee the non-emptiness of $\Omega(\chi^2)$, but not too large to avoid onerous computational burdens due to the large number of integer candidates for which the solution of (6.9) must be evaluated. Setting the value of χ by picking up an integer matrix Z' and computing

$$\chi^2 = C(Z') \tag{6.14}$$

generally leads to unacceptable large values for χ, for which the computational burden is too heavy. This is due to the fact that the matrix $Q_{\hat{R}(Z)}$ is driven be the more precise phase measurements, and the second term of (6.8) largely amplifies the values of χ for any non-correct value of Z. An alternative approach to the extensive search in $\Omega(\chi^2)$ is to make use of approximating functions that are easier to evaluate than $C(Z)$, and a modification of the LAMBDA method is here proposed. In analogy with the bounding functions introduced for the single-baseline ($m = 1$) case in Teunissen (2006), we note that the expression (6.9) can be bounded via the smallest (λ_m) and largest (λ_M) eigenvalues of the matrix $Q^{-1}_{\hat{R}(Z)}$:

$$\begin{gathered} C_1(Z) \le C(Z) \le C_2(Z) \\ C_1(Z) = \left\|vec(Z - \hat{Z})\right\|^2_{Q_{\hat{Z}}} + \lambda_m \sum_{i=1}^{q} \left(\left\|\hat{r}_i(Z)\right\| - 1\right)^2 \\ C_2(Z) = \left\|vec(Z - \hat{Z})\right\|^2_{Q_{\hat{Z}}} + \lambda_M \sum_{i=1}^{q} \left(\left\|\hat{r}_i(Z)\right\| + 1\right)^2 \end{gathered} \tag{6.15}$$

where $\hat{r}_i(Z)$ is the i-th column of $\hat{R}(Z)$ and the inequalities are derived from the rules of the scalar product between vectors. A clever strategy to quicken the search is to make use of these two bounds, and two efficient search strategies for the constrained ILS minimization have been developed (Buist 2007; Giorgi et al. 2008; Giorgi and Buist 2008): the methods were coined the *Expansion* approach and the *Search and Shrink* approach, respectively. The *Expansion* approach works by initially enumerating all the integer matrices contained in a small set of admissible candidates

$$\Omega_{exp}\left(\chi_0^2\right) = \{Z \in \mathbb{Z}^{n\times m} \mid C_1(Z) \le \chi_0^2\} \supseteq \Omega\left(\chi_0^2\right) \tag{6.16}$$

where the scalar χ_0 is initially chosen small enough and iteratively increased until, at step s, the set $\Omega_{exp}\left(\chi_s^2\right)$ turns out to be non-empty: as the evaluation of $C_1(Z)$ only involves the computation of two squared norms, the enumeration proceeds rather quickly. For each of the enumerated integer matrices in $\Omega_{exp}\left(\chi_s^2\right)$, the problem (6.9) is solved and the set $\Omega\left(\chi_s^2\right)$ is evaluated: if it is empty, the scalar χ_s is increased to $\chi_{s+1} > \chi_s$ and the enumeration in $\Omega_{exp}\left(\chi_{s+1}^2\right)$ repeated, otherwise the minimizer of $C(Z)$ is picked up.

A second strategy developed is a *Search and Shrink* approach: a second set is defined as

$$\Omega_{SaS}\left(\chi_0^2\right) = \{Z \in \mathbb{Z}^{n\times m} \mid C_2(Z) \le \chi_0^2\} \subseteq \Omega\left(\chi_0^2\right) \tag{6.17}$$

where χ_0 is chosen large enough to guarantee the non-emptiness of $\Omega_{SaS}(\chi_0^2)$. The search proceeds by iteratively shrinking the set, by means of searching for an integer matrix Z_{s+1} in $\Omega_{SaS}\left(\chi_s^2\right)$ which provides a smaller value for $\chi_{s+1}^2 = C_2(Z_{s+1}) < C_2(Z_s) = \chi_s^2$, until the minimizer of $C_2(Z)$ is found. The minimizer of $C(Z)$, which may differ from the one of $C_2(Z)$, is then extensively searched inside the shrunken set

$$\Omega\left(\overline{\chi}^2\right) = \{Z \in \mathbb{Z}^{n\times m} \mid C(Z) \le \overline{\chi}^2\} \supseteq \Omega_{SaS}\left(\overline{\chi}^2\right) \tag{6.18}$$

where $\overline{\chi}^2 = C_2(\overline{Z})$, being $\overline{Z}$ the minimizer of $C_2(Z)$. The two search strategies provide an efficient alternate way of performing the search for the integer minimizer of (6.8), overtaking both the issues of fixing the initial size of the search space and speeding up the search avoiding the computation of (6.9) a large number of times.

3.3 Solving the Nonlinear Least-Squares Problem

The evaluation of the function $C(Z)$ at a given point Z implies the solution of the nonlinear constrained least squares problem (6.9). Geometrically, it consists to find the closest point between a given data vector $vec(\hat{R}(Z))$ and a curved manifold of dimension $q + 1$ embedded in the $3q$-dimensional space, where the metric is defined by the v-c matrix $Q_{\hat{R}(Z)}$. The manifold, which reflects the nonlinearity of the problem, is defined by the constraints equations $R^T R = I$. Making use of one of the representations that can be employed for the three-dimensional rotations needed to coalesce two orthogonal frames, such as the Gibbs vector, the Direct Cosine Matrix, the Quaternions or the Euler angles (Battin 1987), the vector $vec(\hat{R}(Z) - R)$ can be rewritten as a set of $3q$-nonlinear functions of a vector of independent unknowns γ, for which the orthogonal constraint on $R(\gamma)$ is implicitly fulfilled. The non-linear least-squares problem can then be solved by an iterative technique such as the Gauss-Newton method.

4 Simulation Results

The proposed constrained ILS method was tested with simulated data: the simulation inputs are summarized in Table 6.1. Each of the 24 scenarios was processed with the unconstrained LAMBDA, disregarding the geometrical constraints, and the Constrained LAMBDA method, taking into account the orthogonality on R. The latter was applied on both a single baseline case and a two-baselines case: this to demonstrate the improvement when the number of geometrical constraints increases. Table 6.2 reports for the different methods the single-frequency, single-epoch success rate, which is defined as the ratio of correctly fixed matrix of ambiguities over the set of 10^5 samples simulated. The improvement in success rate was dramatic: especially for the weaker scenarios (lower number of satellite / higher noise levels) the difference between the methods was rather large, e.g. the weakest simulated dataset, with five available satellites and high noise values, showed an increment from a low 3% to 72% for the single baseline case, up to 99.6% for the two-baselines case. As expected, the strengthening of the underlying model due to the embedded geometrical constraints substantially affects the capacity of fixing the correct integer ambiguity matrix: only two baselines were indeed sufficient to obtain single-frequency, single-epoch success rates higher than 99% on all the data sets processed, obtaining a 100% success rate on 20 out of 24 data sets simulated.

Table 6.1 Simulation set up

Frequency	L1
Number of Satellite (PRNs) 5 / 6 / 7 / 8	Corresponding PDOP 4.19 / 2.14 / 1.92 / 1.81
Undifferenced code noise σ_p $[cm]$	30 - 15 - 5
Undifferenced phase noise σ_ϕ $[mm]$	3 - 1
Baselines f_i (x_1, x_2, x_3)	$f_1 = [1, 0, 0]$ m $f_2 = [-0.35, 1.97, 0]$ m
Samples simulated	10^5

Table 6.2 Simulation results: single-frequency, single-epoch success rates for the unconstrained and constrained LAMBDA methods. Success rates higher than 99.9% are stressed

σ_ϕ $[mm]$	3			1		
σ_p $[cm]$	30	15	5	30	15	5
N	*Single-baseline success rate, unconstrained LAMBDA* *Single-baseline success rate, Constrained LAMBDA* *Two-baselines success rate, Constrained LAMBDA*					
5	3.30	19.05	86.67	5.99	26.89	95.37
	72.43	88.86	99.63	96.54	**99.94**	**100**
	99.60	**99.94**	**100**	**100**	**100**	**100**
6	24.83	66.71	96.89	49.13	86.67	**99.99**
	95.75	99.18	99.90	**99.99**	**100**	**100**
	99.99	**100**	**100**	**100**	**100**	**100**
7	50.24	79.69	99.53	74.17	93.27	**100**
	99.34	**99.97**	**100**	**100**	**100**	**100**
	100	**100**	**100**	**100**	**100**	**100**
8	86.17	94.48	**99.99**	**99.97**	**99.99**	**100**
	99.80	**99.99**	**100**	**100**	**100**	**100**
	100	**100**	**100**	**100**	**100**	**100**

Conclusion

The problem of resolving the integer ambiguities which affect the GNSS carrier phase observations is the key to precise relative positioning. The LAMBDA method, which mechanizes the ILS principle, is used to efficiently and reliably fix the ambiguities. When the geometry of the antennae placement is known and constant, nonlinear constraints can be included in the theory, for the purpose of strengthening the model and improving the capacity of fixing the correct integer ambiguities. We proposed in this contribution a model for the GNSS observations which embeds the whole set of nonlinear geometrical constraints arising when considering frame of antennae of invariant relative positions. The cost function to be minimized in a ILS sense has been modified: in order to solve the minimization problem respecting both the integer and orthogonality constraints, a modification of the LAMBDA method is proposed and the integer matrix of ambiguities is searched via one of the two iterative search approaches depicted. Both the *Expansion* and the *Search and Shrink* algorithms can be applied to perform the search, resulting in a faster and more efficient approach than the extensive search. We tested the proposed method on different simulated data sets, investigating the influence of the number of available satellite and the noise levels on the code and phase observations: the difference when using the unconstrained LAMBDA and the Constrained LAMBDA is dramatic, with a large improvement in the capacity of resolving the correct integer matrix, especially for the scenarios characterized by lower number of available satellites/higher noise levels.

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